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Group theory

Garside families in Artin–Tits monoids and low elements in Coxeter groups

Familles de Garside dans les monoïdes d'Artin–Tits et éléments bas d'un groupe de Coxeter

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ABSTRACT

We show that every finitely generated Artin–Tits group admits a finite Garside family, by introducing the notion of a *low element* in a Coxeter group and proving that the family of all low elements in a Coxeter system (W, S) with S finite includes S and is finite and closed under suffix and join with respect to the right weak order.

RÉSUMÉ

Nous montrons que tout groupe d'Artin–Tits finiment engendré possède une famille de Garside finie, en introduisant la notion d'*élément bas* dans un groupe de Coxeter et en prouvant que, si (W, S) est un système de Coxeter avec S fini, l'ensemble des éléments bas de W inclut S et est fini et clos par suffixe et borne supérieure dans l'ordre faible à droite.
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Les groupes d'Artin–Tits sont les groupes définis par des relations de la forme $sts\ldots = tst\ldots$ avec les deux membres de même longueur. Étudiés par J. Tits dans les années 1960, ces groupes sont encore mal compris, et même la décidabilité du problème de mot reste ouverte dans le cas général [15]. Le seul cas bien compris est celui des groupes de type sphérique, où le groupe de Coxeter associé obtenu en ajoutant les relations de torsion $s^2 = 1$ est un groupe fini. Dans ce cas, le groupe d'Artin–Tits est un groupe de Garside et tout élément du groupe possède une décomposition distinguée (« forme normale gloutonne ») en termes des diviseurs d'un certain élément Δ (« élément de Garside ») [14, Chapter 9].

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Il est récemment apparu que de telles décompositions existent dans le contexte plus général des *familles de Garside* : dès que F est une famille de Garside dans un monoïde simplifiable M, le mécanisme de la forme normale gloutonne fonctionne et fournit des décompositions avec de bonnes propriétés [7,8].

Une monoïde d'Artin–Tits de type non sphérique n'est jamais un monoïde de Garside, et ne possède jamais de famille de Garside bornée. Ceci ne dit rien sur des familles de Garside non bornées, en particulier finies, qui sont les plus intéressantes en termes d'applications effectives. Le but de cette note est d'annoncer une démonstration du résultat suivant précédemment conjecturé :

Théorème 1. *Tout monoïde d'Artin–Tits finiment engendré possède une famille de Garside finie.*

La démonstration du théorème 1 repose sur la traduction du problème dans le langage des groupes de Coxeter et l'introduction d'une notion d'élément *bas* définie en termes de l'action sur le système de racines associé. Le théorème 1 découle du Théorème 2 ci-dessous.

Théorème 2. *Pour tout système de Coxeter (W, S) avec S fini, l'ensemble des éléments bas de W inclut S et est fini et clos par borne supérieure par rapport à l'ordre faible à droite et par suffixe.*

Il est standard d'associer à tout système de Coxeter (W, S) un système de racines (Φ, Δ) [13,16] : on a alors une représentation fidèle de W comme groupe de réflexions d'un espace vectoriel V, chaque élément s de S étant associé à la réflexion par rapport à une racine simple α_s. On pose $\Delta = \{\alpha_s \mid s \in S\}$ et $\Phi = W(\Delta)$ (les *racines*), puis, en notant $\mathrm{cone}(X)$ l'ensemble des combinaisons linéaires à coefficients positifs d'éléments de X, on pose, $\Phi^+ = \mathrm{cone}(\Delta) \cap \Phi$ (les racines *positives*), et $\Phi^- = -\Phi^+$ (les racines *négatives*). Pour w dans W, on introduit l'ensemble des inversions à gauche $N(w) := \Phi^+ \cap w(\Phi^-)$. Pour α, β dans Φ, on dit que $\alpha \preccurlyeq \beta$ est vérifié si on a $\forall w \in W\ \ (w(\beta) \in \Phi^- \Rightarrow w(\alpha) \in \Phi^-)$, et on appelle *petite* une racine β ne vérifiant $\alpha \preccurlyeq \beta$ que pour $\alpha = \beta$. Suivant Brink et Howlett [4], l'ensemble Σ des petites racines de Φ inclut Δ et est fini.

