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Group theory

On the generation of discrete and topological Kac–Moody groups

Sur les générateurs des groupes de Kac–Moody topologiques et discrets

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ABSTRACT

This article shows that discrete or topological Kac–Moody groups defined over finite fields are in many cases 2-generated. We provide explicit bounds on the minimal number of generators for arbitrary Kac–Moody groups.

RÉSUMÉ

On montre que les groupes de Kac–Moody topologiques ou discrets définis sur des corps finis sont 2-engendrés dans de nombreux cas. On exhibe ensuite des bornes explicites sur le nombre minimal de générateurs pour un groupe de Kac–Moody arbitraire.

Version française abrégée

On considère des groupes de Kac–Moody sur des corps finis $\mathbb{F}_q$.

Théorème 0.1. *Soit $G = G(q)$ un groupe de Kac–Moody simplement connexe de rang m correspondant à une matrice de Cartan généralisée indécomposable (MCGI) A, défini sur un corps fini $\mathbb{F}_q$, $q = p^a$. Soit $\pi = \{\alpha_1, \dots, \alpha_m\}$ l'ensemble des racines simples de G et soit Δ le diagramme de Dynkin de G dont les sommets sont numérotées par $\alpha_1, \dots, \alpha_m$. Posons que, pour tout sous-ensemble σ de π non vide, $\Delta(\sigma)$ représente le sous-diagramme de Δ engendré par $\alpha_{i_1}, \dots, \alpha_{i_k} \in \pi$ où $\sigma = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$. Soit $d(G)$ le nombre minimal d'éléments de G nécessaires pour générer G. Alors lorsque q est suffisamment grand, on a :*

(i) lorsque $m = 2$, $d(G) \le 3$;
(ii) lorsque G est affine et que $m \ge 3$, $d(G) = 2$;
(iii) lorsque G est strictement hyperbolique (symétrisable) et $m \ge 3$, $d(G) = 2$;
(iv) lorsque G est hyperbolique (symétrisable), $d(G) = 2$ pour $m \ge 5$, et $d(G) \le 3$ si $m = 3$ ou $m = 4$ ($d(G) = 2$ dans au moins 34 des 72 cas) à part peut-être dans trois cas exceptionnels de rang 3 et pour lesquels Δ est de type (∞, ∞, ∞). Dans ces trois cas, $d(G) \le 4$;

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(v) supposons que π peut être découpé en k sous-ensembles mutuellement disjoints π_i, $1 \le i \le k$, tels que $\pi_i = \{\alpha_{i_1}, \dots, \alpha_{i_{l(i)}}\}$ avec $\alpha_{i_j} \in \pi$, $1 \le j \le l(i)$ (où $l(i) = |\pi_i|$) et que pour chaque $i \in \{1, \dots, k-1\}$, on a $\Delta(\pi_i) = \bigsqcup_{j=1}^{s(i)} \Delta_{ij}$ où Δ_{ij} est un diagramme de Dynkin irréductible de type fini (ce qui signifie que $\Delta(\pi_i)$ peut être découpé en $s(i)$ diagrammes de Dynkin de type fini où $s(i) \in \mathbb{N}$ dépend de π_i). Alors :
   (a) si $\Delta(\pi_k) = \bigsqcup_{j=1}^{r} \Delta_{kj}$ où Δ_{kj} est un diagramme de Dynkin irréductible de type fini, alors $d(G) \le 2k$;
   (b) si $\Delta(\pi_k) = \bigsqcup_{j=1}^{r} \Delta_{kj}$ où Δ_{kj} est un diagramme de Dynkin irréductible de rang 2 de type infini, alors $d(G) \le 2k+2$, et, si q est assez grand, $d(G) \le 2k+1$.

Exemple 1. Si Δ est un arbre enraciné fini de profondeur m, $d(G) \le 4$ lorsque $q \ge \sqrt{m}$.

