C. R. Acad. Sci. Paris, Ser. I 353 (2015) 583–588

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis/Functional analysis

Almost commuting functions of almost commuting self-adjoint operators

Fonctions presque commutantes d'opérateurs auto-adjoints presque commutants

Aleksei Aleksandrov [a,b], Vladimir Peller [c]

[a] *St.-Petersburg Branch, Steklov Institute of Mathematics, Fontanka 27, 191023 St. Petersburg, Russia*
[b] *Department of Mathematics and Mechanics, Saint Petersburg State University, 28, Universitetski pr., St. Petersburg, 198504, Russia*
[c] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*

ARTICLE INFO

Article history:
Received 11 December 2014
Accepted 22 April 2015
Available online 11 May 2015

Presented by Gilles Pisier

ABSTRACT

Let A and B be almost commuting (i.e, $AB - BA \in \boldsymbol{S}_1$) self-adjoint operators. We construct a functional calculus $\varphi \mapsto \varphi(A, B)$ for φ in the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$. This functional calculus is linear, the operators $\varphi(A, B)$ and $\psi(A, B)$ almost commute for $\varphi, \psi \in B^1_{\infty,1}(\mathbb{R}^2)$, $\varphi(A, B) = u(A)v(B)$ whenever $\varphi(s, t) = u(s)v(t)$, and the Helton–Howe trace formula holds. The main tool is triple operator integrals.

RÉSUMÉ

On dit que des opérateurs A et B sont presque commutants si leur commutateur $[A, B]$ appartient à la classe trace. Pour des opérateurs A et B auto-adjoints qui presque commutent, nous construisons un calcul fonctionnel $\varphi \mapsto \varphi(A, B)$, $\varphi \in B^1_{\infty,1}(\mathbb{R}^2)$, où $B^1_{\infty,1}(\mathbb{R}^2)$ est la classe de Besov. Ce calcul a les propriétés suivantes : il est linéaire, les opérateurs $\varphi(A, B)$ et $\psi(A, B)$ presque commutent pour toutes les fonctions φ et ψ dans $B^1_{\infty,1}(\mathbb{R}^2)$, $\varphi(A, B) = u(A)v(B)$ si $\varphi(s, t) = u(s)v(t)$, et la formule des traces de Helton et Howe est vraie. L'outil principal est la notion d'intégrales triples opératorielles.

Version française abrégée

Soient A et B des opérateurs auto-adjoints qui *presque commutent*, c'est-à-dire que leur commutateur $[A, B] \stackrel{\text{def}}{=} AB - BA$ appartient à la classe trace $\boldsymbol{S}_1$. Dans [6], on a obtenu la formule suivante :

$$\operatorname{trace}\left(\mathrm{i}\big(\varphi(A,B)\psi(A,B) - \psi(A,B)\varphi(A,B)\big)\right) = \iint\limits_{\mathbb{R}^2} \left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right) \mathrm{d}P$$

E-mail address: peller@math.msu.edu (V. Peller).

http://dx.doi.org/10.1016/j.crma.2015.04.012

pour tous les polynômes φ et ψ de deux variables, où P est une mesure signée boréliienne à support compact. Il se trouve que la mesure P est absolument continue et

$$\mathrm{d}P(x,y)=\frac{1}{2\pi}g(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

où g est la fonction principale de Pincus (voir la version en anglais pour plus d'informations).

Le calcul polynomial $\varphi\mapsto\varphi(A,B)$ a été étendu dans [4] et [12]. Dans [12] on a construit le calcul fonctionnel $\varphi\mapsto\varphi(A,B)$ pour φ appartenant à l'intersection des produit tensoriels projectifs $\mathcal{C}\stackrel{\mathrm{def}}{=}\big(L^\infty(\mathbb{R})\hat{\otimes}B^1_{\infty,1}(\mathbb{R})\big)\bigcap\big(B^1_{\infty,1}(\mathbb{R})\hat{\otimes}L^\infty(\mathbb{R})\big)$. Ce calcul est linéaire, les opérateurs $\varphi(A,B)$ et $\psi(A,B)$ presque commutent pour $\varphi,\psi\in\mathcal{C}$. En outre, si $\varphi(s,t)=u(s)v(t)$, alors $\varphi(A,B)=u(A)v(B)$. Finalement, la formule de Helton et Howe ci-dessus est vraie pour φ et ψ dans $\mathcal{C}$.