Définition 3. Un élément w de W est *bas* s'il existe une famille de petites racines A vérifiant $N(w) = \mathrm{cone}(A) \cap \Phi$. On note L l'ensemble des éléments bas de W.

Dans la démonstration du théorème 2, celle du fait que L inclue S et soit clos par borne supérieure pour l'ordre faible à droite est facile, celui qu'il soit fini découle de la finitude de l'ensemble Σ des petites racines. Enfin, le fait que L soit clos par suffixe est plus délicat, et repose sur celui que l'ensemble des petites racines est *bipodal* au sens où, si Φ' est un système de racines de rang 2 inclus dans Φ qui est maximal et contient une petite racine, alors la base Δ' de Φ' doit être incluse dans Σ.

1. Introduction

Artin–Tits groups, also known as Artin groups, are those groups defined by relations of the form $sts\ldots = tst\ldots$ where both terms have the same length. First investigated by J. Tits in the 1960s, these groups remain incompletely understood, with many open questions, including the decidability of the Word Problem in the general case [15]. The only well-understood case is the one of *spherical type*, namely when the associated Coxeter group, obtained by adding the relations $s^2 = 1$, is finite. A large part of the results in this case follows from the fact that the group is then a Garside group, and the corresponding monoid is a Garside monoid [5,6].

At the heart of the properties of an Artin–Tits monoid of spherical type—and more generally of a Garside monoid—lies the fact that every element of the latter admits a distinguished decomposition ("greedy normal form") involving the divisors of a certain element Δ ("Garside element"), in which each entry is in a sense maximal [14, Chapter 9]. It was recently realized that such distinguished decompositions exist in the more general framework of *Garside families*: whenever F is a Garside family in a left-cancellative monoid M (or category), the mechanism of the greedy normal form works and provides distinguished decompositions with nice properties [7,8]. A Garside monoid corresponds to a Garside family consisting of the divisors of a single element Δ ("bounded Garside family").

An Artin–Tits monoid of non-spherical type is not a Garside monoid: the projection of a possible Garside element to W should be a longest element of W, which cannot exist. This however says nothing about unbounded Garside families in M, in particular about possible finite Garside families, which are the interesting ones in view of effectivity results. The aim of this note is to announce a proof of the following, previously conjectured, statement, which was supported by partial results and computer experiments.

Theorem 1.1. *Every finitely generated Artin–Tits monoid admits a finite Garside family.*

The proof of Theorem 1.1 relies on translating the problem into the language of Coxeter groups and introducing the new notion of a *low element* by looking at the action on the associated root system. Then Theorem 1.1 follows from the next result, which is of independent interest and seems rich in potential further applications.

Table 1
The smallest Garside family F in the Artin–Tits monoid associated with the Coxeter system (W, S); when F is finite, it must consist of all right-divisors of the elements of some minimal finite set E (the "extremal elements").

Type of (W, S)	Spherical	Large no ∞	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{B}_3$	$\widetilde{C}_2$	$\widetilde{C}_3$
$\#E$	1	$3\binom{\#S}{3}$	3	10	35	14	3	12
$\#F$	$\#W$	$O((\#S)^3)$	16	125	1296	315	24	317

Theorem 1.2. *For every Coxeter system (W, S) with S finite, the set of all low elements of W includes S and is finite and closed under join (taken in the right weak order) and suffix.*

2. The Artin–Tits problem

If M is a (left-cancellative) monoid and f, g lie in M, one says that f *left-divides* g or, equivalently, that g is a *right-multiple* of f, written $f \preccurlyeq g$, if $fg' = g$ holds for some g' in M. If there is no nontrivial invertible element in M, that is, if $fg = 1$ holds only for $f = g = 1$, the left-divisibility relation is a partial ordering that is compatible with multiplication on the left.

