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Two functions on $\mathrm{Sp}(g, \mathbb{R})$

Deux fonctions sur $\mathrm{Sp}(g, \mathbb{R})$

Patrick M. Gilmer

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

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ABSTRACT

We consider two functions on $\mathrm{Sp}(g, \mathbb{R})$ with values in the cyclic group of order four $\{\pm 1, \pm i\}$. One was defined by Lion and Vergne. The other is $-i$ raised to the power given by an integer valued function defined by Masbaum and the author (initially on the mapping class group of a surface). We identify these functions when restricted to $\mathrm{Sp}(g, \mathbb{Z})$. We conjecture the identity of these functions on $\mathrm{Sp}(g, \mathbb{R})$.

RÉSUMÉ

Nous considérons deux fonctions sur $\mathrm{Sp}(g, \mathbb{R})$ à valeurs dans le groupe cyclique d'ordre quatre $\{\pm 1, \pm i\}$. L'une a été définie par Lion et Vergne. L'autre est $-i$ élevé à la puissance donnée par une fonction à valeurs entières définie par Masbaum et l'auteur (initialement sur le groupe modulaire d'une surface). Nous montrons que ces deux fonctions coïncident sur $\mathrm{Sp}(g, \mathbb{Z})$. Nous conjecturons qu'elles coïncident sur $\mathrm{Sp}(g, \mathbb{R})$.

1. Introduction

For $f \in \mathrm{Sp}(g, \mathbb{R})$, we describe, in the next two subsections, an invariant $s(f)$ of Lion–Vergne that takes values in $\{\pm 1, \pm i\}$ and an invariant $n(f)$ of Gilmer–Masbaum, which takes values in $\mathbb{Z}$. Our main theorem is the following.

Theorem 1. *For* $f \in \mathrm{Sp}(g, \mathbb{Z})$,

$$s(f) = i^{-n(f)}. \tag{1}$$

In the second section we prove this theorem. We conjecture this theorem also holds for $f \in \mathrm{Sp}(g, \mathbb{R})$. We discuss this conjecture in the third section and prove this conjecture in genus 1. In the fourth section, we prove the square of Eq. (1) for $f \in \mathrm{Sp}(g, \mathbb{R})$. In the final section, we discuss the context of this result and some motivation.

E-mail address: gilmer@math.lsu.edu.

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1.1. *Lion and Vergne's $s(f)$*

Let V be a real vector space equipped with a skew symmetric nonsingular form ω. We refer to V as a symplectic inner product space. A Lagrangian is a subspace λ of V which is equal to its own perpendicular subspace with respect to ω.

An oriented vector space is a vector space equipped with an equivalence class of ordered bases. Two ordered bases are equivalent if the change of basis matrix has positive determinant. To an ordered pair of oriented Lagrangians (λ_1, λ_2) in V, Lion and Vergne associated $\epsilon(\lambda_1, \lambda_2) \in \{-1, 1\}$, as follows.

Define $h_{\lambda_1,\lambda_2} : \lambda_1 \to \lambda_2^*$ by $h_{\lambda_1,\lambda_2}(x)(y) = \omega(x, y)$. The kernel of this map is $\lambda_1 \cap \lambda_2$, which we will denote by κ.

1.1.1. *$\epsilon(\lambda_1, \lambda_2)$ in the case $\kappa = 0$*

In this case, we have that h_{λ_1,λ_2} is invertible. Let $\{a_i\}_{i=1,n}$ be an ordered basis for λ_1, $\{b_i\}_{i=1,n}$ be an ordered basis for λ_2. We let $\{b_i^*\}_{i=1,n}$ be the ordered basis for λ_2^* given by $b_i^*(b_j) = \delta_{ij}$. One defines $\epsilon(\lambda_1, \lambda_2)$ to be one if and only if $\{h_{\lambda_1,\lambda_2}(a_i)\}$ and b_i^* determine the same orientation on λ_2^*. Equivalently $\epsilon(\lambda_1, \lambda_2) = \mathrm{sgn}(\det(\omega(a_i, b_j)))$. Here and below, we let $\mathrm{sgn}(x) = \frac{|x|}{x} \in \{\pm 1\}$ for a non-zero real number x.

