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Calculus of variations

Duality for non-convex variational problems

Dualité pour des problèmes variationnels non convexes

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ABSTRACT

We consider classical problems of the calculus of variations of the kind

$$\mathcal{I}(\Omega) := \inf \left\{ \int_{\Omega} f(u, \nabla u)\, dx + \int_{\Gamma_1} \gamma(u)\, dH^{N-1},\ u = u_0 \text{ on } \Gamma_0 \right\} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, (Γ_0, Γ_1) is a partition of $\partial\Omega$, γ is a Lipschitz function, and $f = f(t, z)$ is an l.s.c. function satisfying suitable growth conditions, which is convex in z, but possibly not in t. We present a new duality theory in which the dual problem reads quite nicely as a linear programming problem. The solvability of such a dual problem is a major issue. It can be achieved in the one-dimensional case, and in higher dimensions under special assumptions on f. Our results apply to phase transition and free-boundary problems.

RÉSUMÉ

On considère des problèmes classiques en calcul de variations de la forme (1), ou Ω est un ouvert borné de $\mathbb{R}^N$, (Γ_0, Γ_1) une partition de $\partial\Omega$, γ une fonction lipschitzienne et $f = f(t, z)$ est une fonction s.c.i. qui vérifie des hypothèses de croissance, avec dépendence convexe en z mais a priori non convexe en t. On présente une nouvelle théorie de dualité, où le problème dual apparaît comme un problème de programmation linéaire. L'existence d'une solution à ce problème constitue une question délicate. Dans cette note, elle est obtenue en dimension un, et en dimension supérieure moyennant quelques hypothèses supplémentaires. Nos résultats s'appliquent à des problèmes de transition de phase et à frontière libre.

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1. Introduction

The aim of this note is to announce a new duality theory for non-convex variational problems of the form

$$\mathcal{I}(\Omega) := \inf \left\{ \int_\Omega f(u, \nabla u)\, dx + \int_{\Gamma_1} \gamma(u)\, dH^{N-1} \ : \ u \in W^{1,p}(\Omega) \,,\ u = u_0 \text{ on } \Gamma_0 \right\}, \tag{2}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with a Lipschitz boundary, (Γ_0, Γ_1) is a partition of $\partial\Omega$, γ is a Lipschitz function, and $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is an l.s.c. integrand, satisfying standard p-growth conditions that ensure the existence of a minimizer in $W^{1,p}(\Omega)$ if $p > 1$ and of a relaxed solution in $BV(\Omega)$ if $p = 1$. The key point is that the integrand $f = f(t, z)$ is assumed to be convex in z, but it may have a *non-convex* dependence in t. While duality theory for convex problems is a widely developed and by now classical topic (see, *e.g.*, [6]), almost no extension is to our knowledge available outside the convex framework (in this direction, let us address the papers [3,7]). The same lack occurs about necessary and sufficient optimality conditions for minimizers that we would like to find under the form of some Euler–Lagrange equation. Trying to fill these gaps has not only its own interest from a theoretical point of view, but can be motivated by several classical optimization problems whose global minimizers so far cannot be characterized by variational methods. In this respect, let us address the following two examples:

– the case when problem (2) describes the configuration of a *Cahn–Hilliard fluid*: let $\Gamma_1 = \partial\Omega$, let $\gamma \in W^{1,\infty}(\mathbb{R})$ represent a wetting function, and take the integrand f of the form

$$f(t, z) := \varepsilon |z|^2 + W(t) - \lambda t,$$

where W is a two-wells potential, ε is a small positive parameter, and λ is a Lagrange multiplier;

– the case when problem (2) corresponds to a *free-boundary problem* for minimal surfaces: let $\Gamma_0 = \partial\Omega$ and $u_0 \equiv 1$, and take the integrand f of the form

$$f(t, z) := \sqrt{1 + |z|^2} + \mathbf{1}_{(0,+\infty)}(t),$$

where $\mathbf{1}_{(0,+\infty)}$ is the function that equals 1 on $(0, +\infty)$ and 0 outside (as variants, one may replace the squareroot term in f by $|z|$ or by $|z|^2$, thus falling upon the free-boundary problem studied in [2]).