Il était aussi démontré dans [12] qu'il est impossible de construire un calcul fonctionnel $\varphi\mapsto\varphi(A,B)$ pour toutes les fonctions φ continûment dérivables qui ait les propriétés ci-dessus.

Le but de cette note est d'améliorer les résultats de [12].

Pour une fonction φ dans la classe de Besov $B^1_{\infty,1}(\mathbb{R}^2)$, nous définissons l'opérateur $\varphi(A,B)$ sous la forme :

$$\varphi(A,B)=\iint f(x,y)\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y),$$

où E_A et E_B sont les mesures spectrales des opérateurs A et B respectivement. La théorie d'intégrales doubles opératorielles a été développée par Birman et Solomyak [3] (voir aussi [10] et [2]).

Le résultat principal de cette note est basé sur la formule suivante :

$$\begin{aligned}\big[\varphi(A,B),Q\big]=&\iiint\frac{\varphi(x,y_1)-\varphi(x,y_2)}{y_1-y_2}\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y_1)[B,Q]\,\mathrm{d}E_B(y_2)\\&+\iiint\frac{\varphi(x_1,y)-\varphi(x_2,y)}{x_1-x_2}\,\mathrm{d}E_A(x_1)[A,Q]\,\mathrm{d}E_A(x_2)\,\mathrm{d}E_B(y),\end{aligned}\tag{1}$$

où $\varphi\in B^1_{\infty,1}(\mathbb{R}^2)$ et A et B sont des opérateurs auto-adjoints pour lesquels les commutateurs $[A,Q]$ et $[B,Q]$ appartiennent à $\boldsymbol{S}_1$.

Les intégrales triples opératorielles

$$\iiint\Phi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)\tag{2}$$

étaient définies dans [14] pour les fonctions Φ dans le produit tensoriel projectif intégral : ici T et R sont des opérateurs bornés, et E_1, E_2, E_3 sont des mesures spectrales. Puis, dans [7], les intégrales triples opératorielles étaient définies pour les fonctions qui appartiennent au produit tensoriel de Haagerup $L^\infty(E_1)\otimes_{\mathrm{h}}L^\infty(E_2)\otimes_{\mathrm{h}}L^\infty(E_2)$ (voir [17]). Il était établi dans [1] que les conditions $T\in\boldsymbol{S}_1$ ou $R\in\boldsymbol{S}_1$ et $\Phi\in L^\infty(E_1)\otimes_{\mathrm{h}}L^\infty(E_2)\otimes_{\mathrm{h}}L^\infty(E_2)$ n'impliquent pas que l'opérateur (2) appartienne à la classe trace.

On a défini dans [1] les produit tensoriels du type de Haagerup $L^\infty(E_1)\otimes_{\mathrm{h}}L^\infty(E_2)\otimes^{\mathrm{h}}L^\infty(E_3)$ et $L^\infty(E_1)\otimes^{\mathrm{h}}L^\infty(E_2)\otimes_{\mathrm{h}}L^\infty(E_3)$ (voir la version en anglais). On a démontré dans [1] que si $T\in\boldsymbol{S}_1$, R est un opérateur borné et $\Phi\in L^\infty(E_1)\otimes_{\mathrm{h}}L^\infty(E_2)\otimes^{\mathrm{h}}L^\infty(E_3)$, alors l'opérateur (2) appartient à $\boldsymbol{S}_1$. De même, si $R\in\boldsymbol{S}_1$, T est un opérateur borné et $\Phi\in L^\infty(E_1)\otimes^{\mathrm{h}}L^\infty(E_2)\otimes_{\mathrm{h}}L^\infty(E_3)$, alors l'opérateur (2) appartient à $\boldsymbol{S}_1$.