1.1.2. *$\epsilon(\lambda_1, \lambda_2)$ in the case $\kappa \neq 0$*

This case is reduced to the case $\kappa = \{0\}$ as follows. We can see that κ is isotropic, and hence $\kappa^\perp/\kappa$ acquires an induced symplectic structure and λ_1/κ, λ_2/κ are Lagrangian subspaces of $\kappa^\perp/\kappa$. Choosing an orientation of κ, we consider the short exact sequences:

$$0 \to \kappa \to \lambda_i \to \lambda_i/\kappa \to 0 \tag{2}$$

and determine an orientation of λ_i/κ, by the rule that an ordered basis for κ followed by the lift to λ_i of an ordered basis for λ_i/κ is an ordered basis for λ_i. Since λ_1/κ and λ_2/κ intersect trivially, $\epsilon(\lambda_1/\kappa, \lambda_2/\kappa)$ is defined, and we may define $\epsilon(\lambda_1, \lambda_2) = \epsilon(\lambda_1/\kappa, \lambda_2/\kappa)$. Here the choice of orientation of κ is not important, as this choice appears twice in this construction.

If λ_1, λ_2 are the same Lagrangian with the same orientation, then the above prescription asks us to compare two orientations on a zero dimensional vector space. This should be interpreted as follows: $\epsilon(\lambda_1, \lambda_2) = 1$. Similarly: if λ_1, λ_2 are the same Lagrangian but with opposite orientations, then we take $\epsilon(\lambda_1, \lambda_2) = -1$.

1.1.3. *Definition of s(f) in terms of ϵ*

Define

$$s(\lambda_1, \lambda_2) = i^{\dim(\lambda_1) - \dim(\lambda_1 \cap \lambda_2)} \epsilon(\lambda_1, \lambda_2).$$

Consider the vector space $\mathbb{R}^{2g}$, with the standard basis denoted by $\{p_1, \cdots p_g, q_1, \cdots q_g\}$ and equipped with the standard symplectic form given by $\omega(p_i, p_j) = \omega(q_i, q_j) = 0$ and $\omega(p_i, q_j) = -\omega(q_i, p_j) = \delta_{ij}$. The Lie group of isometries of this symplectic inner product space is called the symplectic group and is denoted $\mathrm{Sp}(g, \mathbb{R})$. Let λ_0 be the Lagrangian spanned by $\{p_i\}$. If $f \in \mathrm{Sp}(g, \mathbb{R})$, define

$$s(f) = s(\lambda_0, f(\lambda_0)).$$

Here we give λ_0 an arbitrary orientation. Since this orientation enters the computation twice, it does not effect the result.

1.2. *Gilmer–Masbaum's $n(f)$*

Let $f : V \to V$ be an isometry. Turaev ([7], [8, 2.1,2.2]) defined a non-singular bilinear form $\star_f$ on $(f-1)V$ by

$$a \star_f b = \omega((f-1)^{-1}(a), b).$$

Here, $\omega((f-1)^{-1}(a), b)$ means $\omega(x, b)$, where x is any element of $(f-1)^{-1}(a)$.

The determinant of a matrix for $\star_f$ with respect to a basis of $(f-1)V$ will be denoted $\det(\star_f)$. Thus $\mathrm{sgn}[\det(\star_f)]$ will take values in $\{\pm 1\}$. If $f = \mathrm{Id}$, $(f-1)V = 0$, and we let $\mathrm{sgn}[\det(\star_{\mathrm{Id}})] = 1$.

According to [3, Lemma 6.4], if $\lambda \subset V$ is a Lagrangian, then the restriction of the form $\star_f$ to $\lambda \cap (f-1)V$ is symmetric. This form is denoted $\star_{f,\lambda}$. Thus $\star_{f,\lambda}$ has a signature.

In the above situation, one defines

$$n_\lambda(f) = \mathrm{Sign}(\star_{f,\lambda}) - \dim((f-1)V) - \mathrm{sgn}[\det(\star_f)] + 1\ . \tag{3}$$

For $f \in \mathrm{Sp}(g, \mathbb{R})$, let

$$n(f) = n_{\lambda_0}(f).$$

We note that $n_\lambda(f)$ was defined in [3] for f in the mapping class group of a surface (using $H_1(\Sigma_g, \mathbb{Q})$ with a chosen Lagrangian λ of this rational vector space). The terms in the formula for $n_{\lambda_0}(f)$ make perfect sense for $f \in \mathrm{Sp}(g, \mathbb{R})$, so we can make this definition.