Definition 2.1. (See [7,8].) If M is a left-cancellative monoid with no nontrivial invertible element, a *Garside family* of M is a family F containing 1 and such that every element of M admits a F-normal decomposition, meaning a finite sequence $(s_1, \ldots, s_p)$ satisfying $s_1 \cdots s_p = g$ and such that all entries lie in F and $\forall s{\in}F\ \forall f{\in}M\ (s \preccurlyeq f s_i s_{i+1} \Rightarrow s \preccurlyeq f s_i)$ holds for every $i < p$.

(Demanding that a Garside family contains 1 is not necessary, but it is convenient here, and harmless.) The intuition is that a sequence $(s_1, \ldots, s_p)$ is F-normal if every entry s_i lies in F and contains as much as possible of the remainder as it can (whence the word "greedy" often used in this context). In the context of Definition 2.1, the F-normal decomposition is unique (up to adding or deleting final 1s) when it exists and that, if F is a Garside family, $(s_1, \ldots, s_p)$ is F-normal if and only if the simplified condition $\forall s{\in}F\ (s \preccurlyeq s_i s_{i+1} \Rightarrow s \preccurlyeq s_i)$ holds for every $i < p$.

Various characterizations of Garside families are known, depending on the properties of the considered monoid. In the case of the Artin–Tits monoid M associated with a Coxeter system (W, S), the presentation of M by relations in which both members have equal length implies that M is strongly Noetherian, meaning that there exits a map $\lambda : M \to \mathbb{N}$ satisfying $\lambda(g) \neq 0$ for $g \neq 1$ and $\lambda(gh) \geqslant \lambda(g) + \lambda(h)$ for all g, h. On the other hand, by [2, Verkürzungslemma], M admits conditional right-lcms, that is, any two elements of M that admit a common right-multiple admit a right-lcm (least common right-multiple).

Now, by [7, Proposition 3.27], in a left-cancellative monoid M with no nontrivial invertible element that is strongly Noetherian and admits conditional right-lcms, a subfamily F of M is a Garside family if and only if it contains all atoms of M and is closed under right-lcm and right-divisor. Here an element g is called an *atom* if its only left-divisors are 1 and g, and a family F is called closed under right-lcm if the right-lcm of two elements of F belongs to F when it exists; similarly, F is closed under right-divisor if every right-divisor of an element of F belongs to F, where a *right-divisor* of g is any element f such that $g = g'f$ holds for some g'. A consequence is that, under the above assumptions, there exists a smallest Garside family in M, namely the closure of the atoms under right-lcm and right-divisor [7, Corollary 3.28].

Applying this in the case of an Artin–Tits monoid, we obtain:

Proposition 2.2. *A subfamily F of an Artin–Tits monoid M is a Garside family if and only if it contains all atoms of M and is closed under right-lcm and right-divisor. In particular, M admits a unique smallest Garside family, namely the closure of the atoms under right-lcm and right-divisor.*

Corollary 2.3. *An Artin-Tits monoid M with atom set S admits a finite Garside family if and only if the closure of S under right-lcm and right-divisor is finite.*

Table 1 and Proposition 5.1 below give some information about the smallest Garside family in a few Artin–Tits monoids.

3. Translation of the problem to Coxeter groups

The above considerations admit simple counterparts involving Coxeter groups. If M is an Artin–Tits monoid with atom set S, the quotient W of M obtained by adding the relations $s^2 = 1$ to the presentation is a Coxeter group. The canonical projection π from M to W is injective on S and (at the expense of identifying S with its image under π) the pair (W, S) is a Coxeter system. By Matsumoto's lemma, mapping a reduced decomposition of an element of W to the element of M admitting that decomposition provides a set-theoretic section σ of π from W to M, and its image $\underline{W}$ is a copy of W inside M.

