C. R. Acad. Sci. Paris, Ser. I 353 (2015) 291–296

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory

Exceptional parameters of linear mod one transformations and fractional parts $\{\xi(p/q)^n\}$

Paramètres exceptionnels de transformations linéaires mod un et parties fractionnaires $\{\xi(p/q)^n\}$

DoYong Kwon

Department of Mathematics, Chonnam National University, Gwangju 500-757, Republic of Korea

A R T I C L E I N F O

Article history:
Received 22 April 2014
Accepted after revision 28 January 2015
Available online 11 February 2015

Presented by the Editorial Board

A B S T R A C T

We study exceptional parameters of linear mod one transformations. The present note proves that the set of such values has Hausdorff dimension zero. This answers the question posed by Bugeaud.

R É S U M É

Nous étudions des paramètres exceptionnels de transformations linéaires mod un. La présente note prouve que l'ensemble de ces valeurs a zéro pour dimension de Hausdorff. Ceci répond à la question posée par Bugeaud.

1. Introduction

Let $\{\cdot\}$ denote the fractional part. For an interval $[s, s+t) \subset [0, 1)$, and coprime integers p, q with $p > q \geq 2$, we set

$$Z_{p/q}(s, s+t) := \{\xi > 0 : s \leq \{\xi(p/q)^n\} < s+t \text{ for all integers } n \geq 0\}.$$

A well-known connection between Waring's problem and the fractional parts $\{(3/2)^n\}$ leaded Mahler, in [13], to consider a real number $\xi \in Z_{3/2}(0, 1/2)$. Mahler called such a hypothetical number a *Z-number*. He proved that there are at most countably many Z-numbers, but was unable to determine whether $Z_{3/2}(0, 1/2)$ is empty or not. A series of researches since then tell us that, let alone Mahler's original Z-number problem, it is nontrivial enough to determine whether $Z_{p/q}(s, s+\frac{1}{p})$ is empty or not [5,1,3]. The present note is another small step in this direction. It is worthwhile to mention here that if $q = 1$ then the set $Z_{p/1}(s, s+\frac{1}{p})$ is now completely understood thanks to Bugeaud and Dubickas [2].

Let $\tau \in [0, 1)$ be real. For any $k \in \mathbb{Z}$, put

$$\varepsilon_k(\tau) := \lfloor k\tau \rfloor - \lfloor (k-1)\tau \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integral part.

E-mail address: doyong@jnu.ac.kr.

http://dx.doi.org/10.1016/j.crma.2015.01.017
1631-073X/

Proposition 1.1. *(See [1].) Let $p > q \ge 2$ be coprime integers. Then the set $Z_{p/q}(s, s+\frac{1}{p})$ is empty if there exists a reduced rational a/b with $b > a \ge 1$ such that*

$$\frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^b}{1+\frac{q}{p}+\cdots+\left(\frac{q}{p}\right)^{b-1}} \le \{(p-q)s\} \le \frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^{b-1}}{1+\frac{q}{p}+\cdots+\left(\frac{q}{p}\right)^{b-1}}.$$

And the set of s satisfying the inequality is of full Lebesgue measure in $[0, 1-\frac{1}{p}]$. Conversely, if $Z_{p/q}(s, s+\frac{1}{p})$ is nonempty for some $s \in [0, 1-\frac{1}{p}]$, then there exists an irrational $\tau \in (0,1)$ such that

$$\{(p-q)s\} = \frac{p-q}{p}\sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\left(\frac{q}{p}\right)^k.$$

Later, in [3], Dubickas showed that the set $Z_{p/q}(s, s+\frac{1}{p})$ is actually empty for every $s \in [0, 1-\frac{1}{p}]$ provided that p and q satisfy $1 < q < p < q^2$. In particular, the set $Z_{3/2}(s, s+\frac{1}{3})$ is empty for each $s \in [0, \frac{2}{3}]$. In the case of $p > q^2$, the determination of whether or not the set $Z_{p/q}(s, s+\frac{1}{p})$ is empty still remains open.

