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Number theory

On the square-root partition function ☆

Sur la fonction de partition en racines carrées

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ABSTRACT

The well-known partition function $p(n)$, which is the number of solutions of the equation $n = a_1 + \cdots + a_k$ with integers $1 \le a_1 \le \cdots \le a_k$, has a long research history. In this note, we investigate a new partition function. Let $q(n)$ be the number of solutions of the equation $n = [\sqrt{a_1}\,] + \cdots + [\sqrt{a_k}\,]$ with integers $1 \le a_1 \le \cdots \le a_k$, where $[x]$ denotes the integral part of x. We prove that $\exp(c_1 n^{2/3}) \le q(n) \le \exp(c_2 n^{2/3})$ for two explicit positive constants c_1 and c_2.

RÉSUMÉ

La fonction de partition bien connue $p(n)$, qui compte le nombre de solutions de l'équation $n = a_1 + \cdots + a_k$ en entiers $1 \le a_1 \le \cdots \le a_k$, a une longue histoire. Nous étudions dans cette Note une nouvelle fonction de partition. Soit $q(n)$ le nombre de solutions de l'équation $n = [\sqrt{a_1}\,] + \cdots + [\sqrt{a_k}\,]$ en entiers $1 \le a_1 \le \cdots \le a_k$, où $[x]$ désigne la partie entière de x. Nous montrons que $\exp(c_1 n^{2/3} \le q(n) \le \exp(c_2 n^{2/3})$ pour deux constantes positives explicites c_1 et c_2.

1. Introduction

Let $f(n)$ be the number of unordered factorizations of a positive integer n as a product of factors > 1. Let

$$\mathcal{F}(x) = \{m : m \le x, m = f(n) \text{ for some } n\}.$$

In order to obtain the upper bound of $|\mathcal{F}(x)|$, Balasubramanian and Luca [1] introduced the following function: let $q(n)$ be the number of solutions of the equation

$$n = [\sqrt{a_1}\,] + \cdots + [\sqrt{a_k}\,] \tag{1.1}$$

with integers $1 \le a_1 \le \cdots \le a_k$, where $[x]$ denotes the integral part of x.

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For convenience in the future, we call (1.1) a square-root partition of n and $q(n)$ the square-root partition function. The well-known partition function $p(n)$, which is the number of solutions of the equation $n = a_1 + \cdots + a_k$ with integers $1 \le a_1 \le \cdots \le a_k$, has a long research history. We believe that the square-root partition becomes as extensively studied $p(n)$.

Balasubramanian and Luca [1] proved that

$$1 + \sum_{n=1}^{\infty} q(n) z^n = (1-z)^{-3}(1-z^2)^{-5} \cdots (1-z^n)^{-2n-1} \cdots. \tag{1.2}$$

In this note, the following result is proved.

Theorem 1.1. *There exist two explicit positive constants c_1 and c_2 such that*

$$\exp(c_1 n^{2/3}) \le q(n) \le \exp(c_2 n^{2/3})$$

for all integers $n \ge 1$.

Remark 1.2. We may take

$$c_2 = 3e \log 2 + 6e + 1, \quad c_1 = \frac{1}{4} c_2^{-2} \exp(-2c_2).$$

From the proof, it is easy to see that c_1 and c_2 can be improved.

In [1], it is proved that $q(n) \le \exp(5n^{2/3})$. But the proof contains a gap on page 4 of [1]. After our paper was submitted, Balasubramanian has provided us a corrected proof for $q(n) \le \exp(cn^{2/3})$ (private communication). For completeness, we still include our own proof.

2. Proofs

Lemma 2.1. *We have*

$$q(n) \ge q(n-1) + 2n + 1 (n \ge 2), \quad q(n) \ge (n+1)^2 - 1 (n \ge 1).$$

Proof. If $n - 1 = [\sqrt{a_1}] + \cdots + [\sqrt{a_k}]$ is a square-root partition of $n-1$, then $n = [\sqrt{a_0}] + [\sqrt{a_1}] + \cdots + [\sqrt{a_k}]$ $(a_0 = 1)$ is a square root partition of n. Since $n = [\sqrt{b_1}]$ $(n^2 \le b_1 \le n^2 + 2n)$ are $2n+1$ square root partitions of n that cannot be obtained from square root partitions of $n-1$ in the above way, it follows that $q(n) \ge q(n-1) + 2n + 1$ $(n \ge 2)$. Therefore, $q(n) \ge (n+1)^2 - 1$. □

Lemma 2.2. *We have* $\frac{x-1}{x} < \log x < x - 1$ *for* $0 < x < 1$.

Lemma 2.3. *Let* $0 < a < 1$ *and the function* $h(x)$ *be defined as*

$$h(x) = \frac{(2x+1)a^x}{1-a^x}, \quad x \in [1, +\infty).$$

Then the function $h(x)$ *is decreasing on* $[1, +\infty)$.

Proof. We have

$$h'(x) = \frac{a^x}{(1-a^x)^2}(2(1 - a^x + \log a^x) + \log a).$$

It follows from Lemma 2.2 and the fact that $0 < a < 1$ that $h'(x) < 0$ for all $x \in [1, +\infty)$. Thus the function $h(x)$ is decreasing on $[1, +\infty)$. □

Let

$$g(z) = (1-z)^{-3}(1-z^2)^{-5} \cdots (1-z^n)^{-2n-1} \cdots.$$

Lemma 2.4. *We have*

$$\frac{2z}{(1-z)^2} < \log g(z) < (3\log 2 + 6)\frac{z}{(1-z)^2}, \quad 0 < z < 1. \tag{2.1}$$

Proof. We have

$$\log g(z) = -\sum_{n=1}^{\infty}(2n+1)\log(1-z^n).$$

It follows from Lemma 2.2 that

$$\sum_{n=1}^{\infty}(2n+1)z^n < \log g(z) < \sum_{n=1}^{\infty}(2n+1)\frac{z^n}{1-z^n}. \tag{2.2}$$

Since

$$\sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}, \tag{2.3}$$

the lower bound (2.1) follows from the lower bound (2.2).

