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Partial differential equations/Harmonic analysis

Measure boundary value problems for semilinear elliptic equations with critical Hardy potentials

Problèmes aux limites avec données mesures pour des équations semi-linéaires elliptiques avec des potentiels de Hardy critiques

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ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain and $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$ where $d = \text{dist}(.,\partial\Omega)$ and $0 < \kappa \le \frac{1}{4}$. Let $\alpha_\pm = 1 \pm \sqrt{1-4\kappa}$, λ_κ the first eigenvalue of $\mathcal{L}_\kappa$ with corresponding positive eigenfunction ϕ_κ. If g is a continuous nondecreasing function satisfying $\int_1^\infty (g(s)+|g(-s)|)s^{-2\frac{2N-2+\alpha_+}{2N-4+\alpha_+}}ds < \infty$, then for any Radon measures $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$ and $\mu \in \mathfrak{M}(\partial\Omega)$ there exists a unique weak solution to problem $P_{\nu,\mu}$: $\mathcal{L}_\kappa u + g(u) = \nu$ in Ω, $u = \mu$ on $\partial\Omega$. If $g(r) = |r|^{q-1}u$ $(q > 1)$, we prove that, in the supercritical range of q, a necessary and sufficient condition for solving $P_{0,\mu}$ with $\mu > 0$ is that μ is absolutely continuous with respect to the capacity associated with the space $B^{2-\frac{2+\alpha_+}{2q'},q'}(\mathbb{R}^{N-1})$. We also characterize the boundary removable sets in terms of this capacity. In the subcritical range of q we classify the isolated singularities of positive solutions.

RÉSUMÉ

Soient $\Omega \subset \mathbb{R}^N$ un domaine de classe C^2 et $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$ où $d = \text{dist}(.,\partial\Omega)$ et $0 < \kappa \le \frac{1}{4}$. Soient $\alpha_\pm = 1 \pm \sqrt{1-4\kappa}\, y$, λ_κ la première valeur propre de $\mathcal{L}_\kappa$ et ϕ_κ la fonction propre positive correspondante. Si g est une fonction continue croissante vérifiant $\int_1^\infty (g(s)+|g(-s)|)s^{-2\frac{2N-2+\alpha_+}{2N-4+\alpha_+}}ds < \infty$, alors pour toutes mesures de Radon $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$ et $\mu \in \mathfrak{M}(\partial\Omega)$, il existe une unique solution faible au problème $P_{\nu,\mu}$: $\mathcal{L}_\kappa u + g(u) = \nu$ dans Ω, $u = \mu$ sur $\partial\Omega$. Si $g(r) = |r|^{q-1}u$ $(q > 1)$, nous démontrons qu'une condition nécessaire et suffisante pour résoudre $P_{0,\mu}$ avec $\mu > 0$ est que μ soit absolument continue par rapport à la capacité associée à l'espace $B^{2-\frac{2+\alpha_+}{2q'},q'}(\mathbb{R}^{N-1})$. Cette capacité caractérise les

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ensembles éliminables du bord. Dans le cas sous-critique, nous classifions les singularités isolées au bord des solutions positives.

Version française abrégée

Soit Ω un domaine de $\mathbb{R}^N$ de classe C^2. On désigne par $d(x)$ la distance de x à $\partial\Omega$ et on définit l'opérateur de Hardy dans Ω par

$$\mathcal{L}_\kappa u = -\Delta u - \frac{\kappa}{d^2}u, \tag{1}$$

où $0 < \kappa \leq \frac{1}{4}$, et ses exposants caractéristiques

$$\alpha_+ = 1 + \sqrt{1-4\kappa}, \qquad \alpha_- = 1 - \sqrt{1-4\kappa}. \tag{2}$$

On supposera Ω convexe si $\kappa = \frac{1}{4}$ (see [2]). Il est bien connu que, sous ces conditions, $\mathcal{L}_\kappa$ possède une première valeur propre $\lambda_\kappa > 0$ définie par

$$\lambda_\kappa := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_\Omega d^{-2}u^2 \mathrm{d}x}. \tag{3}$$