The approach we adopt in order to deal with this kind of problems is based on the idea of reformulating them in $(N + 1)$ space dimensions; this approach, already exploited in [1] for the Mumford–Shah functional, consists in identifying any admissible function $u : \Omega \to \mathbb{R}$ with the characteristic function of its subgraph, and leads, via a generalized coarea formula, to a dual problem that reads simply as a linear program where competitors are divergence-free vector fields that satisfy pointwise a.e. on $\Omega \times \mathbb{R}$ a convex constraint. Our main result states that there is no duality gap (Theorem 2.1); moreover, we are able to characterize solutions to the primal and to the dual problem through two conditions: they must satisfy on the graph of an optimal function $\bar{u}$ (Theorem 3.1). The existence of a solution to the dual problem, that we call a *calibration*, is a major issue. Our achievements in this direction are collected in Section 3. After exhibiting a calibration in the convex case (when our dual problem reduces to the classical one), we turn to the non-convex setting: for $N = 1$, by suitably perturbing the initial problem and by applying Bellman's dynamic principle to the associated value function, we are able to provide some explicit calibrations; for $N > 1$, we can give a partial existence result that holds under special assumptions on f, and relies on rearrangement techniques for integrals with non-constant densities. As an aside, our results open an innovative perspective, namely the possibility of studying the stability of minimizers of functionals like $\mathcal{I}(\Omega)$ by computing their shape derivative: actually, in the non-convex setting, classical methods for the computation of shape derivatives are completely stuck, whereas we believe that the results presented in this note should allow us to extend successfully the approach recently proposed in [4].

2. The duality result

Let us fix the setting in which we consider the minimization problem $\mathcal{I}(\Omega)$ introduced in (2): we let admissible functions u belong to $W^{1,p}(\Omega)$ for some fixed $p \geq 1$, we take a Lipschitz function $\gamma : \mathbb{R} \to \mathbb{R}$, and we assume that the integrand $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$f \text{ is lower semicontinuous in } (t, z); \tag{3}$$

$$f \text{ is convex in } z; \tag{4}$$

$$\text{there exist } \alpha, \beta, \delta > 0 : \alpha |z|^p - \delta \leq f(t, z) \leq \beta(1 + |z|^p) \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^N. \tag{5}$$

Moreover, we assume that $u_0 \equiv 0$ (this is just for the sake of simplicity, *cf.* Remark 1 below).

Some further notation is needed in order to introduce the dual problem. We define the family of fields

$$K(\Omega) := \Big\{\sigma \in B^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{N+1}) \ : \ \sigma(x,t) \in C(t) \text{ on } \Omega \times \mathbb{R}\Big\}, \tag{6}$$

where $B^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{N+1})$ is the class of bounded Borel regular vector fields from $\Omega \times \mathbb{R}$ into $\mathbb{R}^N$, and $C(t)$ is the convex subset of $\mathbb{R}^{N+1}$ given for every $t \in \mathbb{R}$ by

$$C(t) := \big\{q = (q^x, q^t) : q^t \ge f_z^*(t, q^x)\big\}; \tag{7}$$

here and below f_z^* denotes the Fenchel conjugate of f with respect to z, and variables q in $\mathbb{R}^{N+1}$ are indicated by $(q^x, q^t) \in \mathbb{R}^N \times \mathbb{R}$. Then we set

$$\mathcal{I}^*(\Omega) := \sup_{\sigma \in K(\Omega)} \Big\{ -\int_\Omega \sigma^t(x,0)\,\mathrm{d}x : \operatorname{div}\sigma = 0 \text{ in } \Omega \times \mathbb{R},\ \sigma^x \cdot \nu_\Omega = -\gamma'(t) \text{ a.e. on } \Gamma_1 \times \mathbb{R}\Big\}, \tag{8}$$

where the divergence is intended in distributional sense, $\sigma^t(x,0)$ is meant as the normal trace of σ on $\Omega \times \{0\}$, and ν_Ω denotes the unit outer normal to $\partial\Omega$.

Theorem 2.1. *Under the above assumptions, it holds*

$$\mathcal{I}(\Omega) = \mathcal{I}^*(\Omega). \tag{9}$$

Remark 1. The equality (9) remains true when u has a prescribed trace $u_0 \not\equiv 0$ on Γ_0: one has just to replace the integral $\int_\Omega \sigma^t(x,0)\,\mathrm{d}x$ appearing in the expression of $\mathcal{I}^*(\Omega)$ by $\int_S \sigma \cdot \nu_S \,\mathrm{d}H^N$, being S the graph of a function that agrees with u_0 on Γ_0, and ν_S the unit normal to S oriented upwards. Moreover, by taking $\Gamma_1 = \partial\Omega$ and $\gamma = 0$, we can handle homogeneous Neumann boundary conditions. On the other hand, our duality principle can be easily extended to the case when f (and possibly γ) depend also on the spatial variable x. Possible adaptations of the same idea to variational integrals involving the Hessian of u are not straightforward and will deserve further investigations.