We also consider the free $\mathbb{Z}$ module generated by $\{p_1, \cdots p_g, q_1, \cdots q_g\}$ which we identify with $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$. The form ω restricts to a unimodular $\mathbb{Z}$-valued form. By $\mathrm{Sp}(g, \mathbb{Z})$, we the mean the group of isometries of this symplectic inner product space over $\mathbb{Z}$. We have that $\mathrm{Sp}(g, \mathbb{Z}) \subset \mathrm{Sp}(g, \mathbb{R})$. So we may also restrict s and n to $\mathrm{Sp}(g, \mathbb{Z})$.

2. Comparing characters on a central extension of $\mathrm{Sp}(g, \mathbb{Z})$

Given three Lagrangians λ_1, λ_2, λ_3 of (V, ω), there is a Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}$. This can be defined as the signature of the symmetric bilinear form on $(\lambda_1 + \lambda_2) \cap \lambda_3$ defined by $B(a, b) = \omega(x, b)$ where $a, b \in (\lambda_1 + \lambda_2) \cap \lambda_3$, $x \in \lambda_2$, and $a - x \in \lambda_1$. As noted in [7,8], this is equivalent to the definition given by Kashiwara and used in [4]. As we use both [7,8] and [4], some our results depend on this identification which can be seen for instance using [1, Thm 8.1].

Lion and Vergne [4, 1.6.14] use the Maslov index to specify a certain central extension $\widetilde{\mathrm{Sp}(g, \mathbb{R})}$ by $\mathbb{Z}$ of $\mathrm{Sp}(g, \mathbb{R})$. One defines

$$\widetilde{\mathrm{Sp}(g, \mathbb{R})} = \{(f, m) | f \in \mathrm{Sp}(g, \mathbb{R}), m \in \mathbb{Z}\}$$

with multiplication;

$$(f_1, m_1) \cdot (f_2, m_2) = (f_1 f_2, m_1 + m_2 + \mu(\lambda_0, f_1(\lambda_0), f_1 f_2(\lambda_0))). \tag{4}$$

Thus $\widetilde{\mathrm{Sp}(g, \mathbb{R})}$ is the central extension of $\mathrm{Sp}(g, \mathbb{R})$ specified by the 2-cocycle ν where $\nu(f_1, f_2) = \mu(\lambda_0, f_1(\lambda_0), f_1 f_2(\lambda_0))$. According to [4, 1.7.11], the formula $\mathfrak{s}(f, m) = i^m s(f)$ defines a character on the group $\widetilde{\mathrm{Sp}(g, \mathbb{R})}$.

One can define an extension $\widetilde{\mathrm{Sp}(g, \mathbb{Z})}$ of $\mathrm{Sp}(g, \mathbb{Z})$ by the same procedure as used for $\mathrm{Sp}(g, \mathbb{R})$, and one obtains the pull back by the inclusion $\iota : \mathrm{Sp}(g, \mathbb{Z}) \to \mathrm{Sp}(g, \mathbb{R})$ of the extension $\widetilde{\mathrm{Sp}(g, \mathbb{R})}$ over $\mathrm{Sp}(g, \mathbb{R})$. We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{Sp}(g, \mathbb{Z})} & \longrightarrow & \mathrm{Sp}(g, \mathbb{Z}) & \longrightarrow & 1 \\ & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \tilde\iota} & & \downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{Sp}(g, \mathbb{R})} & \longrightarrow & \mathrm{Sp}(g, \mathbb{R}) & \longrightarrow & 1. \end{array}$$

We define $\mathfrak{r} : \widetilde{\mathrm{Sp}(g, \mathbb{R})} \to \{\pm 1, \pm i\}$ by $\mathfrak{r}(f, m) = i^{n(f) - m}$. We need the following lemma whose proof we delay.

Lemma 2. *The function* $\mathfrak{r} \circ \tilde\iota$ *is a character on* $\widetilde{\mathrm{Sp}(g, \mathbb{Z})}$ *with values in* $\{\pm 1, \pm i\}$.

Proof of Theorem 1 modulo Lemma 2. Given that $\mathfrak{r} \circ \tilde\iota$ and $\mathfrak{s} \circ \tilde\iota$ are characters, it follows that $\mathfrak{t}(m, f) = \mathfrak{r}(f, m)\mathfrak{s}(f, m)$ defines a character on $\widetilde{\mathrm{Sp}(g, \mathbb{Z})}$. This character vanishes on the central element $(1, \mathrm{id}) \in \widetilde{\mathrm{Sp}(g, \mathbb{Z})}$. Thus $\mathfrak{t}$ induces a well defined character on $\mathrm{Sp}(g, \mathbb{Z})$. According to [6, Thm 5.1], if $g \ge 3$, $\mathrm{Sp}(g, \mathbb{Z})$ is perfect. So the induced character is trivial. It follows that $\mathfrak{t}$ is identically one. Thus $s(f) = i^{-n(f)}$ for $f \in \mathrm{Sp}(g, \mathbb{Z})$ if $g \ge 3$. But both s and n remain unchanged upon the stabilization $\mathrm{Sp}(g, \mathbb{Z}) \to \mathrm{Sp}(g + 1, \mathbb{Z})$ given by direct summing a 2×2 identity matrix. Thus $s(f) = i^{-n(f)}$ for low genus as well. □