Si $\varphi\in B^1_{\infty,1}(\mathbb{R}^2)$, alors la fonction $(x_1,x_2,y)\mapsto\big(\varphi(x_1,y)-\varphi(x_2,y)\big)(x_1-x_2)^{-1}$ appartient à l'espace $L^\infty(E_A)\otimes_{\mathrm{h}}L^\infty(E_A)\otimes^{\mathrm{h}}L^\infty(E_B)$ et la fonction $(x,y_1,y_2)\mapsto\big(\varphi(x,y_1)-\varphi(x,y_2)\big)(y_1-y_2)^{-1}$ appartien à $L^\infty(E_A)\otimes_{\mathrm{h}}L^\infty(E_B)\otimes^{\mathrm{h}}L^\infty(E_B)$, voir [1]. Donc les intégrales dans (1) sont bien définies est appartiennent à $\boldsymbol{S}_1$.

La formule (1) implique que, si φ et ψ appartiennent à $B^1_{\infty,1}(\mathbb{R}^2)$ et A et B sont des opérateurs auto-adjoints qui presque commutent, alors

$$\begin{aligned}\big[\varphi(A,B),\psi(A,B)\big]=&\iiint\frac{\varphi(x,y_1)-\varphi(x,y_2)}{y_1-y_2}\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y_1)[B,\psi(A,B)]\,\mathrm{d}E_B(y_2)\\&+\iiint\frac{\varphi(x_1,y)-\varphi(x_2,y)}{x_1-x_2}\,\mathrm{d}E_A(x_1)[A,\psi(A,B)]\,\mathrm{d}E_A(x_2)\,\mathrm{d}E_B(y)\end{aligned}$$

et

$$\big\|\big[\varphi(A,B),\psi(A,B)\big]\big\|_{\boldsymbol{S}_1}\le\mathrm{const}\,\|\varphi\|_{B^1_{\infty,1}(\mathbb{R}^2)}\|\psi\|_{B^1_{\infty,1}(\mathbb{R}^2)}\big\|[A,B]\big\|_{\boldsymbol{S}_1}.$$

Ceci implique que la formule des traces de Helton et Howe est vraie pour toutes les fonctions φ et ψ dans $B^1_{\infty,1}(\mathbb{R}^2)$.

1. Introduction

The spectral theorem allows one, for a pair of commuting (bounded) self-adjoint operators A and B, to construct a linear and multiplicative functional calculus

$$\varphi \mapsto \varphi(A,B) = \int_{\mathbb{R}^2} \varphi(x,y)\,\mathrm{d}E_{A,B}(x,y)$$

for the class of bounded Borel functions on the plane $\mathbb{R}^2$. Here $E_{A,B}$ is the joint spectral measure of the pair(A,B) defined on the Borel subsets of $\mathbb{R}^2$.

If A and B are noncommuting self-adjoint operators, we can define functions of A and B in terms of double operator integrals

$$\varphi(A,B) \stackrel{\text{def}}{=} \iint_{\mathbb{R}^2} \varphi(x,y)\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y) \tag{3}$$

for functions φ that are Schur multipliers with respect to the spectral measure E_A and E_B of the operators A and B. The theory of double operator integrals was developed by Birman and Solomyak [3] (we also refer the reader to [10] and [2] for double operator integrals and Schur multipliers). It was observed in [1] that the Besov space $B^1_{\infty,1}(\mathbb{R}^2)$ of functions on $\mathbb{R}^2$ is contained in the space of Schur multipliers with respect to compactly supported spectral measures on $\mathbb{R}$, and so for $\varphi \in B^1_{\infty,1}(\mathbb{R}^2)$, the operator $\varphi(A,B)$ is well defined by (3) for bounded self-adjoint operators A and B (see [8] for the definition and properties of Besov classes).