La première fonction propre positive associée ϕ_κ n'appartient à $H_0^1(\Omega)$ que si $0 < \kappa < \frac{1}{4}$, et dans tous les cas elle vérifie $\phi_\kappa(x) \sim (d(x))^{\frac{\alpha_+}{2}}$ au voisinage de $\partial\Omega$. On dénote par G_κ et K_κ les noyaux de Green et de Poisson de $\mathcal{L}_\kappa$ dans Ω et par ω^{x_0} la mesure $\mathcal{L}_\kappa$-harmonique dans Ω ($x_0 \in \Omega$). Si g est une fonction continue et croissante sur $\mathbb{R}$ telle que $g(0) \geq 0$, nous étudions tout d'abord le problème $(P_{\nu,\mu})$ suivant :

$$\begin{aligned} &\mathcal{L}_\kappa u + g(u) = \nu \quad \text{in } \Omega, \\ &u = \mu \quad \text{in } \partial\Omega, \end{aligned} \tag{4}$$

où ν, μ sont des mesures de Radon.

Théorème 1. *Supposons que g vérifie*

$$\int_1^\infty \big(g(s) + |g(-s)|\big) s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} \mathrm{d}s < \infty; \tag{5}$$

alors pour toutes mesures de Radon ν et μ dans Ω et $\partial\Omega$ respectivement, ν vérifiant en outre $\int_\Omega \phi_\kappa \mathrm{d}|\nu| < \infty$, il existe une unique fonction $u = u_{\nu,\mu} \in L^1_{\phi_\kappa}(\Omega)$ telle que $g \circ u \in L^1_{\phi_\kappa}(\Omega)$ vérifiant

$$\int_\Omega (u\mathcal{L}_\kappa \zeta + \zeta g \circ u)\mathrm{d}x = \int_\Omega \int_\Omega G_\kappa(x,y)\mathrm{d}\nu(y)\zeta(x)\mathrm{d}x + \int_\Omega \int_{\partial\Omega} K_\kappa(x,y)\mathrm{d}\mu(y)\mathcal{L}_\kappa\zeta(x)\mathrm{d}x \tag{6}$$

pour toute $\zeta \in \mathbf{X}_\kappa(\Omega)$ où

$$\mathbf{X}_\kappa(\Omega) = \big\{\zeta \in H^1_{loc}(\Omega) : (\phi_\kappa)^{-1}\zeta \in H^1_0(\Omega, \phi_\kappa \mathrm{d}x), (\phi_\kappa)^{-1}\mathcal{L}_\kappa\zeta \in L^\infty(\Omega)\big\}. \tag{7}$$

En outre, l'application $(\nu,\mu) \mapsto u_{\nu,\mu}$ de $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ dans $L^1_{\phi_\kappa}(\Omega)$ est croissante et stable pour la convergence faible des mesures.

La démonstration utilise des estimations des noyaux de Green et de Poisson déduits des propriétés de la mesure ω_κ. Dans le cas où $g \circ u = |u|^{q-1}u$, (5) est vérifiée si $0 < q < q_c := \frac{2N+\alpha_+}{2N+\alpha_+-4}$. Dans le cas $q > 1$, nous dénotons par $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$ la capacité associée à l'espace de Besov $B^{2-\frac{2+\alpha_+}{2q'},q'}(\mathbb{R}^{N-1})$ et nous démontrons le théorème qui suit.

Théorème 2. *Soit $q \geq q_c$ et $\nu \in \mathfrak{M}_+(\partial\Omega)$. Alors le problème*

$$\begin{aligned} &\mathcal{L}_\kappa u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \\ &u = \mu \quad \text{in } \partial\Omega \end{aligned} \tag{8}$$

admet une unique solution $u := u_\mu$ si et seulement si pour tout borélien $E \subset \partial\Omega$,

$$C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(E) = 0 \quad \Longrightarrow \quad \mu(E) = 0. \tag{9}$$

Le cas sous-critique $1 < q < q_c$ du problème (8) est traité dans [6]. Nous caractérisons aussi les sous-ensembles du bord éliminables pour l'équation

$$\mathcal{L}_\kappa u + |u|^{q-1}u = 0 \quad \text{in } \Omega. \tag{10}$$

Définissons

$$W(x) = \begin{cases} (d(x))^{\frac{\alpha_-}{2}} & \text{if } 0 < \kappa < \frac{1}{4}, \\ \sqrt{d(x)}\ln|d(x)| & \text{if } \kappa = \frac{1}{4}. \end{cases} \tag{11}$$

Théorème 3. *Soit $q > 1$ et $K \subset \partial\Omega$ un sous-ensemble compact. Toute solution $u \in C(\overline{\Omega} \setminus \{K\})$ de (10) qui vérifie*

$$\lim_{x\to y} \frac{u(x)}{W(x)} = 0 \quad \forall y \in \partial\Omega \setminus \{K\}, \tag{12}$$

est identiquement nulle dans Ω si et seulement si $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(K) = 0$.