For w in W, we denote by $\ell(w)$ the S-length of w in W, that is, the length of a reduced word for w in S. Then the product of two elements f, g of $\underline{W}$ lies in $\underline{W}$ if and only if $\ell(\pi(f)) + \ell(\pi(g)) = \ell(\pi(fg))$ holds in W. We recall that the *(right)* *weak order* on W is the partial order $\leqslant$ on W such that $u \leqslant w$ holds if and only if there exists v in W satisfying $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$, see [1, Chapter 3]. Now, $(W, \leqslant)$ is a complete meet-semilattice [1, Theorem 3.2.1], implying that, if two elements of u, v of W admit a common upper bound with respect to $\leqslant$, they admit a smallest one called *the join* $u \vee v$.

Lemma 3.1. *Assume that (W, S) is a Coxeter system and M is the associated Artin–Tits monoid.*

(i) *The copy $\underline{W}$ of W inside M is a Garside family of M.*
(ii) *If f, g lie in $\underline{W}$, then f left-divides g in M if and only if $\pi(f) \leqslant \pi(g)$ holds in W. Similarly, f right-divides g if and only if a reduced word for $\pi(f)$ is a suffix of a reduced word for $\pi(g)$.*
(iii) *If f, g lie in $\underline{W}$, then f and g have a right-lcm in M if and only if $\pi(f) \vee \pi(g)$ exists in W. In this case the right-lcm of f and g lies in $\underline{W}$ and is the image under σ of $\pi(f) \vee \pi(g)$.*

Proof. Point (i) follows from [7, Proposition 6.27], which says that W embeds in the monoid M' generated by W with the relations $fg = h$ for f, g, h satisfying $\ell(f) + \ell(g) = \ell(h)$ and that its image is a Garside family in M'. By [17], the monoid M' is M, and the image of W is $\underline{W}$. Next, translating the definition of left- and right-divisibility in M directly gives (ii). Finally, the characterization of a Garside family in an Artin–Tits monoid and (i) imply that $\underline{W}$ is closed under right-lcm in M. So, if two elements f, g of $\underline{W}$ admit a common right-multiple, hence a right-lcm, in M, the latter lies in $\underline{W}$, and, by (ii), its projection under π must be the join of $\pi(f)$ and $\pi(g)$, which therefore exists. Conversely, by (ii), if the join exists, its image under σ must be the right-lcm of f and g in $\underline{W}$. So (iii) is true. $\square$

Using the dictionary of Lemma 3.1, we deduce:

Proposition 3.2. *If (W, S) is a Coxeter system and M is the associated Artin–Tits monoid, the projection of the smallest Garside family of M is the smallest subfamily of W that includes S and is closed under join and suffix.*

Thus, in order to prove Theorem 1.1, it is sufficient to show that, if (W, S) is a Coxeter system with S finite, then there exists a finite subset of W that includes S and is closed under join and suffix.

4. Low elements in a Coxeter group

The above result will be established by introducing the notion of a *low element* in a Coxeter group and showing that the family of all low elements has the expected properties (Theorem 1.2).

From now on, (W, S) is a fixed Coxeter system with S finite. Let (Φ, Δ) be a based root system in (V, B) with associated Coxeter system (W, S) as in [13,16]. So, V is a real vector space, B is a symmetric bilinear form on V, and Δ is a subset of V consisting of one element α_s for each s in S (the *simple* roots). The map sending each s in S to the B-*reflection* in α_s extends to a faithful representation of W on V as the subgroup of the orthogonal group $O_B(V)$ generated by these reflections. We set $\Phi = W(\Delta)$ (the *roots*), $\Phi^+ = \mathrm{cone}(\Delta) \cap \Phi$ (the *positive* roots), and $\Phi^- = -\Phi^+$ (the *negative* roots). Here $\mathrm{cone}(X)$ means the set of all nonnegative linear combinations of elements of X.