In this paper we deal with almost commuting self-adjoint operators. *Self-adjoint operators A and B are called almost commuting* if their *commutator* $[A,B] \stackrel{\text{def}}{=} AB - BA$ belongs to a trace class.

In [6], the following trace formula was obtained for bounded almost commuting self-adjoint operators A and B:

$$\operatorname{trace}\left(\mathrm{i}\big(\varphi(A,B)\psi(A,B) - \psi(A,B)\varphi(A,B)\big)\right) = \iint_{\mathbb{R}^2} \left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)\mathrm{d}P, \tag{4}$$

where P is a signed Borel compactly supported measure that corresponds to the pair (A,B). The formula holds for polynomials φ and ψ.

It was shown in [16] that the signed measure P is absolutely continuous with respect to planar Lebesgue measure and

$$\mathrm{d}P(x,y) = \frac{1}{2\pi} g(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where g is the *Pincus principal function*, which was introduced in [15]. We refer the reader to [5] for more detailed information.

In [4] the polynomial functional calculus for almost commuting self-adjoint operators was extended to a functional calculus for the class of functions $\varphi = \mathcal{F}\omega$ that are Fourier transforms of complex Borel measures ω on $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2} (1+|t|)(1+|s|)\,\mathrm{d}|\omega|(s,t) < \infty,$$

and the Helton–Howe trace formula (4) was extended to the class of such functions.

The problem of constructing a rich functional calculus, which would extend the functional calculus constructed in [4] and for which trace formula (4) would still hold was considered in [12]. The problem was to find a big class of functions $\mathcal{C}$ on $\mathbb{R}^2$ and construct a functional calculus $\varphi \mapsto \varphi(A,B)$, $\varphi \in \mathcal{C}$, which has the following properties:

(i) *the functional calculus* $\varphi \mapsto \varphi(A,B)$, $\varphi \in \mathcal{C}$, *is linear*;
(ii) *if* $\varphi(s,t) = u(s)v(t)$, *then* $\varphi(A,B) = u(A)v(B)$;
(iii) *if* φ, $\psi \in \mathcal{C}$, *then* $\varphi(A,B)\psi(A,B) - \psi(A,B)\varphi(A,B) \in \boldsymbol{S}_1$;
(iv) *formula* (4) *holds for arbitrary* φ *and* ψ *in* $\mathcal{C}$.

Note that the right-hand side of (4) makes sense for arbitrary Lipschitz functions φ and ψ. However, it was established in [12] that a functional calculus satisfying (i)–(iii) cannot be defined for all continuously differentiable functions. This was deduced from the trace class criterion for Hankel operators (see [9] and [13]).

On the other hand, in [12] estimates of [10] and [11] were used to construct a functional calculus satisfying (i)–(iv) for the class $\mathcal{C} = \big(L^\infty(\mathbb{R})\hat{\otimes}B^1_{\infty,1}(\mathbb{R})\big)\bigcap\big(B^1_{\infty,1}(\mathbb{R})\hat{\otimes}L^\infty(\mathbb{R})\big)$. Here $\hat{\otimes}$ stands for projective tensor product and $B^1_{\infty,1}(\mathbb{R})$ is a Besov class.

In this paper, we considerably enlarge the class of functions $\big(L^\infty(\mathbb{R})\hat{\otimes}B^1_{\infty,1}(\mathbb{R})\big)\bigcap\big(B^1_{\infty,1}(\mathbb{R})\hat{\otimes}L^\infty(\mathbb{R})\big)$ and construct a functional calculus satisfying (i)–(iv) for the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$ of functions of two variables.

2. Triple operator integrals

Triple operator integrals

$$\iiint \Phi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3) \tag{5}$$

were defined in [14] for spectral measures E_1, E_2, E_3, bounded linear operators T and R, and for functions Φ in the integral projective tensor product $L^\infty(E_1)\hat{\otimes}_{\mathrm{i}}L^\infty(E_2)\hat{\otimes}_{\mathrm{i}}L^\infty(E_3)$.

