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On some class of convex functions

Sur une classe de fonctions convexes

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ABSTRACT

In this paper, we consider the order of starlikeness and strong starlikeness in the class of functions $f(z) = z + a_2 z^2 + \dots$ analytic in $|z| < 1$ in the complex plane and satisfying

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad |z| < 1.$$

RÉSUMÉ

Dans cette Note, nous considérons l'ordre d'étoilement et d'étoilement fort pour la classe de fonctions $f(z) = z + a_2 z^2 + \dots$, analytiques dans le disque unité $\{|z| < 1\}$ du plan complexe et satisfaisant

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad |z| < 1.$$

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. By $\mathcal{S}$ we denote the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is said to be starlike if it maps $\mathbb{D}$ onto a starlike domain with respect to the origin. It is known that it is equivalent to $f \in \mathcal{A}$ and

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$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad z\in\mathbb{D}. \tag{1.2}$$

We denote by $\mathcal{S}^*$ the class of starlike functions.

A set E is said to be convex if and only if it is starlike with respect to each one of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A function $f\in\mathcal{S}$ maps $\mathbb{D}$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\quad z\in\mathbb{D}. \tag{1.3}$$

Such a function f is said to be convex in $\mathbb{D}$ (or briefly convex) and the class of convex functions we denote by $\mathcal{CV}$

In [2], Goodman distinguished in $\mathcal{CV}$ such functions $f(z)$ that have the property that, for every circular arc γ contained in $\mathbb{D}$, with center also in $\mathbb{D}$, the image arc $f(\gamma)$ is a convex arc. He called the family of all such functions uniformly convex and denoted it by $\mathcal{UCV}$. Goodman's idea became the inspiration for introducing new classes of functions. A function $f\in\mathcal{S}^*$ that has the property that, for every circular arc γ contained in $\mathbb{D}$, with center ξ also in $\mathbb{D}$, the image arc $f(\gamma)$ is a starlike arc with respect to $f(\xi)$ is called by Goodman uniformly starlike. The set of all such functions he denoted by $\mathcal{UST}$.

The classes $\mathcal{UCV}$ and $\mathcal{UST}$ introduced by Goodman were also investigated by Rønning [8–10] and by Ma and Minda [5,6]. The class family of uniformly convex functions $\mathcal{UCV}$ may be defined by

$$\mathcal{UCV}=\left\{f\in\mathcal{S}:\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in\mathbb{D}\right\}. \tag{1.4}$$

This class was considered also in [11,12]. In this paper, we introduce the class

$$\mathcal{MN}=\left\{f\in\mathcal{A}:\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf'(z)}{f(z)}-1\right|,\ z\in\mathbb{D}\right\}. \tag{1.5}$$

Hence, $\mathfrak{Re}\left\{1+zf''(z)/f'(z)\right\}>0$, $z\in\mathbb{D}$, and $\mathcal{MN}\subset\mathcal{CV}$.

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β

$$\mathcal{SS}^*(\beta)=\left\{f\in\mathcal{A}:\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\beta\pi}{2},\ z\in\mathbb{U}\right\},\quad \beta\in(0,1)$$

which was introduced in [13] and [1].

In this paper, we consider the order of strong starlikeness in the class of uniformly convex functions. To prove the main results, we also need the following generalization of Nunokawa's lemma [3,4].

Lemma 1.1. *Let $p(z)$ be an analytic function in $|z|<1$ of the form*

$$p(z)=1+\sum_{n=m}^{\infty}c_nz^n,\quad c_m\neq 0,$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\frac{\pi\varphi}{2}\quad \text{for } |z|<|z_0|$$

and

$$|\arg\{p(z_0)\}|=\frac{\pi\varphi}{2}$$

for some $\varphi>0$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=\mathrm{i}\ell\varphi,$$

where

$$\ell\geq\frac{m}{2}\left(a+\frac{1}{a}\right)\geq m\quad \text{when } \arg\{p(z_0)\}=\frac{\pi\varphi}{2} \tag{1.6}$$