We will think of s and n as 1-cochains on $\mathrm{Sp}(g, \mathbb{R})$. We write n as $-j - k$ where j and k are the two 1-cochains (the notation is chosen to be consistent with [3]).

$$j(f) = -\,\mathrm{Sign}(\star_{f, \lambda_0}) \quad \text{and} \quad k(f) = \dim(\mathrm{Image}(f - \mathrm{Id})) + \mathrm{sgn}[\det(\star_f)] - 1 \tag{5}$$

Proposition 3. *(See Turaev [7,8].) Let* $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{R})$,

$$x \star_{f_1, f_2} y = \omega\left((f_1 - 1)^{-1} x + (f_2 - 1)^{-1} x + x, y\right)$$

defines a symmetric bilinear form on $\mathrm{Image}(f_1 - 1) \cap \mathrm{Image}(f_2 - 1)$.

Consider the 2-cochain given by

$$\phi(f_1, f_2) = \mathrm{Sign}(\star_{f_1, f_2}). \tag{6}$$

Recall the coboundary of a 1-cochain c is given by $(\delta c)(g, h) = c(g) + c(h) - c(gh)$. We need:

Theorem 4. *(See Turaev [7,8].)*

$$\delta k \equiv \varphi \pmod 4. \tag{7}$$

Our next result uses some topology. Let Γ_g denote the mapping class group of a closed surface Σ_g of genus g. We may pick an identification of $H_1(\Sigma_g)$ with $\mathbb{Z}^{2g}$ so that the intersection pairing on $H_1(\Sigma_g)$ agrees with ω. Then we have a surjection $h : \Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$ which sends a mapping class $\mathfrak{f}$ to the map it induces on homology. We also identify $H_1(\Sigma_g, \mathbb{R})$ with $\mathbb{R}^{2g}$. We pick a handlebody $\mathcal{H}_g$ with boundary Σ_g such that λ_0 under this identification is the kernel of the map $H_1(\Sigma_g, \mathbb{R}) \to H_1(\mathcal{H}_g, \mathbb{R})$. Proposition 5 is essentially Walker's theorem [10, p. 124] [3, Thm 8.10] together with [3, Prop 8.9] which identifies the signatures of certain manifolds appearing in following proof with $\mathrm{Sign}(\star_{f,\lambda_0})$ for various f.

Proposition 5. *Let $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{Z})$,*

$$\mathrm{Sign}(\star_{f_1,f_2}) - \mu(\lambda_0, f_1(\lambda_0), f_1 f_2(\lambda_0)) + \mathrm{Sign}(\star_{f_1\circ f_2,\lambda_0}) - \mathrm{Sign}(\star_{f_1,\lambda_0}) - \mathrm{Sign}(\star_{f_2,\lambda_0}) = 0. \tag{8}$$

Proof. Given $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{Z})$, we pick $\mathfrak{f}_1, \mathfrak{f}_2 \in \Gamma$, with $h(\mathfrak{f}_i) = f_i$. Then we use $\mathfrak{f}_1$, $\mathfrak{f}_2$ and $\mathcal{H}_g$ to construct five 4-manifolds with boundary as in [3, proof of Thm 8.10]. Using identities appearing in [3], the signatures of each of these manifolds are identified with the terms that appear on the left hand side of Eq. (8). Then we glue together the five 4-manifolds along whole components of their boundaries to obtain a closed 4-manifold. By Novikov additivity, this closed 4-manifold has signature given by the left hand side of Eq. (8). This closed 4-manifold is then shown to be the boundary of a five manifold, as in [3, proof of Thm 8.10], and thus have vanishing signature. □

Because these constructions require that f_1 and f_2 be the maps on the homology of a surface induced by surface automorphisms, the above proof does not extend to the case $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{R})$.