Bugeaud obtained the above proposition by a careful study of linear mod one transformations. For real numbers $\beta > 1$ and $0 \le \alpha < 1$, the linear mod one transformation $f_{\beta,\alpha}$ is defined by

$$f_{\beta,\alpha}(x) := \{\beta x + \alpha\} \text{ for } x \in [0,1).$$

Via setting

$$S_{\beta,\alpha} := \{x \in [0,1) : 0 \le f_{\beta,\alpha}^n(x) < 1/\beta \text{ for all } n \ge 0\},$$

the linear mod one transformation enters the picture as the next proposition says.

Proposition 1.2. *(See [5].) Let $p > q \ge 2$ be coprime integers and $s \in [0, 1-\frac{1}{p}]$. If $S_{p/q,\{(p-q)s\}}$ is a finite set, then the set $Z_{p/q}(s, s+\frac{1}{p})$ is empty.*

Therefore, we have a good reason to specify the following set: for a fixed $\beta > 1$,

$$E_\beta := \{\alpha \in [0,1) : S_{\beta,\alpha} \text{ is an infinite set}\}.$$

Flatto et al. [5] suspected that E_β has Lebesgue measure zero and is a nonempty perfect set. But the perfectness of E_β turns out to be false. For $\beta > 1$, put $\gamma = 1/\beta$. We define intervals $J_b^a(\gamma)$ by $J_1^1(\gamma) := [\gamma, 1)$, and by

$$J_b^a(\gamma) := \left[\frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\gamma^k + \gamma^b}{1+\gamma+\cdots+\gamma^{b-1}}, \frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\gamma^k + \gamma^{b-1}}{1+\gamma+\cdots+\gamma^{b-1}}\right], \text{[1]}$$

for coprime integers $b > a \ge 1$. One observes that if $\beta = p/q$, then the interval $J_b^a(\gamma)$ is given by the inequality in Proposition 1.1.

Proposition 1.3. *(See [1].) For any real $\beta > 1$, the set E_β has Lebesgue measure zero, is uncountable, and is not closed. More precisely,*

$$E_\beta = [0,1) \Big\backslash \bigcup_{\substack{1 \le a \le b \\ \gcd(a,b)=1}} J_b^a(\gamma).$$

Owing to this result, one notes that the set $Z_{p/q}(s, s+\frac{1}{p})$ is empty, possibly except when s lies in a Lebesgue negligible set. In [1, Remark 3], Bugeaud posed a problem to determine the Hausdorff dimension of the set E_β. The present note settles this problem.

Main Theorem. *The set E_β has Hausdorff dimension zero.*

This computation is possible by recognizing the unexpected connection with power series whose coefficient are Sturmian words, which have been investigated by the author [7,8,10,11]. Note that, in the context of a certain generalization of [2], this type of power series appears as well [9].

[1] In [1, Lemma 3], '$\sum_{k=1}^{b-1}$' was used instead of '$\sum_{k=1}^{b-2}$', which, however, makes no difference because $\varepsilon_{-(b-1)}(a/b) = 0$. We adopt here '$\sum_{k=1}^{b-2}$' to make the notation coherent with Proposition 1.1.

2. Preliminaries on Sturmian words

Let $\mathbb{Z}$ (resp. $\mathbb{N}$) be the set of integers (resp. nonnegative integers). For $\alpha, \rho \in [0, 1]$, two arithmetic functions $s_{\alpha,\rho}, s'_{\alpha,\rho} : \mathbb{Z} \to A := \{0, 1\}$ are defined by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$
$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Then two (right) infinite words

$$s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots \text{ and } s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$$

are termed *lower* and *upper mechanical words*, respectively, with *slope* α and *intercept* ρ. The bi-infinite mechanical words are also considered and written as