For α a root, we denote by s_α the B-reflection associated with α. Then, for w in W, the *(left) inversion set $N(w)$ of w* is $\Phi^+ \cap w(\Phi^-)$, which is also $\{\alpha \in \Phi^+ \mid \ell(s_\alpha w) < \ell(w)\}$. Its cardinality is $\ell(w)$.

Lemma 4.1. (i) *For w in W and s in S satisfying $\ell(sw) < \ell(w)$, the element sw is a suffix of a reduced word for w and we have $N(w) = \{\alpha_s\} \sqcup s(N(sw))$ (disjoint union).*

(ii) *The map N is a monomorphism from $(W, \leqslant)$ to $(\mathcal{P}(\Phi^+), \subseteq)$, and $u \leqslant g$ is equivalent to $N(u) \subseteq N(g)$.*

(iii) *For u, v in W such that $u \vee v$ exists, $N(u \vee v) = \mathrm{cone}(N(u) \cup N(v)) \cap \Phi$ holds.*

Points (i) and (ii) can be found in [1, Chapter 3]. Point (iii) then follows by [11, Theorems 11.6 and 1.5]: if $u \vee v$ exists, then $N(u \vee v)$ is the smallest d-closed (i.e., convex) set in Φ^+ containing $N(u) \cup N(v)$, which is by definition $\mathrm{cone}(N(u) \cup N(v)) \cap \Phi$.

Introduced by Brink and Howlett in [4], the *dominance order* is the partial order $\preccurlyeq$ on Φ such that $\alpha \preccurlyeq \beta$ ("β dominates α") means $\forall w \in W \ \ (w(\beta) \in \Phi^- \Rightarrow w(\alpha) \in \Phi^-)$. A positive root β is called *small*[1] when β dominates no other positive root than itself, *i.e.*, if we have $\forall \alpha \in \Phi^+ \ (\alpha \preccurlyeq \beta \Rightarrow \alpha = \beta)$. We denote by Σ the set of small roots. It is proved in [4] that Σ is always finite; on the other hand, Σ admits the following recursive characterization.

Lemma 4.2. *(See [1,4].)* (i) *The set Δ is included in Σ.*

(ii) *For every β in $\Phi^+ \setminus \Delta$, there exists α in Δ satisfying $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\alpha) - 2$, or equivalently, $B(\alpha, \beta) > 0$. Then, for every such α, one has $\beta \in \Sigma$ if and only if $s_\alpha(\beta)$ lies in Σ and $B(\alpha, \beta) < 1$ holds.*

[1] These roots are also called *humble* or *elementary* in the literature. We adopt here the terminology of [1]. See [1, Notes, p. 130] for more details.

The following result is a restatement of a special case of Propositions 1.4 and 3.6 in [10]:

Lemma 4.3. *For w in W, let $N^1(w) := \{\alpha \in \Phi^+ \mid \ell(s_\alpha w) = \ell(w) - 1\} \subseteq N(w)$. Then $N^1(w)$ is the set of all α in Φ^+ such that the cone of $\{\alpha\}$ is an extreme ray of the polyhedral cone spanned by $N(w)$.*

Definition 4.4. An element w of W is *low* if $N^1(w) \subseteq \Sigma$ holds, *i.e.*, if we have $N(w) = \mathrm{cone}(A) \cap \Phi$ for some family of small roots A. We denote by L the set of low elements of W.

We can now sketch the proof of Theorem 1.2; a complete proof of this theorem can be found in [12].