Corollaire 0.2. *Soit G un groupe de Kac–Moody minimal défini sur un corps $\mathbb{F}_q$, avec $q = p^a$ et $p \ge \max_{i \ne j} |a_{ij}|$ (où $A = (a_{ij})$ est la MCGI de G). Soit $\overline{G}$ le groupe de Kac–Moody topologique correspondant à G. Alors les conclusions du Théorème 0.1 sont vraies si on remplace G par $\overline{G}$ et si $d(\overline{G})$ représente le nombre minimal de générateurs topologiques de $\overline{G}$.*

1. Introduction

It is a well-known result that every non-Abelian finite simple group can be generated by only two elements (cf. [2]). It is interesting to see whether this statement is true for other classes of simple groups. For example, non-affine Kac–Moody groups (over finite fields) are known to be simple [6]. How many generators do they require? In this article, we discuss the generation of Kac–Moody groups $G(q)$ defined over finite fields $\mathbb{F}_q$ and show that it is often the case that they too are 2-generated.

Kac–Moody groups over arbitrary fields were defined by J. Tits [16]. In [1], Abramenko and Muhlherr have shown that with some restrictions (if the groups are 2-spherical, with some mild bounds on the size of $\mathbb{F}_q$), Kac–Moody groups over $\mathbb{F}_q$ are finitely presented with the number of generators depending on q and the Lie rank of $G(q)$.[1] In [4], the author has shown that the family of affine Kac–Moody groups over $\mathbb{F}_q$ (of rank at least 3) possesses bounded presentations: there exists $C > 0$ such that if $G(q)$ is an affine Kac–Moody group of rank at least 3 corresponding to an indecomposable generalised Cartan matrix (IGCM) and $q \ge 4$, then $G(q)$ has a presentation with $d(G)$ generators and $r(G)$ relations satisfying $d(G) + r(G) \le C$. Related results for other Kac–Moody groups over finite fields were also proved in [4]. As a consequence, the number of generators of a 2-spherical Kac–Moody group is independent of q and depends on the type of Dynkin diagram of $G(q)$ rather than on the rank of G. We make use of this observation to provide bounds on the minimal number of generators of $G(q)$.

Theorem 1.1. *Let $G = G(q)$ be a simply connected Kac–Moody group of rank m corresponding to an IGCM A and defined over a finite field $\mathbb{F}_q$. Let $\pi = \{\alpha_1, \dots, \alpha_m\}$ be the set of simple roots of G and Δ be the Dynkin diagram of G whose vertices are labelled by $\alpha_1, \dots, \alpha_m$. Suppose further that for any non-empty subset σ of π, $\Delta(\sigma)$ denotes the subdiagram of Δ spanned by $\alpha_{i_1}, \dots, \alpha_{i_k} \in \pi$ where $\sigma = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$. Let $d(G)$ denote the minimal number of elements of G that are required to generate G. Then for q large enough there holds:*

(i) if $m = 2$, then $d(G) \le 3$;
(ii) if G is affine with $m \ge 3$, then $d(G) = 2$;
(iii) if G is (symmetrizable) strictly hyperbolic and $m \ge 3$, then $d(G) = 2$;
(iv) if G is (symmetrizable) hyperbolic, then if $m \ge 5$, then $d(G) = 2$, and if $m = 3$ or 4, then $d(G) \le 3$ (with $d(G) = 2$ in at least 34 out of 72 cases) with the possible exception of three rank-3 diagrams with Δ of type (∞, ∞, ∞). In each one of those three cases, $d(G) \le 4$;
(v) suppose that we may subdivide π into k mutually disjoint subsets π_i, $1 \le i \le k$, such that each $\pi_i = \{\alpha_{i_1}, \dots, \alpha_{i_{l(i)}}\}$ for some $\alpha_{i_j} \in \pi$, $1 \le j \le l(i)$ (with $l(i) = |\pi_i|$) and for each $i \in \{1, \dots, k-1\}$, $\Delta(\pi_i) = \bigsqcup_{j=1}^{s(i)} \Delta_{ij}$ with Δ_{ij} an irreducible Dynkin diagram of finite type (i.e., $\Delta(\pi_i)$ can be partitioned into $s(i)$ disjoint Dynkin diagrams of finite type for some $s(i) \in \mathbb{N}$ depending on π_i). Then
   (a) if $\Delta(\pi_k) = \bigsqcup_{j=1}^{s(k)} \Delta_{kj}$ with Δ_{kj} an irreducible Dynkin diagram of finite type, then $d(G) \le 2k$;
   (b) if $\Delta(\pi_k) = \bigsqcup_{j=1}^{s(k)} \Delta_{kj}$ with Δ_{kj} an irreducible Dynkin diagram of rank 2 of infinite type, then $d(G) \le 2k+2$ (and if we increase q, $d(G) \le 2k+1$).