Now we prove the upper bound (2.1).

We choose an integer M such that $z^{M+1} \le \frac{1}{2} < z^M$. Then, by Lemma 2.2, we have

$$M < -\frac{\log 2}{\log z} < \frac{\log 2}{1-z}. \tag{2.4}$$

By Lemma 2.3 and [2.2–2.4], we have

$$\begin{aligned}
\log g(z) &< \sum_{n=1}^{\infty}(2n+1)\frac{z^n}{1-z^n} \\
&= \sum_{n=1}^{M}(2n+1)\frac{z^n}{1-z^n} + \sum_{n=M+1}^{\infty}(2n+1)\frac{z^n}{1-z^n} \\
&\le M\frac{(2\times 1+1)z^1}{1-z^1} + \frac{1}{1-z^{M+1}}\sum_{n=M+1}^{\infty}(2n+1)z^n \\
&\le M\frac{3z}{1-z} + 6\sum_{n=1}^{\infty} nz^n \\
&\le (3\log 2)\frac{z}{(1-z)^2} + 6\frac{z}{(1-z)^2} \\
&= (3\log 2 + 6)\frac{z}{(1-z)^2}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. The conclusion is trivial for $n=1$. So we may assume that $n \ge 2$. By (1.2), Lemma 2.4 and Lemma 2.2, for $0 < z < 1$, we have

$$\log(q(n)z^n) \le \log g(z) \le (3\log 2 + 6)\frac{z}{(1-z)^2} < \frac{3\log 2 + 6}{z(-\log z)^2}.$$

That is,

$$\log q(n) \le \frac{3\log 2 + 6}{z(-\log z)^2} - n\log z, \quad 0 < z < 1.$$

We choose $z_1 = \exp(-n^{-1/3})$. Then $z_1 \ge e^{-1}$ and

$$\log q(n) \le \frac{3\log 2 + 6}{e^{-1}(-\log z_1)^2} - n\log z_1 = c_2 n^{2/3}, \tag{2.5}$$

where $c_2 = 3e\log 2 + 6e + 1$.

We will use this upper bound to give a lower bound of $\log q(n)$. This is the main augment of the proof.

For $0 < z < 1$, by Lemma 2.1 and (2.5), we have

$$g(z) = 1 + \sum_{k=1}^{n-1} q(k)z^k + \sum_{k=n}^{\infty} q(k)z^k \le nq(n) + \sum_{k=n}^{\infty}\exp(c_2 k^{2/3})z^k.$$

We choose $z_2 = \exp(-2c_2 n^{-1/3})$. Then

$$\exp(c_2 k^{2/3}) z_2^{k/2} = \exp(c_2(k^{2/3} - kn^{-1/3})) \le 1, \quad k \ge n.$$

Thus, by Lemma 2.1, $c_2 > 6$ and $e^x - 1 > x$ for $x > 0$, we have

$$\begin{aligned} g(z_2) &\le nq(n) + \sum_{k=n}^{\infty} z_2^{k/2} = nq(n) + \frac{(\sqrt{z_2})^n}{1-\sqrt{z_2}} \\ &= nq(n) + \frac{\exp(-c_2 n^{2/3})}{1-\exp(-c_2 n^{-1/3})} \\ &= nq(n) + \frac{\exp(-c_2(n^{2/3} - n^{-1/3}))}{\exp(c_2 n^{-1/3}) - 1} \\ &< nq(n) + \frac{1}{6} n^{1/3} \exp(-6(n^{2/3} - n^{-1/3})) \\ &< nq(n) + 1 < 2nq(n) \le q(n)^2. \end{aligned}$$

It follows from Lemmas 2.4 and (2.2) that

$$\log q(n) \ge \frac{1}{2} \log g(z_2) \ge \frac{z_2}{(1-z_2)^2} \ge \frac{z_2}{(\log z_2)^2} \ge \left(\frac{1}{4} c_2^{-2} \exp(-2c_2)\right) n^{2/3}. \qquad \square$$

Remark 2.5. Let $\tau(a)$ be a real function. Let $q_\tau(n)$ be the number of the equation

$$n = [\tau(a_1)] + [\tau(a_2)] + \cdots + [\tau(a_k)] \tag{2.6}$$

with integers $1 \le a_1 \le a_2 \le \cdots \le a_k$. We call (2.6) a τ partition of n and $q_\tau(n)$ the τ partition function. Similar to the arguments in this paper with slight changes, for $\tau(a) = a^\beta$, where β is any fixed real number with $0 < \beta < 1$, there exist two positive constants $c_1 = c_1(\tau)$ and $c_2 = c_2(\tau)$ such that

$$\exp(c_1 n^{1/(\beta+1)}) \le q_\tau(n) \le \exp(c_2 n^{1/(\beta+1)}).$$

References

[1] R. Balasubramanian, F. Luca, On the number of factorizations of an integer, Integers 11 (2011) A12, 5 p.