and

$$\ell\leq-\frac{m}{2}\left(a+\frac{1}{a}\right)\leq -m\quad \text{when } \arg\{p(z_0)\}=-\frac{\pi\varphi}{2}, \tag{1.7}$$

where

$$\{p(z_0)\}^{1/\varphi}=\pm\mathrm{i}a,\quad \text{and}\quad a>0.$$

2. Main result

At first we look for a such that $g(z) = z + az^2$ belongs to the class $\mathcal{MN}$. Because $\mathcal{MN} \subset \mathcal{CV}$, then $|a| \le 1/4$. The condition (1.5) becomes

$$\mathfrak{Re}\left\{\frac{1+4az}{1+2az}\right\} > \left|\frac{az}{1+az}\right|, \quad z \in \mathbb{D}. \tag{2.1}$$

If we denote $az = re^{i\alpha}$, $0 \le r \le 1/4$, then (2.1) becomes

$$\frac{1+6r\cos\alpha+8r^2}{1+4r\cos\alpha+4r^2} > \frac{r}{\sqrt{1+2r\cos\alpha+r^2}}, \quad 0 \le r \le 1/4,\ |\alpha| \le \pi. \tag{2.2}$$

The calculations show that (2.2) is not satisfied for $r = 1/4$ and $\alpha = \pi$, hence the class $\mathcal{MN}$ is a proper subset of the class of convex functions $\mathcal{CV}$. For $0 \le r \le 1/4$ the left-hand side of (2.2) increases with respect to $\cos\alpha$, while the right-hand side decreases with respect to $\cos\alpha$. Therefore, if

$$\frac{1-6r+8r^2}{1-4r+4r^2} > \frac{r}{\sqrt{1-2r+r^2}}, \quad 0 \le r \le 1/4, \tag{2.3}$$

then $g(z) = z + az^2$, $|a| \le r$, belongs to the class $\mathcal{MN}$. The calculations in (2.3) yield

$$1 - 6r + 6r^2 > 0, \quad 0 \le r \le 1/4. \tag{2.4}$$

It is satisfied for $r \in [0, (3-\sqrt{3})/6)$. Therefore, if $|a| \le (3-\sqrt{3})/6 = 0.21\ldots$, then $g(z) = z + az^2$ belongs to the class $\mathcal{MN}$.

Theorem 2.1. *If $f(z) \in \mathcal{MN}$, then we have*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{4} \quad (z \in \mathbb{D}), \tag{2.5}$$

or $f(z)$ is strongly starlike of order 1/2.

Proof. If we denote $p(z) = zf'(z)/f(z)$, then the inequality in (1.5) becomes

$$\mathfrak{Re}\left\{p(z) + zp'(z)/p(z)\right\} > |p(z) - 1| \quad (z \in \mathbb{D}). \tag{2.6}$$

If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \frac{\pi}{4} \quad \text{for } |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{4},$$

then from Lemma 1.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell/2,$$

where $p^2(z_0) = \pm ia$, $a > 0$ and

$$\ell : \begin{cases} \ell \ge \left(a + \frac{1}{a}\right)/2 & \text{when } \arg\{p(z_0)\} = \frac{\pi}{4}, \\ \ell \le -\left(a + \frac{1}{a}\right)/2 & \text{when } \arg\{p(z_0)\} = -\frac{\pi}{4}. \end{cases} \tag{2.7}$$