The bound $d(G) = 2$ is optimal and was obtained in cases (*ii*), (*iii*) and part of (*iv*). Note that the bound $d(G) \le 2m$ follows from $(v)(a)$. Below are few examples of application of $(v)(a)$.

[1] An existence of finite generating set of $G(q)$ can be derived directly from the original presentation of $G(q)$.

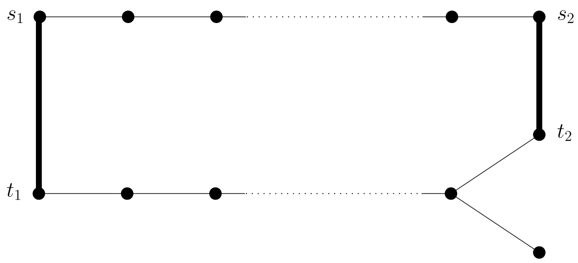
Example 1. If the nodes of Δ can be partitioned into two disjoint subsets π_1 and π_2 such that for every two-element subset $\{\alpha_{i_s}, \alpha_{i_t}\} \subset \pi_i$, $\Delta(\{\alpha_{i_s}, \alpha_{i_t}\})$ is of type $A_1 \times A_1$ (i.e., α_{i_s} and α_{i_t} are not connected in Δ), then for q large enough, $d(G) \le 4$.

The partition corresponding to Example 1 can often be obtained, one possible obstacle being the existence of many cycles of length 3 in Δ. Example 2 is a special case of Example 1.

Example 2. If Δ is a finite rooted tree and has rank m, then $d(G) \le 4$ provided that $q \ge \sqrt{m}$.

The following example illustrates the fact that infinite subdiagrams of Δ can sometimes be ignored.

Example 3. If Δ is the diagram below, then using an appropriate partitioning of Δ we immediately obtain that $d(G) \le 4$. In fact, using methods employed in Section 2, we can easily obtain $d(G) \le 3$.



The groups discussed so far are often called the *minimal* Kac–Moody groups. They are discrete infinite groups. In recent years, there has been a significant progress in the study of topological Kac–Moody groups. Those are either completions of minimal Kac–Moody groups $G(q)$, $q = p^a$, achieved by various methods (e.g., a completion of Carbone and Garland $G^{c\lambda}$ obtained via methods of representation theory, a Caprace–Rémy–Ronan completion G^{crr} obtained via geometric methods) or a topological group $G^{\mathrm{ma+}}$ explicitly constructed by Mathieu. All of these are discussed in details in a recent paper of Rousseau [15]. There it is further shown that provided that p is large enough, $G^{\mathrm{ma+}} \twoheadrightarrow G^{c\lambda} \twoheadrightarrow G^{\mathrm{crr}}$ and $G(q)$ is dense in each of those topological groups. In [5], it was shown that under the same restriction on p (and modulo the centres), $G^{\mathrm{ma+}} \cong G^{c\lambda} \cong G^{\mathrm{crr}}$. Thus one can simply talk about a topological Kac–Moody group $\overline{G} = \overline{G}(q)$ that corresponds to $G = G(q)$ without any ambiguity. We now observe that, since for p large enough, $G(q)$ is dense in $\overline{G}(q)$, an immediate consequence of Theorem 1.1 is a bound on the number of (topological) generators of $\overline{G}(q)$.

Corollary 1.2. *Let G be a minimal Kac–Moody group defined over the field $\mathbb{F}_q$, with $q = p^a$ and $p \ge \max_{i \ne j} |a_{ij}|$ (where $A = (a_{ij})$ is the IGCM of G). Let $\overline{G}$ denote the topological Kac–Moody group corresponding to G. Then Theorem 1.1 holds if we replace G by $\overline{G}$, and $d(\overline{G})$ stands for the minimal number of topological generators of $\overline{G}$.*

In a proof of our results we make an extensive use of a result of Guralnick and Kantor regarding the generation of finite groups of Lie type: see their Corollary to Theorem I on p. 745 of [11]. We will refer to it as Corollary 1 in [11]. We also use recent estimates obtained by Menezes, Quick and Roney-Dougal [14].

