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Boundary asymptotics of the relative Bergman kernel metric for elliptic curves

Asymptotique au bord de la métrique du noyau de Bergman relatif pour des courbes elliptiques

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ABSTRACT

For a family of compact Riemann surfaces, we study the asymptotic behaviors of the relative Bergman kernel metric near the boundaries of the moduli spaces. We have shown that the relative Bergman kernel metric on a family of elliptic curves has hyperbolic growth at the node. The proof relies largely on the elliptic function theory.

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RÉSUMÉ

Pour une famille de surfaces de Riemann compactes, nous étudions les comportements asymptotiques de la métrique du noyau relatif de Bergman à proximité des frontières des espaces de modules. Nous montrons que la métrique du noyau relatif de Bergman sur une famille de courbes elliptiques a une croissance hyperbolique au point singulier. La preuve est principalement basée sur la théorie des fonctions elliptiques.

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1. Introduction

On a connected complex manifold, the Bergman kernel is a reproducing kernel of the space of L^2 holomorphic top-forms. It is a canonical volume form determined by the complex structure and plays big roles in many deep results in complex geometry, such as the so-called partial C^0 estimates by Tian [11] and Donaldson & Sun [6]. As the complex structure changes, the variation of the Bergman kernels was initially studied by Maitani & Yamaguchi [9], who obtained the following theorem.

Theorem 1.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}_z \times \mathbb{C}_t$ with a smooth boundary. Let $B_t(z)$ be the Bergman kernel function of $\Omega_t := \Omega \cap (\mathbb{C}_z \times \{t\})$. Then $\log B_t(z)$ is a plurisubharmonic function on Ω.*

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Later, Berndtsson [2] generalized this result to the higher dimensional cases.

Theorem 1.2. *Let D be a pseudoconvex domain in $\mathbb{C}^n_z \times \mathbb{C}^k_t$, and let Φ be a plurisubharmonic function on D. For each t, set $D_t := D \cap (\mathbb{C}^n_z \times \{t\})$ and $\Phi_t := \Phi|_{D_t}$. Let $B_t(z)$ be the Bergman kernel of the Hilbert space $A^2(D_t, \Phi_t) := \{f \in \mathcal{O}(D_t) | \int_{D_t} e^{-\Phi_t}|f|^2 < +\infty\}$. Then $\log B_t(z)$ is a plurisubharmonic function on D.*

After that, the cases of arbitrary dimensional Stein manifolds and complex projective algebraic manifolds were decisively solved by Berndtsson [3], Tsuji [12] and Berndtsson & Păun [5]. Recently it has been shown by Guan & Zhou [8] and Berndtsson & Lempert [4] that this log-plurisubharmonic variation of Bergman kernels is intimately related to the extension of holomorphic functions with (optimal) L^2 estimates, which is originally due to Ohsawa & Takegoshi [10].

For a compact manifold X, let L be a positive line bundle equipped with a Hermitian metric h and let $\{s_1, \ldots, s_N\}$ be an orthonormal basis of $H^0(X, L)$. Then the Bergman kernel for L over X is defined as

$$B := \sum_{j=1}^{N} |s_i|_h^2. \tag{1}$$

As a special case of a holomorphic family $\{X_\lambda\}$ of compact Riemann surfaces, the Bergman kernel B_λ for the canonical bundle can be written as $B_\lambda = k_\lambda(z) dz \wedge d\bar{z}$, under some local coordinate z. Then the above log-plurisubharmonic variation results guarantee the following semi-positivity:

$$\sqrt{-1}\,\partial_\lambda \bar{\partial}_\lambda \log k_\lambda(z) \geq 0. \tag{2}$$

Still, the asymptotic behavior of the left-hand side of (2) is not yet fully understood in the limiting case, i.e., when λ tends to the boundary of the moduli space. In this paper, we compute its asymptotic behavior via elliptic functions, as a Legendre family of elliptic curves degenerate. The main result is as follows.