Proof (sketch). First, the fact that L is finite follows from the finiteness of Σ and Lemma 4.1(ii): there is only a finite number of subsets of Σ, hence a finite number of low elements, since the map N is injective. Then, the fact that L includes S follows from the fact that, for s in S, we have $N(s) = \{\alpha_s\} \subseteq \Sigma$. Now, assume that we have $N(u) = \mathrm{cone}(A) \cap \Phi$ and $N(v) = \mathrm{cone}(B) \cap \Phi$ with $A, B \subseteq \Sigma$. By definition of the conic closure, we have $\mathrm{cone}(\mathrm{cone}(A) \cup \mathrm{cone}(B)) = \mathrm{cone}(A \cup B)$. By Lemma 4.1(iii), we deduce

$$\mathrm{cone}\big(N(u \vee v)\big) \cap \Phi = \mathrm{cone}\big(\mathrm{cone}(A) \cup \mathrm{cone}(B)\big) \cap \Phi = \mathrm{cone}(A \cup B) \cap \Phi :$$

as $A \cup B$ is included in Σ, we conclude that $u \vee v$ lies in L, so L is closed under join.

The difficult part is to show that L is closed under suffix, and here we only sketch the argument. Recall first that a *maximal rank* 2 *root subsystem of* Φ is a set Φ' of the form $\Phi' = P \cap \Phi$ where P is a plane in V intersecting Φ^+ in at least two roots (maximality refers to the fact that every reflection subgroup of W generated by two reflections has a rank 2 root system that is contained in such a maximal rank 2 root subsystem). The cone spanned by $\Phi' \cap \Phi^+$ has then a basis Δ' of cardinality 2 included in $\Phi' \cap \Phi^+$, and then one has $P \cap \Phi^+ = \mathrm{cone}(\Delta') \cap \Phi$.

(i) We start by showing that Σ is *bipodal*, meaning that, for every small root β and for every maximal rank-2 root subsystem Φ' of Φ with basis Δ' satisfying $\beta \in \Phi' \setminus \Delta'$, we must have $\Delta' \subseteq \Sigma$. To prove this, we note that, for every α in Δ satisfying $B(\alpha, \beta) > 0$, the reflection subgroup generated by reflections in $\Delta' \cup \{\alpha\}$ is of rank at most three and β is a small root for its corresponding root subsystem. Using this observation and Lemma 4.2, one reduces by induction on $\ell(s_\beta)$ to the case of root systems of rank three. Then one checks the result in rank three using the explicit descriptions of small roots in [3].

(ii) Now, consider w and s as in Lemma 4.1(i). Write $s = s_\alpha$ with $\alpha \in \Delta$. For every β in $\Phi^+ \setminus \{\alpha\}$, let $f_\alpha(\beta)$ be the simple root different from α in the standard simple system of the maximal rank-2 root subsystem $\Phi \cap P$, where P is the plane spanned by α and β: in other words, we have $f_\alpha(\beta) \in \Phi^+$ and $P \cap \Phi^+ = \mathrm{cone}(\{\alpha, f_\alpha(\beta)\}) \cap \Phi^+$. Then we show that $N^1(sw)$ is included in $\{s(\beta), f_\alpha(\beta) \mid \beta \in N^1_w \setminus \{\alpha\}\}$. To prove this, one uses Lemma 4.3 to reformulate it as a statement in terms of Bruhat order, and checks that statement using standard properties from [9] of cosets of (maximal dihedral) reflection subgroups.

(iii) Finally, to show that L is closed under suffix, it is enough to show that, for w in L (*i.e.*, for $N^1(w) \subseteq \Sigma$) and s in S satisfying $\ell(sw) < \ell(w)$, the element sw also lies in L. Write $s = s_\alpha$ with α in Δ. By (ii), it is sufficient to show that $s(\beta)$ and $f_\alpha(\beta)$ are small for every β in $N^1(w) \setminus \{\alpha\}$. But, by (i), $f_\alpha(\beta)$ is small since β is small. On the other hand, assume that $s(\beta)$ is not small. Then, Lemma 4.2 implies $B(\alpha, \beta) \leqslant -1$. But then the subgroup generated by s and s_β is infinite dihedral with α and β as its simple roots, so its positive system $\Phi^+ \cap \mathrm{cone}(\{\alpha, \beta\})$, which is infinite, must be included in $N(w)$, which is finite. This contradiction shows that $s(\beta)$ must be small, and completes the proof. □

A more general notion of *n-low element* is introduced and studied in [12] using a notion of n-small roots. The 0-low elements are the low elements as defined here. We conjecture that, for every n, the n-low elements give rise to a Garside family in the associated Artin–Tits monoid. We know that they form a finite set, closed under join. To conclude, it would suffice to prove that the set of n-small roots is bipodal for every n, which is conjectured in general and proved in some cases including affine Weyl groups.