$$\overleftrightarrow{s_{\alpha,\rho}} := \cdots s_{\alpha,\rho}(-1)\underline{s_{\alpha,\rho}(0)}s_{\alpha,\rho}(1)\cdots, \quad \overleftrightarrow{s'_{\alpha,\rho}} := \cdots s'_{\alpha,\rho}(-1)\underline{s'_{\alpha,\rho}(0)}s'_{\alpha,\rho}(1)\cdots,$$

where the zeroth letters are marked with underlines. If α is irrational, then $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are called *Sturmian words*. The case of $\rho = 0$ is of our special interest, which is stated as a lemma below. For any finite word $u \in A^*$, we write $\widetilde{u}$ for its reversal, and a word u satisfying $\widetilde{u} = u$ is said to be a *palindrome*. We mean by $u^\omega := uuu\cdots$ (resp. ${}^\omega u := \cdots uuu$) the right (resp. left) infinite word infinitely concatenated by u. If $s \in A^{\mathbb{N}}$ is a right infinite word, then $\widetilde{s}$ is the left infinite word that is defined in a natural manner.

Lemma 2.1. *Let $\alpha \in [0, 1]$ be real.*

(a) *If α is irrational, then there exists a right infinite word $c_\alpha \in A^{\mathbb{N}}$ such that*
$$\overleftrightarrow{s_{\alpha,0}} = \widetilde{c_\alpha}1\underline{0}c_\alpha, \text{ and } \overleftrightarrow{s'_{\alpha,0}} = \widetilde{c_\alpha}0\underline{1}c_\alpha.$$

(b) *If $\alpha = a/b$ is rational with $\gcd(a, b) = 1$, then there exists a palindrome $z_{a,b} \in A^{b-2}$ of length $b - 2$ such that*
$$\overleftrightarrow{s_{\alpha,0}} = {}^\omega(0z_{a,b}1)\underline{0}(z_{a,b}10)^\omega, \text{ and } \overleftrightarrow{s'_{\alpha,0}} = {}^\omega(1z_{a,b}0)\underline{1}(z_{a,b}01)^\omega.$$

Proof. See [12]. □

Remark 1.

(i) In the lemma, c_α and $z_{a,b}$ are called the *characteristic word* and the *central word* respectively in the literature.
(ii) If $b = 2$, then $z_{a,b}$ is the empty word. In case $b = 1$, we adopt convention that $\overleftrightarrow{s_{0,0}} = \overleftrightarrow{s'_{0,0}} = {}^\omega 0\underline{0}0^\omega$ and $\overleftrightarrow{s_{1,0}} = \overleftrightarrow{s'_{1,0}} = {}^\omega 1\underline{1}1^\omega$.

For $\beta > 1$, let $(\cdot)_\beta$ send each infinite word $a_0a_1\cdots \in A^{\mathbb{N}}$ to a real number $\sum_{i=0}^{\infty} a_i/\beta^{i+1}$. Then a real function $\mu_\beta : [0, 1] \to \mathbb{R}$ is defined by

$$\mu_\beta(x) := (s'_{x,0})_\beta = \sum_{i=0}^{\infty} \frac{s'_{x,0}(i)}{\beta^{i+1}}.$$

A detailed real analysis on μ_β was pursued in [10,11]. However, the following results are enough for our purpose.