Remark 2. Proving the existence of a solution to the dual problem $\mathcal{I}^*(\Omega)$ in a reasonable space of fields appears as a quite challenging task. The main difficulty comes from the fact that there is no control on the positive part of σ_n^t, being σ_n a maximizing sequence.

Remark 3. In case $\mathcal{I}(\Omega)$ admits a solution $\bar{u} \in L^\infty(\Omega)$, it is enough to impose the divergence constraint appearing in the dual problem in a bounded subset of $\Omega \times \mathbb{R}$. Precisely, for any $M > \|u\|_\infty$, it holds:

$$\mathcal{I}(\Omega) = \mathcal{I}^*_M(\Omega) := \sup_{\sigma \in K(\Omega)} \Big\{ -\int_\Omega \sigma^t(x,0)\,dx : \operatorname{div}\sigma = 0 \text{ in } \Omega \times (-M,M),\ \sigma^x \cdot \nu_\Omega = -\gamma'(t) \text{ a.e. on } \Gamma_1 \times (-M,M)\Big\}.$$

Sketch of proof of Theorem 2.1. The inequality $\mathcal{I}^*(\Omega) \le \mathcal{I}(\Omega)$ is straightforward: for any pair (u, σ) admissible respectively in the primal and in the dual problem, since σ is divergence free, we may apply the Gauss–Green Theorem on the region $\Delta := \{(x, su(x)) : x \in \Omega,\ s \in (0,1)\}$ so that

$$-\int_\Omega \sigma^t(x,0)\,\mathrm{d}x = \int_\Sigma \sigma^x \cdot \nu_\Omega \,\mathrm{d}H^N + \int_{G_u} \sigma \cdot \nu_u \,\mathrm{d}H^N$$

where G_u denotes the graph of u, ν_u is the unit outer normal to Δ on G_u, and $\Sigma := \partial\Delta \cap (\Gamma_1 \times \mathbb{R})$. The first integral at the r.h.s. of the above equality can be computed as

$$\int_\Sigma \sigma^x \cdot \nu_\Omega \,\mathrm{d}H^N = \int_{\Gamma_1} \Big(\int_0^{u(x)} \gamma'(t)\,\mathrm{d}t\Big)\,\mathrm{d}H^{N-1} = \int_{\Gamma_1} \gamma(u)\,\mathrm{d}H^{N-1},$$

while the second one can be estimated by using the Fenchel inequality and the constraint $\sigma(x,t) \in C(t)$:

$$\begin{aligned}\int_{G_u} \sigma \cdot \nu_u \,\mathrm{d}H^N &= \int_\Omega \big[\sigma^x(x, u(x)) \cdot \nabla u - \sigma^t(x, u(x))\big]\,\mathrm{d}x \\ &\le \int_\Omega \big[f_z^*(u(x), \sigma^x(x,u(x))) + f(x, \nabla u(x)) - \sigma^t(x,u(x))\big]\,\mathrm{d}x \le \int_\Omega f(u, \nabla u)\,\mathrm{d}x.\end{aligned}$$

We have thus obtained that $-\int_\Omega \sigma^t(x,0)\,dx \le \int_\Omega f(u,\nabla u)\,dx + \int_{\Gamma_1}\gamma(u)\,dH^{N-1}$, which implies the inequality $\mathcal{I}^*(\Omega) \le \mathcal{I}(\Omega)$ by the arbitrariness of u and σ.

The converse inequality $\mathcal{I}(\Omega) \le \mathcal{I}^*(\Omega)$ is much more delicate, and we limit ourselves to sketch it, working for simplicity in the pure homogeneous Dirichlet case, *i.e.* $\Gamma_0 = \partial\Omega$. We proceed in two steps:

Step 1. We reformulate problem $\mathcal{I}(\Omega)$ in $(N+1)$ space dimensions as

$$\mathcal{I}(\Omega) = \inf\Big\{ \int_{\Omega\times\mathbb{R}} h_f(t, D\mathbf{1}_u) \ :\ u \in BV(\Omega),\ u = 0 \text{ on } \partial\Omega \Big\},$$

where $\mathbf{1}_u$ is the characteristic function of the subgraph of u, and the integrand $h_f = h_f(t,p)$ equals $-p^t f\big(t, -p^x/p^t\big)$ if $p^t < 0$, $+\infty$ if $p^t > 0$ or $p^t = 0$, $p^x \neq 0$, and 0 if $p^t = |p^x| = 0$. Then we show that