Finally let us remark that while the statement and the proof of our result deals with the so-called split Kac–Moody groups, it can be generalised to the case of almost split Kac–Moody groups as defined by Hee [12]. To do so, one needs to modify the proof given in Section 2 by using instead the so-called twisted Dynkin diagrams of those groups. The remaining ingredients of the proof coming from finite group theory then apply in the same way.

2. Outline of a proof

Let $G = G(q)$ be a simply connected Kac–Moody group. Let A be its IGCM of size m and $\alpha_1, \ldots, \alpha_m$ its fundamental roots. In the next paragraph, we will assume Proposition 2.1 of [9] that defines a simply connected Kac–Moody group via its presentation.

The group G is generated by its root elements $x_\alpha(u)$, $\alpha \in \Phi$ (the set of real roots), $u \in \mathbb{F}_q$. For each $u \in \mathbb{F}_q$ and each $1 \le i \le m$, write $x_i(u) = x_{\alpha_i}(u)$ and $x_{-i}(u) = x_{-\alpha_i}(u)$. Then for each $a \in \mathbb{F}_q^*$ and $1 \le i \le m$, put $n_i(a) = x_i(a)x_{-i}(a^{-1})x_i(a)$, $n_i = n_i(1)$, and let $h_i(a) = n_i(a)n_i^{-1}$. For $\alpha \in \Phi$, $X_\alpha := \langle x_\alpha(u), u \in \mathbb{F}_q \rangle \cong (\mathbb{F}_q, +)$ and $M_\alpha := \langle X_\alpha, X_{-\alpha} \rangle \cong A_1(q)$. In particular, $X_i := \langle x_i(u), u \in \mathbb{F}_q \rangle$ and $M_i := \langle X_i, X_{-i} \rangle$. Moreover, G is a group with a BN-pair, (B, N) where N is generated by a subgroup T and elements n_i, $1 \le i \le m$, and $T = \langle h_i(a), a \in \mathbb{F}_q^*, 1 \le i \le m \rangle \cong C_{q-1}^m$ is a torus of G. Remark that T normalises each M_i, $1 \le i \le m$. Also, $N/T \cong W$, the Weyl group of G, and as each $n_i \in M_i$ projects onto a generator w_i of W, we obtain the first basic ingredient of our proof.

Lemma 2.1. *If we have generated all M_i, $1 \le i \le m$, we have generated G.*

Notice that the notations above work just as well for finite groups of Lie type that can be thought of as the special case of Kac–Moody groups over $\mathbb{F}_q$ where A is a Cartan matrix.

Lemma 2.2. *Let $\Sigma(q)$ be a finite (quasi-) simple group of Lie type that is defined over $\mathbb{F}_q$ and corresponding to a root system $\Sigma = A_2, C_2$ or G_2. Let α_1 and α_2 be the fundamental roots of Σ with $|\alpha_1| \le |\alpha_2|$. Then $\Sigma(q)$ is generated by M_1 and n_2.*

Proof. This is achieved by an easy calculation. □

In the future, we will denote by M_{ij} the semi-simple subgroup of G that corresponds to $\Delta(\{\alpha_i, \alpha_j\})$. We now prove our main result. We do it in several steps.

Proposition 2.3. *Let G be an affine simply connected Kac–Moody group of rank $(m+1) \ge 3$, corresponding to an IGCM, defined over a field $\mathbb{F}_q$ with q large enough. Then $d(G) = 2$.*

Proof. For the affine groups, we use the notations from the book of Carter [8]. In particular, we denote the fundamental roots of G by $\alpha_0, \dots, \alpha_m$. For the type $\widetilde{C}'_m$, we use the description given on p. 585 of [8].