Nous montrons que, si $q > 1$, toute solution positive de (10) dans Ω admet une trace au bord représentée par une mesure de Borel régulière. En supposant que $0 \in \partial\Omega$ et $1 < q < q_c$, nous étudions aussi le comportement au voisinage de 0 des solutions positives de (10) qui vérifient (12) avec $K = \{0\}$.

English version

Let Ω be a bounded C^2 domain in $\mathbb{R}^N$, $N \geq 2$ and $d(x) = \text{dist}(x, \Omega)$. We define λ_Ω by (3). It is well known that $\lambda_\Omega \in (0, \frac{1}{4}]$. Also we define the Hardy operator $\mathcal{L}_\kappa$ in Ω by (1) with $0 < \kappa < \lambda_\Omega$ if $\lambda_\Omega < \frac{1}{4}$ or $0 < \kappa \leq \frac{1}{4}$ if $\lambda_\Omega = \frac{1}{4}$ and the characteristic exponents by (2). We assume that Ω is convex if $\kappa = \frac{1}{4}$. It is well known that $\mathcal{L}_\kappa$ possesses a first eigenvalue $\lambda_\kappa > 0$ defined by

$$\lambda_\kappa := \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega d^{-2}u^2 dx}{\int_\Omega u^2 dx}, \tag{13}$$

see [3]. The first positive eigenfunction $\phi_\kappa > 0$ may or may not belong to $H^1_0(\Omega)$ according $0 < \kappa < \frac{1}{4}$ or $\kappa = \frac{1}{4}$, and $\phi_\kappa(x) \sim (d(x))^{\frac{\alpha_+}{2}}$, $|\nabla\phi_\kappa(x)| \sim (d(x))^{\frac{\alpha_+}{2}-1}$ as $d(x) \to 0$. Let $G_\kappa(x, y)$ (resp. $K_\kappa(x, y)$) be the Green (resp. Poisson) kernel of $\mathcal{L}_\kappa$, then

$$G_\kappa(x, y) \sim \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{(d(x))^{\frac{\alpha_+}{2}}(d(y))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}}\right\} \quad \forall (x, y) \in \Omega \times \Omega, x \neq y, \tag{14}$$

see [4], and finally [5]

$$K_\kappa(x, y) \sim \frac{(d(x))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \quad \forall (x, y) \in \Omega \times \partial\Omega. \tag{15}$$

The corresponding Green and Poisson operators are denoted by $\mathbb{G}_\kappa[.]$ and $\mathbb{K}_\kappa[.]$. We first consider the boundary value problem (4) where g is a continuous nondecreasing function such that $g(0) \geq 0$ and ν and μ are Radon measures in Ω and $\partial\Omega$, respectively. We say that g is a subcritical nonlinearity if it satisfies (5).

Theorem 1. *Assume that g is a subcritical nonlinearity. Then for all $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ there exists a unique function $u = u_{\nu,\mu} \in L^1_{\phi_\kappa}(\Omega)$ such that $g \circ u \in L^1_{\phi_\kappa}(\Omega)$ verifying (6) for all ζ in the space of test functions $\mathbf{X}_\kappa(\Omega)$ defined by (7). Furthermore the mapping $(\nu, \mu) \mapsto u_{\nu,\mu}$ from $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ into $L^1_{\phi_\kappa}(\Omega)$ is nondecreasing and stable for the weak convergence of measures.*

When $g(u) = |u|^{q-1}u$ with $q > 0$, the inequality (6) means

$$0 < q < q_c := \frac{2N + \alpha_+}{2N + \alpha_+ - 4}. \tag{16}$$

When $q \geq q_c$ not all the measures μ are eligible for solving (8). We denote by $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$ the capacity associated with the Besov space $B^{2-\frac{2+\alpha_+}{2q'},q'}(\mathbb{R}^{N-1})$.