5. Descriptions of $\pi(F)$ and L in some special cases

We keep the same notation, and describe the image $\pi(F)$ of the smallest Garside family F of M in a few cases. By Proposition 3.2, $\pi(F)$ is the closure of S under join and suffix in W. We denote the Coxeter matrix of (W, S) as $(m_{s,t})_{s,t \in S}$, and write $[s,t]_k$ for the alternating product $stst\cdots$ with k factors, $k \geqslant 1$. It is well-known that the *standard dihedral parabolic* subgroup $W_{\{s,t\}}$ consists of the identity together with the elements $[s,t]_k$ and $[t,s]_k$, $k \geqslant 1$. Moreover, $W_{\{s,t\}}$ is finite if and only if $m_{s,t}$ is finite and, in this case, the longest element is $[t,s]_{m_{s,t}} = [s,t]_{m_{s,t}}$.

Proposition 5.1. (i) *If M is an Artin–Tits monoid of spherical type, then we have $\pi(F) = L = W$.*

(ii) *If M is an Artin–Tits monoid of large type (i.e., $m_{s,t} \geqslant 3$ holds for all $s \neq t$), then we have $\pi(F) = L = X$, where X is the union of all finite standard parabolic subgroups of W (each being of rank at most two) together with all elements $t[r,s]_{m_{r,s}}$ with r, s, t distinct in S and $m_{r,s}$, $m_{s,t}$, $m_{t,r}$ all finite.*

(iii) *If M is a right-angled Artin–Tits monoid (i.e., $m_{s,t} \in \{2, \infty\}$ holds for all $s \neq t$), then we have $\pi(F) = L = X$, where X is the union of all finite standard parabolic subgroups of W (which are of the form W_I where I is a set of pairwise commuting simple reflections).*

Proof (sketch). First, $\sigma(L)$ is a Garside family in M by Theorem 1.2, which implies $F \subseteq \sigma(L)$, whence $\pi(F) \subseteq L$ in every case. Hence, for (i), it suffices to show $W \subseteq \pi(F)$ and, for (ii) and (iii), it suffices to show $X \subseteq \pi(F)$ and $L \subseteq X$.

(i) Here, $\pi(F)$ contains the join of all elements of S, which is the longest element w_0 of W, and every element of W is a suffix of w_0. We deduce $W \subseteq \pi(F)$, as required—and the argument shows that, even if W is infinite, $W_I \subseteq \pi(F)$ holds for every finite standard parabolic subgroup W_I of W.

(ii) First, $S \subseteq \pi(F)$ holds by definition. Next, for r, s distinct in S with $m_{r,s}$ finite, the subgroup $W_{\{r,s\}}$ is finite and by the remark above we have $W_{\{r,s\}} \subseteq \pi(F)$. Finally, for r, s, t pairwise distinct in S with $m_{r,s}$, $m_{s,t}$ and $m_{t,r}$ all finite, as just seen, tr and ts lie in $\pi(F)$, hence so does their join, which is $t[r, s]_{m_{r,s}}$. This shows $X \subseteq \pi(F)$.