Later on the definition of triple operator integrals was extended in [7] to the class of functions Φ in the Haagerup tensor product $L^\infty(E_1)\otimes_{\mathrm{h}} L^\infty(E_2)\otimes_{\mathrm{h}} L^\infty(E_3)$ (see [17] for information about Haagerup tensor products). However, it was shown in [1] that unlike in the case $\Phi\in L^\infty(E_1)\hat{\otimes}_{\mathrm{i}}L^\infty(E_2)\hat{\otimes}_{\mathrm{i}}L^\infty(E_3)$, the condition in which Φ belongs to $L^\infty(E_1)\otimes_{\mathrm{h}} L^\infty(E_2)\otimes_{\mathrm{h}} L^\infty(E_3)$ does not guarantee that if one of the operators T and R is of trace class, then the triple operator integral (5) belongs to $\boldsymbol{S}_1$.

In [1] the following Haagerup-like tensor products were introduced.

Definition. A function Ψ is said to belong to the tensor product $L^\infty(E_1)\otimes_{\mathrm{h}} L^\infty(E_2)\otimes^{\mathrm{h}} L^\infty(E_3)$ if it admits a representation

$$\Psi(x_1,x_2,x_3)=\sum_{j,k\ge 0}\alpha_j(x_1)\beta_k(x_2)\gamma_{jk}(x_3) \tag{6}$$

with $\{\alpha_j\}_{j\ge0},\ \{\beta_k\}_{k\ge0}\in L^\infty(\ell^2)$ and $\{\gamma_{jk}\}_{j,k\ge0}\in L^\infty(\mathcal{B})$, where $\mathcal{B}$ is the space of bounded operators on ℓ^2.

For a bounded linear operator R and for a trace class operator T, the triple operator integral

$$W=\iiint \Psi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)$$

was defined in [1] as the following continuous linear functional on the class of compact operators:

$$Q\mapsto \operatorname{trace}\left(\left(\iiint \Psi(x_1,x_2,x_3)\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)Q\,\mathrm{d}E_1(x_1)\right)T\right) \tag{7}$$

and it was shown that

$$\|W\|_{\boldsymbol{S}_1}\le\|\Psi\|_{L^\infty\otimes_{\mathrm{h}}L^\infty\otimes^{\mathrm{h}}L^\infty}\|T\|_{\boldsymbol{S}_1}\|R\|,$$

where $\|\Psi\|_{L^\infty\otimes_{\mathrm{h}}L^\infty\otimes^{\mathrm{h}}L^\infty}$ is the infimum of $\|\{\alpha_j\}_{j\ge0}\|_{L^\infty(\ell^2)}\|\{\beta_k\}_{k\ge0}\|_{L^\infty(\ell^2)}\|\{\gamma_{jk}\}_{j,k\ge0}\|_{L^\infty(\mathcal{B})}$ over all representations in (6).

Similarly, the tensor product $L^\infty(E_1)\otimes^{\mathrm{h}} L^\infty(E_2)\otimes_{\mathrm{h}} L^\infty(E_3)$ was defined in [1] as the class of functions Ψ of the form

$$\Psi(x_1,x_2,x_3)=\sum_{j,k\ge 0}\alpha_{jk}(x_1)\beta_j(x_2)\gamma_k(x_3),$$

where $\{\beta_j\}_{j\ge0},\ \{\gamma_k\}_{k\ge0}\in L^\infty(\ell^2)$, $\{\alpha_{jk}\}_{j,k\ge0}\in L^\infty(\mathcal{B})$. If T is a bounded linear operator, and $R\in\boldsymbol{S}_1$, then the continuous linear functional

$$Q\mapsto \operatorname{trace}\left(\left(\iiint \Psi(x_1,x_2,x_3)\,\mathrm{d}E_3(x_3)Q\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)\right)R\right)$$

on the class of compact operators determines a trace class operator

$$W\stackrel{\mathrm{def}}{=}\iiint \Psi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)$$

and

$$\|W\|_{\boldsymbol{S}_1}\le\|\Psi\|_{L^\infty\otimes^{\mathrm{h}}L^\infty\otimes_{\mathrm{h}}L^\infty}\|T\|\cdot\|R\|_{\boldsymbol{S}_1}.$$