Theorem 2. *Let $q>1$ and $\mu\in\mathfrak{M}_+(\partial\Omega)$. Then problem (8) admits a solution if and only if μ is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$, i.e. for any Borel set $E\subset\partial\Omega$, implication (9) holds.*

We also characterize the boundary removable sets for (10).

Theorem 3. *Let $q>1$ and $K\subset\partial\Omega$ be compact. Any $u\in C(\overline{\Omega}\setminus\{K\})$ solution to (10) that verifies (12) is identically zero in Ω if and only if $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(K)=0$.*

Note that the subcritical case of problem (8), i.e. $1<q<q_c$, is considered in [6], in that there is no condition on μ. In the same article, it is proved that when $q\geq q_c$, isolated singularities are removable. Large solutions are treated in [1].

When $1<q<q_c$, only the empty set has zero capacity. There exist singular solutions to (10) with an isolated singularity on the boundary, either solutions $u_{k\delta_a}$ of (8) with $\mu=k\delta_a$ for $k>0$ and $a\in\partial\Omega$ or solutions $u_a=\lim_{k\to\infty}u_{k\delta_a}$. This very singular solution is described by considering the following problem on the half upper-sphere $S^{N-1}_+=\{x=(x_1,\ldots,x_N)\in\mathbb{R}^N:|x|=1,\,x_N>0\}$

$$\begin{aligned}&-\Delta'\omega-\ell_{N,q,\kappa}\omega-\frac{\kappa}{(\mathbf{e}_N.\sigma)^2}\omega+|\omega|^{q-1}\omega=0\quad\text{in }S^{N-1}_+,\\&\omega=0\quad\text{in }\partial S^{N-1}_+,\end{aligned}\tag{17}$$

where Δ' is the Laplace–Beltrami operator on S^{N-1}, $(\mathbf{e}_1,\ldots,\mathbf{e}_N)$ is the canonic basis in $\mathbb{R}^N$, $\sigma=\frac{x}{|x|}$ and

$$\ell_{N,q}=\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1}-N\right).$$

The spherical Hardy operator $\omega\mapsto\mathcal{L}'_\kappa:=-\Delta'\omega-\frac{\kappa}{(\mathbf{e}_N.\sigma)^2}\omega$ on S^{N-1}_+ admits a first eigenvalue μ_κ defined by

$$\mu_{\kappa,1}=\inf_{\psi\in H^1_0(S^{N-1}_+)\setminus\{0\}}\frac{\int_{S^{N-1}_+}(|\nabla'\psi|^2-\kappa(\mathbf{e}_N.\sigma)^{-2}\omega^2)\mathrm{d}S}{\int_\Omega(\mathbf{e}_N.\sigma)^{-2}\psi^2\mathrm{d}S}.\tag{18}$$

We prove that $\mu_{\kappa,1}=\frac{\alpha_+}{2}(N+\frac{\alpha_+}{2}-2)$ with corresponding positive eigenfunction $\rho_\kappa=(\mathbf{e}_N.\sigma)^{\frac{\alpha_+}{2}}$. There exists a second eigenvalue $\mu_{\kappa,2}=\mu_{\kappa,1}+N+\alpha_+-1$ with $N-1$ independent eigenfunctions $\rho_{\kappa,j}=(\mathbf{e}_N.\sigma)^{\frac{\alpha_+}{2}}\mathbf{e}_j.\sigma$ for $j=1,\ldots,N-1$. We denote by $\mathcal{E}_\kappa$ the set of functions ω such that $\rho_\kappa^{-1}\omega\in L^{q+1}_{\rho_\kappa^{q+1}}(S^{N-1}_+)\cap H^1_0(S^{N-1}_+,\rho_\kappa^2\mathrm{d}S)$ that satisfy (17), and by $\mathcal{E}^+_\kappa$ the set of positive solutions.

Theorem 4. *I - If $q\geq q_c$, $\mathcal{E}_\kappa=\{\emptyset\}$.*
II - If $1<q<q_c$, $\mathcal{E}^+_\kappa=\{0,\omega_\kappa\}$ where ω_κ is the unique positive solution to (17).
III - If $q_e\leq q<q_c$, $\mathcal{E}_\kappa=\{0,\omega_\kappa,-\omega_\kappa\}$ where $q_e:=\frac{2N+2+\alpha_+}{2N-2+\alpha_+}$.