Now we show $L \subseteq X$. First, by [3], the full subgraph of the Coxeter graph on the support of a small root contains no cycle or infinite bond. Hence, in large type, the small reflections (meaning the reflections in a small root) are precisely the reflections in the finite standard parabolic subgroups. Assume that r, s, t are pairwise distinct in S. We claim that an element of L cannot admit a reduced expression of the form $w = ut[r, s]_k$ with $u \neq t \in S$ and $2 \leqslant k \leqslant m_{r,s}$. Indeed, assume $w \in L$. For any reduced expression $r_n \cdots r_1$ of w and $1 \leqslant i \leqslant n$, we define $t_i := r_n \cdots r_{i+1} r_i r_{i+1} \cdots r_n$, a reflection with $\ell(t_i w) < \ell(w)$. By Lemma 4.3, t_i is a small reflection if $\ell(t_i w) = \ell(w) - 1$ holds. So, here, t_{n-2}, *i.e.*, $utrtu$, must be a small reflection, which forces $u = r$ (and $m_{r,t} < \infty$). Also, t_1 must be a small reflection. Now, we have $t_1 = utvtu = rtvtr$ with $v = [r, s]_{2k-1}$, and v is a reflection of $W_{\{r,s\}}$ unequal to r. Since r, s and t are pairwise non-commuting, Matsumoto's Lemma implies, first, $\ell(tvt) = \ell(v) + 2$ and, then, $\ell(t_1) = \ell(tvt) + 2$, by considering the cases $v = s$ and $\ell(v) > 1$ separately. Hence the smallest standard parabolic subgroup containing t_1 is $W_{\{r,s,t\}}$ of rank 3, a contradiction.

Similar (but simpler) arguments show that an element of L cannot admit a reduced expression of the form $t[r, s]_k$ with r, s, t distinct in S and $2 \leqslant k < m_{r,s}$, or $[r, s]_k$ with r, s distinct in S and $2 \leqslant k < m_{r,s} = \infty$. Now, every element w of W has a unique decomposition $w = uv$ with $u \in W$ satisfying $\ell(us) = \ell(ur) > \ell(u)$ and $v \in W_{\{r,s\}}$, see [1, Proposition 2.4.4 (i)]. So, as L is closed under suffix, no element of any of the three types excluded above can be a suffix of an element of L, and $L \subseteq X$ follows.

(iii) The proof is similar to (and simpler than) that of (ii). First, since small roots cannot have any infinite bonds in their supports, the set of small reflections is precisely S. Then, for $I \subseteq S$ consisting of commuting simple reflections, the parabolic subgroup W_I is finite. Hence such W_I are contained in $\pi(F)$, which implies $X \subseteq \pi(F)$.

For $L \subseteq X$, suppose $w = ur_1 \cdots r_n$ where $u, r_1, \ldots, r_n$ are distinct pairwise commuting simple reflections such that u does not commute with all of $r_1, \ldots, r_n$, and let i be minimal with $ur_i \neq r_i u$. If w were low, arguing as in (ii), we would deduce that $ur_1 \cdots r_i \cdots r_1 u$, *i.e.*, $ur_i u$, is a small reflection, so lies in S, leading to a contradiction with $ur_i \neq r_i u$. Hence $ur_1 \cdots r_n$ is not low and, again, one easily deduces $L \subseteq X$. □

In type $\widetilde{\mathrm{C}}_2$, with Coxeter graph $\sigma_1 = \sigma_2 = \sigma_3$, one finds $|L| = 25$ and $|\pi(F)| = |F| = 24$: here $\sigma_1\sigma_3\sigma_2$ is low, but does not lie in $\pi(F)$. Hence $\pi(F) = L$ need not hold in general.

However, for type $\widetilde{\mathrm{A}}_n$, the equality $\pi(F) = L$ holds and we have $|L| = (n+2)^n$. Indeed, while preparing this manuscript, the third author (CH), together with P. Nadeau and N. Williams, built two automata recognizing the language of reduced words with respective state sets $\pi(F)$ and L. It is easy to see that $\pi(F)$ and L inject into the state set of the *canonical automaton* defined by Brink and Howlett, see [1, p. 120]. Now, in type $\widetilde{\mathrm{A}}_n$, Eriksson showed that the latter is minimal [1, p. 125], so $\pi(F)$ and L must share its cardinality, which is $(n+2)^n$. A more direct proof would be desirable.

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