Suppose first that G is neither of type $\widetilde{C}^t_m$, nor of type $\widetilde{A}_2$. Choose i so that α_0 and α_i are not joined by an edge in Δ. Take an element $x = n_0 x_i \in G$ with $x_i \in M_i$ chosen so that if p is odd, $1 \ne x_i \in X_i$, while if $p = 2$, $x_i \in M_i$ of order $(q+1)$. Since $(o(n_0), o(x_i)) = 1$ and $[n_0, x_i] = 1$, we have that $1 \ne (n_0 x_i)^{o(n_0)} = x_i^{o(n_0)} \in M_i$ and $1 \ne (n_0 x_i)^{o(x_i)} = n_0^{o(x_i)} \in M_0$. Now consider the subgroup G_0 of G that corresponds to the Dynkin subdiagram $\Delta(\pi_0)$ where $\pi_0 = \pi - \{\alpha_0\} = \{\alpha_1, \dots, \alpha_m\}$. Notice that G_0 is a finite (possibly quasi-) simple group. By Corollary 1 of [11], there exists $y \in G_0$ such that G_0 is generated by $x_i^{o(n_0)}$ and y. Let $j \in \{1, 2, \dots, m\}$ be such that α_j and α_0 are joined in Δ (e.g., $j = 1$ for $\widetilde{A}_n$, $\widetilde{F}_4$; $j = 2$ for $\widetilde{B}_n$, etc.). Notice that $G_0 \ge M_j$ for every such j. Consider M_{0j}. We have $M_{0j} \ge M_0$ and by Lemma 2.2, $M_{0j} = \langle M_j, n_0^{o(x_i)} \rangle$. Since $\langle G_0, M_{0j} \rangle \ge \langle M_i, 0 \le i \le m \rangle = G$, we obtain $G = \langle x, y \rangle$.

Suppose now that G is of type $\widetilde{C}^t_m$ with $m \ge 3$. Take $x = h_0(u) n_1 x_m$ where $u^2 \ne \pm 1$ and $x_m \in M_m$ of odd order s co-prime to $t := o(h_0(u^2) h_1(-u^2))$. Notice that as $m \ge 3$, $[h_0(u) n_1, x_m] = 1$. Then $x^2 = h_0(u) h_0(u)^{n_1} n_1^2 x_m^2 = h_0(u) h_0(u) h_1(u^{-A_{01}}) h_1(-1) x_m^2 = h_0(u^2) h_1(-u^2) x_m^2$. An explicit calculation shows that $x^{2s} = h_0(u^{2s}) h_1((-u^2)^s)$ induces a non-trivial inner-diagonal automorphism on M_0. Thus by Corollary 1 of [11], there exists $y_0 \in M_0$ such that $\langle x^{2s}, y_0 \rangle \ge M_0$. On the other hand, $1 \ne x^{2t} = x_m^{2t} \in M_m$. Let $H \le G$ corresponding to $\Delta(\{\alpha_2, \dots, \alpha_m\})$. Again by Corollary 1 of [11], there exists $y_m \in H$ such that $\langle x_m^{2t}, y_m \rangle = H$. Take $y = y_0 y_m$. Clearly $[y_0, y_m] = 1$, $[y_0, H] = 1$ and $[y_m, M_0] = 1$. It follows that $\langle x, y \rangle \ge \langle x^{2s}, y_0 y_m \rangle \ge M_0$ and $\langle x, y \rangle \ge \langle x^{2t}, y_0 y_m \rangle \ge H$. In particular, $h_0(u), x_m \in \langle x, y \rangle$, and so $n_1 \in \langle x, y \rangle$. But by Lemma 2.2, $\langle M_0, n_1 \rangle = M_{01} \ge M_1$, and so $G = \langle x, y \rangle$.

If G is of type $\widetilde{C}^t_2$, take $x = h_0(u_0) h_2(u_2) n_1$ with $o(h_0(u_0))$ and $o(h_2(u_2))$ as large as possible and such that $u_0^2 u_2^{-2} \ne -1$. Then $x^2 = h_0(u_0) h_2(u_2) h_0(u_0)^{n_1} h_2(u_2)^{n_1} n_1^2 = h_0(u_0^2) h_2(u_2^2) h_1(-u_0^2 u_2^2)$. Now choose $y_0 \in M_0 - T$ of order $q-1$ if q is even and $(q-1)/|Z(M_0)|$ if q is odd, and $y_2 \in M_2$ of order $q+1$ if q is even and $(q+1)/|Z(M_2)|$ if q is odd. A celebrated theorem of Dickson (cf. 6.5.1 of [10]) implies that $\langle x^2, y_i^{o(y_j)} \rangle \ge M_i$, $\{i, j\} = \{0, 2\}$. Take $y = y_0 y_2$. It follows that $\langle x, y \rangle$ contains M_0 and M_2; in particular, $n_1 \in \langle x, y \rangle$. Now Lemma 2.2 implies that $\langle x, y \rangle \ge \langle M_0, n_1 \rangle \ge M_1$. Thus $G = \langle x, y \rangle$.