This allows us to describe the isolated boundary singularities of positive solutions to (10). Assume $0\in\partial\Omega$ and $\mathbf{e}_N$ is the outward normal unit vector to $\partial\Omega$ at 0.

Theorem 5. *Assume, $1<q<q_c$ and $u\in C(\overline{\Omega}\setminus\{0\})$ is a positive solution to (10) that verifies (12) with $K=\{0\}$. Then*

(i) *either there exists $k\geq0$ such that $u=u_{k\delta_0}$ and $\lim_{|x|\to0}|x|^{N+\frac{\alpha_+}{2}-2}u(x)=c_Nk(\mathbf{e}_N.\frac{x}{|x|})^{\frac{\alpha_+}{2}}$,*
(ii) *or $\lim_{|x|\to0}|x|^{\frac{2}{q-1}}u(x)=\omega_\kappa(\frac{x}{|x|})$.*

The above two convergences hold locally uniformly on S^{N-1}_+.

We can also define a boundary trace of any positive solution u to (10). For $\delta>0$ small enough, we denote by $\omega^{x_0}_{\Omega'_\delta}$ the harmonic measure relative to the operator $\mathcal{L}_\kappa$ in $\Omega'_\delta=\{x\in\Omega:d(x)>\delta\}$ where $x_0\in\Omega$ (with $d(x_0)\geq\delta_1>\delta$) and set $\Sigma_\delta=\partial\Omega'_\delta$.

Theorem 6. *Assume $q>1$ and $u\in C(\overline{\Omega}\setminus\{0\})$ is a positive solution to (10) in Ω. Then for any $y\in\partial\Omega$, the following dichotomy occurs*

(i) *either there exist an open subset $U\subset\mathbb{R}^N$ containing y and a positive Radon measure λ_U on $\partial\Omega\cap U$ such that*

$$\lim_{\delta\to 0}\int_{\Sigma_\delta\cap U} Z(x)u(x)\mathrm{d}\omega^{x_0}_{\Omega'_\delta}=\int_{\partial\Omega\cap U} Z\mathrm{d}\lambda_U \quad \forall Z\in C_0(U). \tag{19}$$

(ii) *Or for any open subset $U\subset\mathbb{R}^N$ containing y, there holds*

$$\lim_{\delta\to 0}\int_{\Sigma_\delta\cap U} u(x)\mathrm{d}\omega^{x_0}_{\Omega'_\delta}=\infty. \tag{20}$$

The set $\mathcal{R}_u$ of y such that (i) holds is relatively open in $\partial\Omega$ and it carries a positive Radon measure μ_u such that (19) occurs with U replaced by $\mathcal{R}_u$ and λ_U by μ_u; its complement $\mathcal{S}_u$ in $\partial\Omega$ has the property that (20) occurs for any open subset U such that $U\cap\mathcal{S}_u\neq\{\emptyset\}$.

Abridged proof of Theorem 1. Let $(\nu,\mu)\in\mathfrak{M}_{\phi_\kappa}(\Omega)\times\mathfrak{M}(\partial\Omega)$. For $\lambda>0$ we set

$$E_\lambda(\nu)=\left\{x\in\Omega:\mathbb{G}_\kappa\left[|\nu|\right](x)>\lambda\right\},\qquad \mathcal{E}_\lambda(\nu)=\int_{E_\lambda(\nu)}\phi_\kappa\,\mathrm{d}x, \tag{21}$$

and

$$F_\lambda(\nu)=\left\{x\in\Omega:\mathbb{K}_\kappa\left[|\mu|\right](x)>\lambda\right\},\qquad \mathcal{F}_\lambda(\mu)=\int_{E_\lambda(\nu)}\mathrm{d}x, \tag{22}$$

and prove

$$\mathcal{E}_\lambda(\nu)+\mathcal{F}_\lambda(\mu)\leq c\left(\frac{\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}+\|\mu\|_{\mathfrak{M}(\partial\Omega)}}{\lambda}\right)^{\frac{2N+\alpha_+}{2N+\alpha_+-4}}. \tag{23}$$