$$\mathcal{I}(\Omega) = \mathcal{J}(\Omega) := \inf\Big\{ \int_{\Omega\times\mathbb{R}} h_f(t, Dv) \ :\ v \in L^1_{\mathrm{loc}}(\Omega\times\mathbb{R}),\ v(x,t) - H(-t) \in BV(\Omega\times\mathbb{R}) \Big\}, \tag{10}$$

being H the Heavyside function. The proof of the above equality is based on the following identity, obtained through a new *generalized coarea formula* (here $\mathbf{1}_{\{v>s\}}$ is the characteristic function of $\{v > s\}$):

$$\int_{\Omega\times\mathbb{R}} h_f(t, Dv) = \int_0^{+\infty} ds \int_{\Omega\times\mathbb{R}} h_f(t, D\mathbf{1}_{\{v>s\}}).$$

Step 2. The integrand $h_f(t,\cdot)$ is precisely the support function of the convex set $C(t)$ introduced in (7), *i.e.*, $h_f(t,p) = \sup\{p\cdot q \ :\ q \in C(t)\}$. Then, by exploiting an *inf–sup commutation argument*, we show that $\mathcal{J}(\Omega) \le \mathcal{I}^*(\Omega)$. Combined with (10), this yields $\mathcal{I}(\Omega) \le \mathcal{I}^*(\Omega)$ as required. □

3. Optimality conditions

In the light of Theorem 2.1, solutions to primal and dual problems can be easily characterized as follows:

Theorem 3.1. *Under the same assumptions of Theorem 2.1, a function u and a field σ admissible respectively for $\mathcal{I}(\Omega)$ and $\mathcal{I}^*(\Omega)$ are optimal for such problems if and only if they satisfy*

$$\sigma^x(x,u(x)) \in \partial_z f(u(x), \nabla u(x)) \quad \text{and} \quad \sigma^t(x,u(x)) = f_z^*(u(x), \sigma^x(x,u(x))) \quad \text{a.e. in } \Omega. \tag{11}$$

If u is a candidate minimizer for problem $\mathcal{I}(\Omega)$, we call a *calibration for u* any vector field σ, admissible for problem $\mathcal{I}^*(\Omega)$, which satisfies conditions (11): since they essentially determine σ on the graph of u (actually in a unique way as soon as f is differentiable in z), the challenge in the construction of a calibration is to find a divergence-free extension of σ outside the graph of u, which satisfies also the pointwise constraint $\sigma(x,t) \in C(t)$ on $\Omega\times\mathbb{R}$.

Let us point out that, if u belongs to $L^\infty(\Omega)$, it is enough to construct this extension in $\Omega\times(-M,M)$, being $M > \|u\|_\infty$, since one can simply take $\sigma \equiv 0$ outside (indeed, note that 0 belongs to the convex set $C(t)$).

In the remaining of this note, we present the construction of a calibration in the following situations: the case when f is globally convex, the case when $N = 1$, and finally the general case when $N > 1$ and f is non-convex in t (but satisfies some additional assumptions).

3.1. Calibrations in the convex case

Assume that f is convex in (t,z) and $\gamma = 0$. Assume further that $\mathcal{I}(\Omega)$ admits a solution $\bar{u} \in L^\infty(\Omega)$.

By classical results of duality theory in the convex framework (see, *e.g.*, [6]), there exists $\bar{\eta}$ satisfying the optimality conditions $\bar{\eta} \in \partial_z f(\bar{u}, \nabla\bar{u})$ and $\operatorname{div}\bar{\eta} \in \partial_t f(\bar{u}, \nabla\bar{u})$ a.e. in Ω.

Then it is easy to check that a calibration $\sigma = (\sigma^x, \sigma^t)$ for $\bar{u}$ is given by $\sigma = 0$ outside $\Omega\times(-M,M)$ and

$$\sigma^x(x,t) = \bar{\eta}(x), \qquad \sigma^t(x,t) = f_z^*(\bar{u}, \bar{\eta}) - (\operatorname{div}\bar{\eta})\big(t - \bar{u}(x)\big) \quad \text{in } \Omega\times(-M,M). \tag{12}$$

3.2. Calibrations in the one dimensional case

Example 1. Let $N = 1$, $\Omega = (0,h)$, $\Gamma_0 = \{0,h\}$ and $u_0 = 0$. Then, by writing any divergence-free vector field σ under the form $(\partial_t w, -\partial_x w)$, our duality principle reads

$$\inf_u \Big\{ \int_0^h f(u,u')\,ds : u(0) = u(h) = 0 \Big\} = \sup_w \Big\{ w(h,0) - w(0,0) : -\partial_x w \ge f_z^*(t, \partial_t w) \Big\}.$$