3. Commutators of functions of almost commuting self-adjoint operators

Given a differentiable function φ on $\mathbb{R}^2$, we define the divided differences $\mathfrak{D}_1 f$ and $\mathfrak{D}_2 f$ on $\mathbb{R}^3$ by

$$(\mathfrak{D}_1\varphi)(x_1,x_2,y)\stackrel{\mathrm{def}}{=}\frac{\varphi(x_1,y)-\varphi(x_2,y)}{x_1-x_2}\quad\text{and}\quad(\mathfrak{D}_2\varphi)(x,y_1,y_2)=\frac{\varphi(x,y_1)-\varphi(x,y_2)}{y_1-y_2}.$$

It was shown in [1] that if φ is a bounded function on $\mathbb{R}^2$ whose Fourier transform is supported in the ball $\{\xi \in \mathbb{R}^2 : \|\xi\| \le 1\}$, then

$$(\mathfrak{D}_1\varphi)(x_1, x_2, y) = \sum_{j,k\in\mathbb{Z}} \frac{\sin(x_1 - j\pi)}{x_1 - j\pi} \cdot \frac{\sin(x_2 - k\pi)}{x_2 - k\pi} \cdot \frac{\varphi(j\pi, y) - \varphi(k\pi, y)}{j\pi - k\pi},$$

$$\sum_{j\in\mathbb{Z}} \frac{\sin^2(x_1 - j\pi)}{(x_1 - j\pi)^2} = \sum_{k\in\mathbb{Z}} \frac{\sin^2(x_2 - k\pi)}{(x_2 - k\pi)^2} = 1, \quad x_1\, x_2 \in \mathbb{R},$$

and

$$\sup_{y\in\mathbb{R}} \left\| \left\{ \frac{\varphi(j\pi, y) - \varphi(k\pi, y)}{j\pi - k\pi} \right\}_{j,k\in\mathbb{Z}} \right\|_{\mathcal{B}} \le \text{const}\, \|f\|_{L^\infty(\mathbb{R})}.$$

It follows that for $\varphi \in B^1_{\infty,1}(\mathbb{R}^2)$,

$$\mathfrak{D}_1\varphi \in L^\infty(\mathbb{R}) \otimes_{\mathrm{h}} L^\infty(\mathbb{R}) \otimes^{\mathrm{h}} L^\infty(\mathbb{R}) \quad \text{and} \quad \|\varphi\|_{L^\infty(\mathbb{R})\otimes_{\mathrm{h}} L^\infty(\mathbb{R})\otimes^{\mathrm{h}} L^\infty(\mathbb{R})} \le \text{const}\, \|\varphi\|_{B^1_{\infty,1}}.$$

Similarly, for $\varphi \in B^1_{\infty,1}(\mathbb{R}^2)$,

$$\mathfrak{D}_2\varphi \in L^\infty(\mathbb{R}) \otimes^{\mathrm{h}} L^\infty(\mathbb{R}) \otimes_{\mathrm{h}} L^\infty(\mathbb{R}) \quad \text{and} \quad \|\varphi\|_{L^\infty(\mathbb{R})\otimes^{\mathrm{h}} L^\infty(\mathbb{R})\otimes_{\mathrm{h}} L^\infty(\mathbb{R})} \le \text{const}\, \|\varphi\|_{B^1_{\infty,1}}.$$