If g satisfies (5) and $\{(\nu_n,\mu_n)\}$ is a sequence of smooth functions that converges in the weak-star topology of measures to (ν,μ), then the corresponding solutions $\{u_{\nu_n,\mu_n}\}$ to problem P_{ν_n,μ_n} defined in (4) converge to some u and $\{g\circ u_{\nu_n,\mu_n}\}$ converges to $g\circ u$ in $L^1_{\phi_\kappa}$ by Vitali's convergence theorem. This implies $u=u_{\nu,\mu}$. Uniqueness holds by adapting Brezis estimates and using monotonicity. □

Abridged proof of Theorem 2. Using estimate (15) and the harmonic lifting in Besov spaces introduced in [9, Sect. 3] we prove that for any $\mu\in\mathfrak{M}(\partial\Omega)$ there holds

$$\frac{1}{c}\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}}\leq\int_\Omega\left(\mathbb{K}_\kappa\left[|\mu|\right]\right)^q\phi_\kappa\,\mathrm{d}x\leq c\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}}, \tag{24}$$

for some $c=c(\Omega,\kappa,q)>0$. If the above quantity is finite, we can solve (8) with such a μ. If $\mu\in B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)\cap\mathfrak{M}_+(\partial\Omega)$, it is absolutely continuous with respect to the capacity $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$. Finally, if $\mu\in\mathfrak{M}_+(\partial\Omega)$ is absolutely continuous with respect to the capacity $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$, there exists an increasing sequence $\{\mu_n\}\subset B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)\cap\mathfrak{M}_+(\partial\Omega)$ that converges to μ. This implies that u_{μ_n} converges to u_μ in $L^q_{\phi_\kappa}(\Omega)$.

Conversely, if $\mu\in\mathfrak{M}_+(\partial\Omega)$ is such that there exists a solution u_μ to (8), we use a variant of the *optimal lifting* $R[.]$ defined in [7, Sect. 1] to prove that for any $\eta\in C^2(\partial\Omega)$ such that $0\leq\eta\leq 1$ there holds

$$\int_{\partial\Omega}\eta\mathrm{d}\mu\leq c\int_\Omega u^q\zeta\,\mathrm{d}x+c\left(\int_\Omega u^q\zeta\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_\Omega\phi_\kappa\,\mathrm{d}x+\|\eta\|^{q'}_{B^{2-\frac{2+\alpha_+}{2q'},q'}}\right)^{\frac{1}{q'}}. \tag{25}$$

Here $\zeta=\phi_\kappa(R[\eta])^{q'}$ and $R:C^2(\partial\Omega)\mapsto C^2(\overline{\Omega})$ is a linear mapping that satisfies $0\leq\eta\leq 1\Longrightarrow 0\leq R[\eta]\leq 1$ and $R[\eta]\lfloor_{\partial\Omega}=\eta$. If $K\subset\partial\Omega$ is a compact set with zero $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q}$-capacity, there exists a sequence $\{\eta_n\}\subset C^2(\partial\Omega)$ such that $0\leq\eta_n\leq 1$, $\eta_n=1$ on K and $\|\eta_n\|^{q'}_{B^{2-\frac{2+\alpha_+}{2q'},q'}}\to 0$. This implies $\phi_\kappa(R[\eta_n])^{q'}\to 0$ and finally $\mu(K)=0$. □

Abridged proof of Theorem 3. If $K\subset\partial\Omega$ is compact with $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q}(K)>0$, its capacitary measure μ_K belongs to $B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)\cap\mathfrak{M}_+(\partial\Omega)$. Thus u_{μ_K} exists and K is not removable. Conversely, by using again optimal lifting, and test functions of the form $\phi_\kappa(R[1-\eta])^{2q'}$ where $0 \le \eta \le 1$ and $\eta = 1$ in a neighborhood of K, we prove first that $u \in L^q_{\phi_\kappa}(\Omega)$ and finally that $u = 0$. $\square$

Abridged proof of Theorems 4–5. Existence is obtained in minimizing the functional $\mathcal{J}_\kappa$ defined over $L^{q+1}_{\rho_\kappa^{q+1}}(S^{N-1}_+) \cap H^1_0(S^{N-1}_+, \rho_\kappa^2 \mathrm{d}S)$ by

$$\mathcal{J}_\kappa(w) := \int\limits_{S^{N-1}_+} \left(\left|\nabla' w\right|^2 - (\ell_{N,q} - \mu_{\kappa,1}) w^2 + \frac{2}{q+1} \rho_\kappa^{q-1} |w|^{q+1} \right) \rho_\kappa^2 \mathrm{d}S. \tag{26}$$