Actually this equality can be directly obtained by using Bellman's optimality principle (see, *e.g.*, [5]), which also allows us to check that a calibration is given by the rotated gradient $(\partial_t V, -\partial_x V)$ of the *value function* $V(x,t) := \inf\Big\{ \int_0^x f(u,u')\,ds : u(0)=0,\ u(x)=t \Big\}$. It is worth noticing that when f is convex, such calibration may happen to be different from the one given in (12) (this can be checked by taking for instance $h=1$, $f(t,z)=\frac{1}{2}\big(|t-1|^2+|z|^2\big)$, and making explicit computations).

Example 2. Let $N=1$, $\Omega=(0,h)$, $\Gamma_0=\{0\}$, $f(t,z)=\frac{|z|^2}{2}$, and $u_0=0$. Then, in a similar way to the situation above, it holds

$$\inf_u \Big\{ \int_0^h \frac{|u'|^2}{2}\,ds + \gamma(u(h)) : u(0)=0 \Big\} = \sup_w \Big\{ w(h,0)-w(0,0) : -\partial_x w \ge \frac{|\partial_t w|^2}{2},\ w(h,t) = -\gamma(t) \Big\},$$

and a calibration is given by $\sigma=(-\partial_t W, \partial_x W)$, with $W(x,t) := \inf\Big\{ \int_x^h \frac{|u'|^2}{2}\,ds + \gamma(u(h)) : u(x)=t \Big\}$. We remark that, in this example, having a calibration at disposal allows us to distinguish local from global minimizers. Namely, if one takes for instance $\gamma(t) = -(1-|t-2|)_+$, the local minimizer $u=0$ turns out to be global only for $h<1$, whereas for $h\ge 1$ the unique global minimizer is $u(s)=\frac{2s}{h}$; incidentally, by this way one sees that $\mathcal{I}(\cdot)$ is not differentiable as a shape functional, as $\mathcal{I}((0,h)) = \min\big\{\frac{2}{h}-1, 0\big\}$.

3.3. Calibrations in the general case: a partial result

In higher dimension and outside the convex framework, we can prove the following existence result:

Theorem 3.2. *Assume that $\Gamma_0=\partial\Omega$, and that $f(t,z)=\varphi(z)+g(t)$, with φ and g satisfying:*

(i) *φ is convex;*
(ii) *there exist $\alpha,\beta,\delta>0$ such that $\alpha|z|-\delta \le \varphi(z) \le \beta(1+|z|)$ $\forall (t,z)\in\mathbb{R}\times\mathbb{R}^N$;*
(iii) *there exists $K>0$ such that $\varphi^*(z)\le K$ $\forall z^*\in \mathrm{dom}(\varphi^*)$;*
(iv) *$g:\mathbb{R}\to[0,+\infty)$ is bounded, lower semicontinuous, and nondecreasing.*

Then there exists a calibration; more precisely, the supremum $\mathcal{I}^*(\Omega)$ in (8) is attained at some field $\overline{\sigma}$ which belongs to the bounded *convex set $\widetilde{C}(t) := \big\{ q=(q^x,q^t) : |q^t|+\varphi^*(q^x)\le g(t)\big\}$.*

We point out that the above result allows to cover for instance the free-boundary problem mentioned in the introduction, when $\Gamma_0=\partial\Omega$, $u_0\equiv 1$, and $f(t,z)=\sqrt{1+|z|^2}+\mathbf{1}_{(0,+\infty)}(t)$.

Its proof is quite delicate and is based on the idea of considering a modified dual problem $\widetilde{\mathcal{I}}^*(\Omega)$, which is distinguished from $\mathcal{I}^*(\Omega)$ only for the different pointwise constraint imposed on the admissible fields σ: they must take values into the bounded set $\widetilde{C}(t)$ introduced in the statement above, in place of the set $C(t)$ defined in (7). Thanks to some rearrangement results for integral functionals with non-constant densities proved in [8] (applied to the suitable modification $\widetilde{\mathcal{J}}(\Omega)$ of the functional $\mathcal{J}(\Omega)$ in (10)), it turns out that $\widetilde{\mathcal{I}}^*(\Omega)$ agrees with $\mathcal{I}^*(\Omega)$, and both are attained at some field $\overline{\sigma}$ belonging a.e. to $\widetilde{C}(t)$.

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