Finally let G be of type $\widetilde{A}_2$. Take $x = n_0 h_1(u)$ with $u^3 \ne \pm 1$. Then $x^2 = h_1(u)^{n_0} n_0^2 h_1(u) = h_1(u) h_0(u^{-A_{10}}) h_0(-1) h_1(u) = h_1(u^2) h_0(-u)$. An explicit calculation shows that x^2 acts non-trivially on M_{12} and so by Corollary 1 of [11], there exists $y \in M_{12}$ such that $\langle x^2, y \rangle \ge M_{12}$. In particular, $M_i \le \langle x, y \rangle$ for $i = 1, 2$, and so $n_0 \in \langle x, y \rangle$. But by Lemma 2.2, $\langle M_1, n_0 \rangle = M_{01} \ge M_0$. Therefore $G = \langle x, y \rangle$. □

Proposition 2.4. *Let G be a simply connected Kac–Moody group of rank 2 defined over a field $\mathbb{F}_q$. Then $d(G) \le 3$.*

Proof. We label the simple roots by α_1 and α_2. Choose $1 \ne x = h_1(u) h_2(v) \in T$ that induces non-trivial inner-diagonal automorphisms on both M_1 and M_2. Now use Corollary 1 of [11] to choose $y_i \in M_i$ so that $\langle x, y_i \rangle \ge M_i$, $i = 1, 2$. The result follows immediately. □

Proposition 2.5. *Let G be a simply connected strictly hyperbolic (symmetrizable) Kac–Moody group of rank at least 3. Then if q is large enough, $d(G) = 2$.*

Proof. We use the list of diagrams and notations as in Table 2 of [3]. If G is of type BG_3, BG'_3, GG_3 or $G'G3$, choose $x = h_1(u) n_2 h_3(v)$ with appropriately chosen $u, v \in \mathbb{F}_q^*$ and $y_i \in M_i$ for $i \in \{1, 3\}$ so that $(o(y_1), o(y_3)) = 1$ and $\langle x^2, y_i^{o(y_j)} \rangle \ge M_i$, $\{i, j\} = \{1, 3\}$. Let $y = y_1 y_3$. Then $\langle x, y \rangle$ contains M_1, M_3 and n_2. Apply Lemma 2.2 to conclude that $M_{12} = \langle M_1, n_2 \rangle \le \langle x, y \rangle$. As $M_1 \le M_{12}$, the result follows.

If G is of type CG'_3, CG_3, $G'G'_3$, choose $x = n_1 h_3(v)$ with appropriately chosen $v \in \mathbb{F}_q^*$ and $y \in M_2$ such that $\langle x^2, y \rangle \ge M_{23}$. Since $h_3(v) \in M_{23}$ and n_1 and M_2 generate M_{12}, we have that $G = \langle x, y \rangle$.

If G is of type $AD_3^{(2)}$, AGG_3, $AC_2^{(1)}$ or $AG'G'_3$, choose $x = n_1h_2(u)$ and $y \in M_{23}$ such that $\langle x^2, y\rangle \geq M_{23}$. Now use the fact that $h_2(u) \in M_{23}$ and that $\langle n_1, M_2\rangle = M_{12}$ to conclude that $G = \langle x, y\rangle$.

Finally, if G is of type $AC_3^{(1)}$, take $x = n_1h_4(u)$ and $y \in M_{234}$ such that $\langle x^2, y\rangle \geq M_{234}$ (such a y exists by Corollary 1 of [11]). Since $\langle n_1, M_4\rangle = M_{14}$ while $M_4 \leq M_{234}$, we conclude that $\langle x, y\rangle = G$. □

The proof of part (iv) of Theorem 1.1 for the hyperbolic groups follows by similar tricks and calculations done for every single group on the list of 130 diagrams (cf. tables of Section 7 of [7]). The proof of part $(v)(a)$ and $(v)(b)$ of Theorem 1.1 are obvious if one uses an observation (cf. Lemma 5 of [13]) that two elements generate a product of finite simple groups $H_1^{m_1} \times \ldots \times H_n^{m_n}$ ($H_i \ncong H_j$, $i \neq j$) if and only if their projections into each $H_i^{m_i}$ generate it, and from the estimates (see Corollary 1.4 of [14]) on the number h in a direct product H^h (H is a finite simple group) for which it is possible to be generated by 2 elements.

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