A non-trivial minimizer w exists if $\ell_{N,q} > \mu_{\kappa,1}$ (defined by (18)), i.e. $1 < q < q_c$, and $\omega = \rho_\kappa w$ satisfies (17). Nonexistence is standard since $\mu_{\kappa,1} < \ell_{N,q}$ if and only if $1 < q < q_c$. For uniqueness, we assume that ω_j $(j = 1, 2)$ are positive solutions to (17) and we set $w_j = \frac{\omega_j}{\rho_\kappa}$. Then

$$-\mathrm{div}'\left(\rho_\kappa^2 \nabla' w_j\right) + (\mu_{\kappa,1} - \ell_{N,q}) \rho_\kappa^2 w_j + \rho_\kappa^{q+1} w_j^q = 0 \quad \text{on } S^{N-1}_+.$$

Since $w_j \sim \rho_\kappa^{\frac{\alpha_+}{2}}$ and $|\nabla' w_j| \sim \rho_\kappa^{\frac{\alpha_+}{2}-1}$ near ∂S^{N-1}_+, we use Green's formula and get

$$\int\limits_{S^{N-1}_+} \left(\left(\frac{\nabla' w_1}{w_1} - \frac{\nabla' w_2}{w_2} \right) . \nabla'\left(w_1^2 - w_2^2\right) + \rho_\kappa^{q-1}\left(w_1^{q-1} - w_2^{q-1}\right)\left(w_1^2 - w_2^2\right) \right) \rho_\kappa^2 \mathrm{d}S = 0,$$

thus $w_1 = w_2$. For statement III we prove, with the method used in [11, Th 3.1], that any solution ω depends only on the azimuthal angle $\theta \in [0, \frac{\pi}{2}]$, and then that the resulting ODE verified by ω admits only constant-sign solutions. For Theorem 5, we construct a barrier function as in [8, Appendix] and obtain:

$$u(x) \le c|x|^{-\frac{2}{q-1}+\frac{\alpha_+}{2}} \left(d(x)\right)^{\frac{\alpha_+}{2}} \quad \forall x \in \Omega. \tag{27}$$

With this estimate, we adapt the scaling method developed in [10, Sect. 3.3] to obtain the classification result. The details of this Note are presented in [5]. $\square$

References

[1] C. Bandle, V. Moroz, W. Reichel, Boundary blow up type sub-solutions to semilinear elliptic equations with Hardy potential, J. Lond. Math. Soc. 2 (2008) 503–523.

[2] H. Brezis, M. Marcus, Hardy's inequalities revisited, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 25 (1997) 217–237.

[3] J. Davila, L. Dupaigne, Hardy-type inequalities, J. Eur. Math. Soc. 6 (2004) 335–365.

[4] S. Filippas, L. Moschini, A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains, Commun. Math. Phys. 273 (2007) 237–281.

[5] K. Gkikas, L. Véron, Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials, arXiv:1410.1176v2, 2014.

[6] M. Marcus, P.T. Nguyen, Moderate solutions of semilinear elliptic equations with Hardy potential, arXiv:1407.3572v1, 2014.

[7] M. Marcus, L. Véron, Removable singularities and boundary trace, J. Math. Pures Appl. 80 (2001) 879–900.

[8] M. Marcus, L. Véron, The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption, Commun. Pure Appl. Math. 56 (2003) 689–731.

[9] M. Marcus, L. Véron, Boundary trace of positive solutions of supercritical semilinear elliptic equations in dihedral domains, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (2015), http://dx.doi.org/10.2422/2036-2145.201310_011, in press.

[10] Phuoc T. Nguyen, L. Véron, Boundary singularities of solutions to elliptic viscous Hamilton–Jacobi equations, J. Funct. Anal. 263 (2012) 1487–1538.

[11] L. Véron, Geometric invariance of singular solutions of some nonlinear partial differential equations, Indiana Univ. Math. J. 38 (1989) 75–100.