Proposition 2.2. *For any fixed $\beta > 1$, the function $\mu_\beta(x)$ is strictly increasing. If $\alpha \in [0, 1]$ is irrational, then the function $\mu_\beta(x)$ is continuous at $x = \alpha$. On the other hand, at rational α_0, $\mu_\beta(x)$ is left-continuous but not right-continuous.*

Proof. See [11, Lemma 3.1]. □

Proposition 2.3. *Let $\beta > 1$ be fixed, and suppose that $\alpha = a/b \in [0, 1]$ is rational with $\gcd(a, b) = 1$.*

(a) *The right limits of $\mu_\beta(x)$ at rational points are given by $\mu_\beta(0+) = (1(0^\omega))_\beta = \frac{1}{\beta}$, and by*
$$\mu_\beta(\alpha+) = (1(z_{a,b}10)^\omega)_\beta.$$

(b) $\mu_\beta(0+) - \mu_\beta(0) = \frac{1}{\beta}$, *and*
$$\mu_\beta(\alpha+) - \mu_\beta(\alpha) = (1(z_{a,b}10)^\omega)_\beta - ((1z_{a,b}0)^\omega)_\beta = \frac{\beta - 1}{\beta(\beta^b - 1)}.$$

(c) $$\sum_{\substack{0\le a/b<1\\ \gcd(a,b)=1}} \frac{\beta-1}{\beta(\beta^b-1)} = \frac{1}{\beta-1} = \mu_\beta(1),$$

where $\frac{\beta-1}{\beta(\beta^b-1)} = \frac{1}{\beta}$ *when* $a/b = 0/1 = 0$.

Proof. In [10], see Lemma 2.3 for (a), Theorem 2.4 for (b), and Theorem 3.1 for (c). □

Proposition 2.3 says that μ_β is a pure jump distribution. Noting that

$$\cdots\varepsilon_{-1}(\tau)\underline{\varepsilon_0(\tau)}\varepsilon_1(\tau)\cdots = \begin{cases} \widetilde{c_\tau}\underline{1}0c_\tau & \text{if } \tau \text{ is irrational,} \\ {}^\omega(10z_{a,b})\underline{1}(0z_{a,b}1)^\omega & \text{if } \tau = a/b \text{ with } \gcd(a,b)=1, \end{cases}$$

Proposition 1.1 can be restated more neatly.

Lemma 2.4. *Let $p > q \ge 2$ be coprime integers and put $\beta = p/q$. Then the set $Z_{p/q}(s, s+\frac{1}{p})$ is empty if there exists a reduced rational a/b with $b > a \ge 1$ such that*

$$\mu_\beta(a/b) \le \frac{q\{(p-q)s\}}{p-q} + \frac{q}{p} \le \mu_\beta((a/b)+).$$

Conversely, if $Z_{p/q}(s, s+\frac{1}{p})$ is nonempty for some $s \in [0, 1-\frac{1}{p}]$, then there exists an irrational $\tau \in (0,1)$ such that

$$\frac{q\{(p-q)s\}}{p-q} + \frac{q}{p} = \mu_\beta(\tau).$$

Proof. Divided by $\beta - 1$, the inequality in Proposition 1.1 becomes

$$(0(z_{a,b}01)^\omega)_\beta \le \frac{\{(p-q)s\}}{\beta-1} \le (0(z_{a,b}10)^\omega)_\beta,$$

which is, in turn, followed by

$$\mu_\beta(a/b) = ((1z_{a,b}0)^\omega)_\beta \le \frac{\{(p-q)s\}}{\beta-1} + \frac{1}{\beta} \le (1(z_{a,b}10)^\omega)_\beta = \mu_\beta((a/b)+).$$

For the case where $Z_{p/q}(s, s+\frac{1}{p})$ is nonempty, a similar argument shows that

$$\frac{\{(p-q)s\}}{\beta-1} + \frac{1}{\beta} = (1c_\tau)_\beta = \mu_\beta(\tau). \qquad \square$$

3. Proof of the main theorem

Let $\widetilde{E_\beta} := \mu_\beta([0,1]\setminus\mathbb{Q})$ be the image of μ_β at irrational points. Lemma 2.4 tells us that $E_\beta\setminus\{0\}$ is obtained by translation and then dilation of $\widetilde{E_\beta}$. More precisely, the same argument as in the proof of Lemma 2.4 derives

$$\widetilde{E_\beta} = \frac{1}{\beta-1}E_\beta + \frac{1}{\beta}.$$

So we compute the Hausdorff dimension of $\widetilde{E_\beta}$. Among diverse techniques for calculating Hausdorff dimensions is the following.