Theorem 1.3. *Let B_λ denote the Bergman kernel of the elliptic curve $X_\lambda := \{y^2 = x(x-1)(x-\lambda)\}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. In local coordinate z, write $B_\lambda = k_\lambda(z)\, dz \wedge d\bar{z}$. Then as $\lambda \to 0$, it has*

(i) $$\log k_\lambda(z) \sim -\log(-\log|\lambda|),$$

(ii) $$\sqrt{-1}\,\partial_\lambda \bar{\partial}_\lambda \log k_\lambda(z) \sim \frac{\sqrt{-1}\, d\lambda \wedge d\bar{\lambda}}{4|\lambda|^2 (\log|\lambda|)^2}.$$

Here $g(\lambda) \sim h(\lambda)$ means that the quotient of two functions $g(\lambda)$ and $h(\lambda)$ tends to 1 as $\lambda \to 0$. Note that when $\lambda \to 0$, X_λ degenerates to a singular curve $X_0 := \{y^2 = x^2(x-1)\}$. This theorem demonstrates that the Levi form of the relative Bergman kernel metric has hyperbolic growth near the node. In comparison, the Poincaré hyperbolic metric on the punctured unit disk has exactly the same asymptotic behavior near the origin. Two key ingredients involved here are the Weierstrass-$\wp$ function's coordinate parameterization and the elliptic modular lambda function's Taylor expansion.

2. Proof of the main theorem

Proof. From [1, p. 281], we know that the elliptic modular lambda function $\lambda = \lambda(\tau)$ effects a one-to-one conformal mapping of the region $\Omega := \{\tau \in \mathbb{C} | 0 < \operatorname{Re} \tau < 1, |\tau - \frac{1}{2}| > \frac{1}{2}, \operatorname{Im} \tau > 0\}$ onto the upper half plane $\mathbb{H}$. Also, this mapping extends continuously to the boundary in such a way that $\tau = \infty$ corresponds to $\lambda = 0$. Let Ω' be the reflection of Ω with respect to the imaginary axis, then Ω and Ω' together correspond to $\mathbb{C} \setminus \{0, 1\}$. In other words, $\operatorname{Im} \tau \to +\infty$ corresponds to $\lambda \to 0$. Since λ is conformal, so is its inverse function $\tau = \lambda^{-1} : \mathbb{C} \setminus \{0, 1\} \to \Omega \cup \Omega'$. Thus, for any fixed $\lambda \in \mathbb{C} \setminus \{0, 1\}$, there exists a complex number $\tau \in \Omega \cup \Omega' \subset \mathbb{H}$. Using 1 and this τ ($\operatorname{Im} \tau > 0$) as a lattice, one can get a complex torus, denoted by $X_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

For $z \in X_\tau$, the Weierstrass-$\wp$ function with respect to the lattice $(1, \tau)$ is defined to be

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum ranges over all $\omega = n_1 + n_2\tau$ except 0 ($n_1, n_2 \in \mathbb{Z}$).

Letting $e_1 := \wp(\frac{1}{2})$, $e_2 := \wp(\frac{\tau}{2})$, $e_3 := \wp(\frac{1+\tau}{2})$, then according to [1, p. 277], we know that the Weierstrass-$\wp$ function satisfies:

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Now change the variables, by setting:

$$\begin{cases} x = \frac{\wp(z)-e_2}{e_1-e_2} \\ y = \frac{\wp'(z)}{2(e_1-e_2)^{\frac{3}{2}}} \end{cases}.$$

Then it is easy to check that

$$y^2 = x(x-1)\left(x - \frac{e_3-e_2}{e_1-e_2}\right).$$

Actually, $\lambda(\tau) := \frac{e_3-e_2}{e_1-e_2}$ is just the definition of the elliptic modular lambda function. Another standard characterization of the elliptic modular lambda function $\lambda(\tau)$ is using $q := \exp(\pi i \tau)$ ($q \to 0$ as $\operatorname{Im}\tau \to +\infty$) to write it as

$$\lambda(\tau) = 16q - 128q^2 + 704q^3 - 3072q^4 + 11\,488q^5 - \ldots = 16q - 128q^2 + O(q^3). \tag{3}$$

Therefore, the complex torus $X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ can be identified with an elliptic curve

$$X_\lambda := \left\{y^2 = x(x-1)(x-\lambda)\right\}.$$

So later we will not distinguish X_τ and X_λ and their Bergman kernels. By definition (1), the Bergman kernel B_τ of the canonical bundle on X_τ can be simply written as $B_\tau = \frac{1}{\operatorname{Im}\tau} dz \wedge d\bar{z}$ under the local coordinate z. This means that $k_\lambda(z) = \frac{1}{\operatorname{Im}\tau}$. Now, we are able to analyze the asymptotic behaviors of B_τ as $\operatorname{Im}\tau \to +\infty$ (or equivalently the asymptotic behaviors of B_λ as $\lambda \to 0$):

Step 1: We check the conclusion (i).