For the case $\arg\{p(z_0)\} = \pi/4$, we have

$$\mathfrak{Re}\left\{p(z_0) + z_0 p'(z_0)/p(z_0)\right\} = \mathfrak{Re}\left\{(ia)^{1/2} + i\ell/2\right\} = \sqrt{a}\cos(\pi/4) = \sqrt{a/2}, \tag{2.8}$$

and

$$\begin{aligned}|p(z_0)-1| &= \left|(\mathrm{i}a)^{1/2}-1\right| \\ &= \left|\sqrt{a}\cos(\pi/4)-1+\mathrm{i}\sqrt{a}\sin(\pi/4)\right| \\ &= \sqrt{(\sqrt{a}\cos(\pi/4)-1)^2+(\sqrt{a}\sin(\pi/4))^2} \\ &= \sqrt{a+1-2\sqrt{a}\cos(\pi/4)} \\ &= \sqrt{a+1-\sqrt{2a}}. \end{aligned} \tag{2.9}$$

From the evident inequality $a/2 \le a+1-\sqrt{2a}$, $a>0$, and by (2.8) and (2.9), we obtain:

$$\mathfrak{Re}\left\{p(z_0)+z_0p'(z_0)/p(z_0)\right\} \le |p(z_0)-1|$$

which contradicts (2.6). Therefore,

$$|\arg\{p(z)\}| < \frac{\pi}{4} \quad \text{for } |z|<1.$$

For the case $\arg\{p(z_0)\} = -\pi/4$, applying the same method as the one above we will get a contradiction. In this way, we have proved that f is strongly starlike of order $1/2$. □

We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some holomorphic function ω such that $\omega(0)=0$ and $|\omega(z)| \le z$ for all $z \in \mathbb{D}$.

Theorem 2.2. *If $f(z) \in \mathcal{MN}$, then we have*

$$\frac{zf'(z)}{f(z)} \prec q(z) = \frac{1}{g(z)} \quad (z \in \mathbb{D}), \tag{2.10}$$

where

$$g(z) = 2\left(\frac{1-z}{z}\right)^2\left(\log(1-z)+\frac{z}{1-z}\right) = 1-\frac{2}{3}z-\frac{1}{6}z^2+\cdots \quad (z\in\mathbb{D}). \tag{2.11}$$

Proof. If we denote $p(z) = zf'(z)/f(z)$, then from (1.5) we obtain

$$\mathfrak{Re}\left\{p(z)+\frac{zp'(z)}{p(z)}\right\} > |p(z)-1| \ge \mathfrak{Re}\{1-p(z)\} \quad (z\in\mathbb{D}), \tag{2.12}$$

hence

$$\mathfrak{Re}\left\{p(z)+\frac{zp'(z)}{2p(z)}\right\} > \frac{1}{2} \quad (z\in\mathbb{D}). \tag{2.13}$$

The inequality (2.13) is equivalent to the following Briot–Boquet differential subordination

$$p(z)+\frac{z\,p'(z)}{2\,p(z)} \prec \frac{1}{1-z} \quad (z\in\mathbb{D}). \tag{2.14}$$

Applying Theorem 3.3d, [7, p. 109], we conclude that

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{2\int_0^1\left(\frac{1-z}{1-tz}\right)^2 t\,\mathrm{d}t} \\ &= \{{}_2F_1(2,1;3;z/(z-1))\}^{-1}. \end{aligned}$$

The calculation yields (2.11). □

Corollary 2.3. *If $f(z) \in \mathcal{MN}$, then we have*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{g1(-1)} = \alpha_0 := \frac{1}{4(\log 4-1)} = 0.647\ldots, \quad (z\in\mathbb{D}) \tag{2.15}$$

or $\mathcal{MN} \subset \mathcal{S}^*(\alpha_0)$, where $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α.

Proof. By Theorem 3.3d, [7, p. 109], the function

$$q(z) = \frac{1}{g(z)} \quad (z \in \mathbb{D}) \tag{2.16}$$

is the univalent solution to the Briot–Boquet differential subordination (2.14). Moreover, by the calculation result [7, p. 106], we have

$$\min_{|z|=1} \mathfrak{Re}\{q_1(z)\} = q_1(-1) = \alpha_0. \tag{2.17}$$

Therefore, by Theorem 2.2 we conclude that $\mathcal{MN} \subset \mathcal{S}^*(\alpha_0)$. □

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