Lemma 3.1. *(See [4].) Suppose that a set S can be covered by n_k sets of diameter at most δ_k with $\lim_{k\to\infty}\delta_k = 0$. Then*

$$\dim_H S \le \liminf_{k\to\infty} \frac{\log n_k}{-\log\delta_k}.$$

We recall the Farey sequence. The reader is referred to [6] for details. For positive integer k, the *Farey sequence* F_k of order k is the sequence of reduced fractions $a/b \in [0,1]$ with $b \le k$, arranged in increasing order. For example,

$$F_1 = \{0/1, 1/1\},\ F_2 = \{0/1, 1/2, 1/1\},\ F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}, \ldots.$$

The next lemma is one of folklores.

Lemma 3.2. *(See [6].) The number of terms in F_k fulfills*

$$|F_k| - 1 = \Phi(k) := \sum_{i=1}^{k} \varphi(i) = \frac{3k^2}{\pi^2} + O(k\log k),$$

where φ is the Euler totient function.

We are now in a position to prove our main theorem.

Proof of Main Theorem. We construct a series of collections of sets that cover $\widetilde{E_\beta}$. For each integer $k \geq 1$, let

$$F_k = \{0 = r_1 < r_2 < \cdots < r_{\Phi(k)} < r_{\Phi(k)+1} = 1\}$$

be the Farey sequence of order k. Then a set $[0, \frac{1}{\beta-1}) \setminus \bigcup_{i=1}^{\Phi(k)} [\mu_\beta(r_i), \mu_\beta(r_i+)]$ covers $\widetilde{E_\beta}$, and consists of $n_k := \Phi(k)$ intervals. Suppose $r_i = a_i/b_i$. Then Proposition 2.3 shows that the diameter of each interval is less than or equal to

$$\delta_k := \frac{1}{\beta-1} - \sum_{i=1}^{\Phi(k)} \frac{\beta-1}{\beta(\beta^{b_i}-1)} = \frac{1}{\beta(\beta-1)} - \sum_{b=2}^{k} \frac{\varphi(b)(\beta-1)}{\beta(\beta^b-1)},$$

and also that δ_k tends to zero as k approaches infinity. Let us pick a number θ in the interval $(1, \beta)$. We claim that $\delta_k \leq \theta^{-k}$ for all sufficiently large k. Appealing to Proposition 2.3 again, one deduces that

$$\delta_k = \sum_{b=k+1}^{\infty} \frac{\varphi(b)(\beta-1)}{\beta(\beta^b-1)} \leq \frac{\beta-1}{\beta} \sum_{b=k+1}^{\infty} \frac{b-1}{\beta^{b-1}} = \frac{\beta-1}{\beta} \sum_{b=k}^{\infty} \frac{b}{\beta^b} = \frac{k(\beta-1)+1}{(\beta-1)\beta^k},$$

where $\varphi(b) \leq b-1$, and where $\beta^b - 1 \geq \beta^{b-1}$ as long as $k \geq -\log_\beta(\beta-1)$. Now the claim follows. We compute the Hausdorff dimension of $\widetilde{E_\beta}$:

$$\dim_H \widetilde{E_\beta} \leq \lim_{k\to\infty} \frac{\log n_k}{-\log \delta_k} \leq \lim_{k\to\infty} \frac{\log(3k^2/\pi^2 + O(k\log k))}{k\log\theta} = 0. \qquad \square$$

That the diameter of each interval in $[0, \frac{1}{\beta-1}) \setminus \bigcup_{i=1}^{\Phi(k)} [\mu_\beta(r_i), \mu_\beta(r_i+)]$ is less than or equal to δ_k is a very rough estimate. Moreover, δ_k tends to zero very quickly. Consequently, the above proof urges us to introduce 'thinner' measures than the Hausdorff measure.