Theorem 3.1. *Let A and B be self-adjoint operators and let Q be a bounded linear operator such that $[A, Q] \in \boldsymbol{S}_1$ and $[B, Q] \in \boldsymbol{S}_1$. Suppose that $\varphi \in B^1_{\infty,1}(\mathbb{R}^2)$. Then $\big[\varphi(A, B), Q\big] \in \boldsymbol{S}_1$,*

$$\begin{aligned}\big[\varphi(A, B), Q\big] = & \iiint \frac{\varphi(x, y_1) - \varphi(x, y_2)}{y_1 - y_2}\, \mathrm{d}E_A(x)\, \mathrm{d}E_B(y_1)[B, Q]\, \mathrm{d}E_B(y_2) \\ & + \iiint \frac{\varphi(x_1, y) - \varphi(x_2, y)}{x_1 - x_2}\, \mathrm{d}E_A(x_1)[A, Q]\, \mathrm{d}E_A(x_2)\, \mathrm{d}E_B(y)\end{aligned} \tag{8}$$

and

$$\big\|\big[\varphi(A, B), Q\big]\big\|_{\boldsymbol{S}_1} \le \text{const}\, \|\varphi\|_{B^1_{\infty,1}(\mathbb{R}^2)} \big(\big\|[A, Q]\big\|_{\boldsymbol{S}_1} + \big\|[B, Q]\big\|_{\boldsymbol{S}_1}\big).$$

To obtain the main result of the paper, we apply Theorem 3.1 in the case $Q = \psi(A, B)$, where $\psi \in B^1_{\infty,1}(\mathbb{R}^2)$.

Theorem 3.2. *Let A and B be almost commuting self-adjoint operators and let φ and ψ be functions in the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$. Then*

$$\begin{aligned}\big[\varphi(A, B), \psi(A, B)\big] = & \iiint \frac{\varphi(x, y_1) - \varphi(x, y_2)}{y_1 - y_2}\, \mathrm{d}E_A(x)\, \mathrm{d}E_B(y_1)[B, \psi(A, B)]\, \mathrm{d}E_B(y_2) \\ & + \iiint \frac{\varphi(x_1, y) - \varphi(x_2, y)}{x_1 - x_2}\, \mathrm{d}E_A(x_1)[A, \psi(A, B)]\, \mathrm{d}E_A(x_2)\, \mathrm{d}E_B(y)\end{aligned} \tag{9}$$

and

$$\big\|\big[\varphi(A, B), \psi(A, B)\big]\big\|_{\boldsymbol{S}_1} \le \text{const}\, \|\varphi\|_{B^1_{\infty,1}(\mathbb{R}^2)} \|\psi\|_{B^1_{\infty,1}(\mathbb{R}^2)} \big\|[A, B]\big\|_{\boldsymbol{S}_1}. \tag{10}$$

Note that the right-hand side of inequality (10) does not involve the norms of A or B. Thus formulae (8) and (9) allow us to consider commutators $\big[f(A, B), g(A, B)\big]$ even for unbounded self-adjoint operators A and B with trace class commutator $[A, B]$, though the functions $f(A, B)$ and $g(A, B)$ of A and B are not necessarily defined for f and g in $B^1_{\infty,1}(\mathbb{R}^2)$.

Let us also mention that in [12] it was proved that for almost commuting self-adjoint operators A and B, the functional calculus $\varphi \mapsto \varphi(A, B)$, $\varphi \in \big(L^\infty(\mathbb{R})\hat{\otimes} B^1_{\infty,1}(\mathbb{R})\big) \bigcap \big(B^1_{\infty,1}(\mathbb{R})\hat{\otimes} L^\infty(\mathbb{R})\big)$, is *almost multiplicative*, i.e.,

$$(\varphi\psi)(A, B) - \varphi(A, B)\psi(A, B) \in \boldsymbol{S}_1, \quad \varphi, \psi \in \big(L^\infty(\mathbb{R})\hat{\otimes} B^1_{\infty,1}(\mathbb{R})\big) \bigcap \big(B^1_{\infty,1}(\mathbb{R})\hat{\otimes} L^\infty(\mathbb{R})\big).$$

It would be interesting to find out whether the functional calculus $\varphi \mapsto \varphi(A, B)$, $\varphi \in B^1_{\infty,1}(\mathbb{R}^2) \bigcap L^\infty(\mathbb{R}^2)$, is also almost multiplicative.