Proof of Lemma 2. The claim is easily seen to be equivalent to the following identity involving 2-cocycles of $\mathrm{Sp}(g, \mathbb{Z})$:

$$\mu(\lambda_0, f_1(\lambda_0), f_1 f_2(\lambda_0)) + \delta n_{\lambda_0}(f_1, f_2) = 0 \pmod 4.$$

Using Eqs. (5), (6), (7) and (8), and letting $\equiv$ denote equality modulo 4,

$$\delta(n_{\lambda_0})(f_1, f_2) = -\delta(j)(f_1, f_2) - \delta(k)(f_1, f_2) \tag{9}$$

$$\equiv \mathrm{Sign}(\star_{f_1,\lambda_0}) + \mathrm{Sign}(\star_{f_2,\lambda_0}) - \mathrm{Sign}(\star_{f_1\circ f_2,\lambda_0}) - \mathrm{Sign}(\star_{f_1,f_2}) \tag{10}$$

$$= -\mu(\lambda_0, f_1(\lambda_0), f_1 f_2(\lambda_0)). \quad \square \tag{11}$$

We remark that Lemma 2 and its proof are closely related to [3, Thm 6.6] and its proof.

3. On the conjecture that Theorem 1 holds for $f \in \mathrm{Sp}(g, \mathbb{R})$

By the argument for Lemma 2, we have:

Lemma 6. *If Eq. (8) holds modulo four for all $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{R})$, then $\mathfrak{r}$ is a character with values in $\{\pm 1, \pm i\}$ on the group $\widetilde{\mathrm{Sp}(g, \mathbb{R})}$.*

Proposition 7. *If Eq. (8) holds modulo four for all $f_1, f_2 \in \mathrm{Sp}(g, \mathbb{R})$, then Eq. (1) holds for all f in $\mathrm{Sp}(g, \mathbb{R})$.*

Proof. We use essentially the same argument as in the proof of Theorem 1 except we do not need to stabilize as $\mathrm{Sp}(g, \mathbb{R})$ is perfect even for low g. □

Proposition 8. *Eq. (1) holds for all f in $\mathrm{Sp}(1, \mathbb{R})$.*

Proof. One easily has [4, 1.8.4] that, if $a \neq 0$, then $s\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) = \mathrm{sgn}(a)$, and if $c \neq 0$, then $s\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathrm{sgn}(c)i$. To complete the proof, one only needs to calculate n modulo 4 explicitly in these cases. □

4. The square of Eq. (1)

We can obtain that the square of Eq. (1) is valid for $f \in \mathrm{Sp}(g, \mathbb{R})$.

Proposition 9. *If* $f \in \mathrm{Sp}(g, \mathbb{R})$, $(s(f))^2 = (-1)^{n(f)}$.

Proof. From the definition of s, one easily has that

$$(s(f))^2 = (-1)^{g+\dim(\lambda_0 \cap f(\lambda_0))}.$$

From the definition of n_{λ_0}, one easily has that

$$(-1)^{n(f)} = (-1)^{\mathrm{Sign}(\star_{f,\lambda_0}) - \dim(\mathrm{Image}(f-1))}.$$

By [3, Proposition 7.3] (whose proof is valid for $f \in \mathrm{Sp}(g, \mathbb{R})$),

$$g + \dim(\lambda_0 \cap f(\lambda_0)) = \mathrm{Sign}(\star_{f,\lambda_0}) - \dim(\mathrm{Image}(f-1)) \pmod 2. \qquad \square$$

5. Final comments

Central extensions of the mapping class group are used to upgrade projective representations arising in topological quantum field theory (TQFT) to honest representations [10,5]. More generally an extension of the three dimensional cobordism category is used to remove the projective ambiguity of TQFT maps induced by more general cobordisms than mapping cylinders [10,9]. An index two subcategory of the extended cobordism category [2] proved useful in demonstrating that certain projective modules associated with surfaces by an integral version TQFT are free. In [3], the function n was defined in order to describe an index four subgroup of the extended mapping class group. This allowed Masbaum and the author to define modular representations of the unextended mapping class group. In [3, Remark 7.5], it is asked whether there is a corresponding index four subcategory of the 3-dimensional extended cobordism category. As s gives a very different description of this same index four subgroup, it is plausible to hope that Theorem 1 might help answer this question. As a tentative step in this direction, Wang [11] makes use of Theorem 1 to define a version of n for connected extended cobordisms which have been further enhanced with a choice of orientation for the Lagrangians that are part of the extended structure. This version of n agrees with n when applied to mapping cylinders.

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