For any set $U \subset \mathbb{R}^n$, we write $|U| := \sup\{|x-y| : x, y \in U\}$. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and increasing function. For any set $S \subset \mathbb{R}^n$, let us define

$$\mathcal{H}^h_\delta(S) := \inf\left\{\sum_{i=1}^{\infty} h(|U_i|) : S \subset \bigcup_{i=1}^{\infty} U_i,\ |U_i| \leq \delta\right\}.$$

Then $\mathcal{H}^h(S) := \lim_{\delta\searrow 0} \mathcal{H}^h_\delta(S)$ is known to be a measure. We have proved, in the above, that $\mathcal{H}^h(\widetilde{E_\beta}) = 0$ whenever $h(t) = t^s$ with $s > 0$. On the other hand, if we set $h(t) := (-\log t)^{-2-\varepsilon}$ for $\varepsilon > 0$, then $\mathcal{H}^h$ is a much thinner measure than any s-dimensional Hausdorff measure. Since $\mathcal{H}^h_{\delta_k}(\widetilde{E_\beta}) \leq n_k h(\delta_k)$, one deduces

$$\mathcal{H}^h(\widetilde{E_\beta}) \leq \lim_{k\to\infty} n_k h(\delta_k) \leq \lim_{k\to\infty} \frac{3k^2/\pi^2 + O(k\log k)}{(k\log\theta)^{2+\varepsilon}} = 0.$$

It seems to be a challenge to find a function h for which $0 < \mathcal{H}^h(\widetilde{E_\beta}) < \infty$. Such h is called an *(exact) dimension function* in the literature. See [4, Section 2.5].

Acknowledgements

The referee's comments and suggestions are gratefully acknowledged, which enhanced readability and motivated the author to consider dimension functions. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2007508).

References

[1] Y. Bugeaud, Linear mod one transformations and the distribution of fractional parts $\{\xi(p/q)^n\}$, Acta Arith. 114 (4) (2004) 301–311.
[2] Y. Bugeaud, A. Dubickas, Fractional parts of powers and Sturmian words, C. R. Math. Acad. Sci. Paris, Ser. I 341 (2) (2005) 69–74.
[3] A. Dubickas, Powers of a rational number modulo 1 cannot lie in a small interval, Acta Arith. 137 (3) (2009) 233–239.

[4] K. Falconer, Fractal Geometry. Mathematical Foundations and Applications, 2 edn., John Wiley & Sons, Inc., Hoboken, NJ, USA, 2003.
[5] L. Flatto, J.C. Lagarias, A.D. Pollington, On the range of fractional parts $\{\xi(p/q)^n\}$, Acta Arith. 70 (2) (1995) 125–147.
[6] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, 6th edn., Oxford University Press, Oxford, 2008.
[7] D.Y. Kwon, A devil's staircase from rotations and irrationality measures for Liouville numbers, Math. Proc. Camb. Philos. Soc. 145 (3) (2008) 739–756.
[8] D.Y. Kwon, The natural extensions of β-transformations which generalize Baker's transformations, Nonlinearity 22 (2) (2009) 301–310.
[9] D.Y. Kwon, The orbit of a β-transformation cannot lie in a small interval, J. Korean Math. Soc. 49 (4) (2012) 867–879.
[10] D.Y. Kwon, Moments of discrete measures with dense jumps induced by β-expansions, J. Math. Anal. Appl. 399 (1) (2013) 1–11.
[11] D.Y. Kwon, A two-dimensional singular function via Sturmian words in base β, J. Number Theory 133 (11) (2013) 3982–3994.
[12] M. Lothaire, Algebraic Combinatorics on Words, Cambridge University Press, Cambridge, 2002.
[13] K. Mahler, An unsolved problem on the powers of 3/2, J. Aust. Math. Soc. 8 (1968) 313–321.