4. An extension of the Helton–Howe trace formula

In this section we use the results of the previous section to extend the Helton–Howe trace formula.

Theorem 4.1. *Let A and B be almost commuting self-adjoint operators and let φ and ψ be functions in the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$. Then the following formula holds:*

$$\operatorname{trace}\left(\mathrm{i}\big(\varphi(A,B)\psi(A,B)-\psi(A,B)\varphi(A,B)\big)\right)=\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2}\left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)g(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{11}$$

where g is the Pincus principal function associated with the operators A and B.

It would be interesting to extend the Pincus principal function to the case of unbounded self-adjoint operators with trace class commutators and extend formula (11) to unbounded almost commuting operators.

References

[1] A.B. Aleksandrov, F.L. Nazarov, V.V. Peller, Functions of perturbed noncommuting self-adjoint operators, C. R. Acad. Sci. Paris, Ser. I (2015), in press.
[2] A.B. Aleksandrov, V.V. Peller, Operator Hölder–Zygmund functions, Adv. Math. 224 (2010) 910–966.
[3] M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals, in: Problems of Math. Phys., vol. 1, Leningrad University, 1966, pp. 33–67 (Russian). English translation: Topics in Mathematical Physics, vol. 1, Consultants Bureau Plenum Publishing Corporation, New York, 1967, pp. 25–54.
[4] R.W. Carey, J.D. Pincus, Mosaics, principal functions and mean motion in von Neumann algebras, Acta Math. 138 (1977) 153–218.
[5] K.F. Clancey, Seminormal Operators, Lecture Notes in Mathematics, vol. 742, Springer-Verlag, Berlin, 1979.
[6] J.W. Helton, R. Howe, Integral operators, commutators, traces, index, and homology, in: Lecture Notes in Mathematics, vol. 345, Springer-Verlag, New York, 1973, pp. 141–209.
[7] K. Juschenko, I.G. Todorov, L. Turowska, Multidimensional operator multipliers, Trans. Amer. Math. Soc. 361 (2009) 4683–4720.
[8] J. Peetre, New Thoughts on Besov Spaces, Duke Univ. Press, Durham, NC, 1976.
[9] V.V. Peller, Hankel operators of class $\boldsymbol{S}_p$ and their applications (rational approximation, Gaussian processes, the problem of majorizing operators), Mat. Sb. 113 (1980) 538–581. English translation in Math. USSR Sbornik 41 (1982) 443–479.
[10] V.V. Peller, Hankel operators in the theory of perturbations of unitary and self-adjoint operators, Funkc. Anal. Prilozh. 19 (2) (1985) 37–51 (Russian). English translation: Funct. Anal. Appl. 19 (1985) 111–123.
[11] V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators, in: Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Applied Mathematics, vol. 122, Dekker, New York, 1990, pp. 529–544.
[12] V.V. Peller, Functional calculus for a pair of almost commuting self-adjoint operators, J. Funct. Anal. 112 (1993) 325–345.
[13] V.V. Peller, Hankel Operators and Their Applications, Springer-Verlag, New York, 2003.
[14] V.V. Peller, Multiple operator integrals and higher operator derivatives, J. Funct. Anal. 233 (2006) 515–544.
[15] J.D. Pincus, Commutators and systems of singular integral equations, I, Acta Math. 121 (1968) 219–249.
[16] J.D. Pincus, On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutator, Stony Brook, preprint, 1972.
[17] G. Pisier, Introduction to Operator Space Theory, London Math. Society Lect. Notes Series, vol. 294, Cambridge University Press, Cambridge, UK, 2003.