From $q := \exp(\pi i \tau)$, it follows that $|q| = \exp(-\pi \cdot \operatorname{Im}(\tau))$ and also $\operatorname{Im}\tau = \frac{\log|q|}{-\pi}$. Therefore, it has

$$\begin{aligned} &\log k_\lambda(z) \\ &\quad = \log \frac{1}{\operatorname{Im}\tau} \\ &\quad = -\log \operatorname{Im}\tau \\ &\quad = -\log(-\frac{\log|q|}{\pi}). \end{aligned}$$

According to (3) as $\operatorname{Im}\tau \to +\infty$ ($q \to 0$), it has $|\lambda| = |16q - 128q^2 + O(q^3)| \sim 16|q| \to 0$. So, we get as $\lambda \to 0$ that

$$\log k_\lambda(z) \sim -\log(-\log|\lambda|).$$

Step 2: To check the conclusion (ii), we need to compute $\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z)$.

Taking the partial derivatives, one knows that

$$\begin{aligned} &\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z) \\ &\quad = \sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log \frac{1}{\operatorname{Im}\tau} \\ &\quad = -\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log \operatorname{Im}\tau \\ &\quad = -\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log\left(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}\right) \\ &\quad = -\sqrt{-1}\partial_\lambda\left(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}\left(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}\right)\right) \wedge d\bar{\lambda}. \end{aligned}$$

Since τ being holomorphic implies that $\frac{\partial\tau}{\partial\bar{\lambda}} = 0$, it has

$$\begin{aligned} &\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z) \\ &\quad = -\sqrt{-1}\partial_\lambda\left(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}\left(\frac{-\bar{\tau}}{2\sqrt{-1}}\right)\right) \wedge d\bar{\lambda} \\ &\quad = \sqrt{-1}\partial_\lambda\left(\frac{\bar{\tau}'}{\tau-\bar{\tau}}\right) \wedge d\bar{\lambda} \\ &\quad = \sqrt{-1}\frac{\frac{\partial\bar{\tau}'}{\partial\lambda}\cdot(\tau-\bar{\tau}) - \bar{\tau}'\frac{\partial(\tau-\bar{\tau})}{\partial\lambda}}{(\tau-\bar{\tau})^2} d\lambda \wedge d\bar{\lambda} \end{aligned}$$

$$= \sqrt{-1}\frac{0\cdot(\tau-\bar{\tau}) - \bar{\tau}'\frac{\partial(\tau)}{\partial\lambda}}{(\tau-\bar{\tau})^2}\,\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}$$
$$= \sqrt{-1}\frac{-|\tau'|^2}{(\tau-\bar{\tau})^2}\,\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda},$$

where $\tau' = \tau'(\lambda)$ is the derivative of τ with respect to λ. Since $(\tau-\bar{\tau})^2 = -4(\operatorname{Im}\tau)^2 \le 0$, one has:

$$\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z) = \sqrt{-1}\left(\frac{|\tau'|}{2\cdot\operatorname{Im}\tau}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda} \ge 0.$$

Thus, the semi-positivity as stated in (2) can be proved. By the Inverse Function Theorem, we know that $\tau'(b) = (\lambda^{-1})'(b) = \frac{1}{\lambda'(a)}$ holds for any $b = \lambda(a)$, where λ' is the derivative of λ with respect to τ. Therefore, one has:

$$\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda = \sqrt{-1}\left(\frac{1}{2\cdot\operatorname{Im}\tau\cdot|\lambda'(\tau)|}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}. \tag{4}$$

From (3) again, one can compute that $\lambda'(\tau) = \frac{\partial\lambda}{\partial q}\cdot\frac{\partial q}{\partial\tau} = (16 - 256q + \mathrm{O}(q^2))\cdot q\cdot\sqrt{-1}\pi$. As $\lambda\to 0$ (or equivalently $\operatorname{Im}\tau\to+\infty$ or $q\to 0$), it follows that

$$|\lambda'(\tau)| \sim 16\pi|q| \sim \pi|\lambda|.$$

Substituting it into (4), we will have:

$$\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z)$$
$$= \sqrt{-1}\left(\frac{1}{2\cdot\frac{\log|q|}{-\pi}\cdot|\lambda'(\tau)|}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}$$
$$\sim \sqrt{-1}\left(\frac{1}{2\cdot\frac{\log|q|}{-\pi}\cdot\pi|\lambda|}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}$$
$$= \sqrt{-1}\left(\frac{1}{-2|\lambda|\cdot\log|q|}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}$$
$$\sim \sqrt{-1}\left(\frac{1}{-2|\lambda|\cdot\log|\lambda|}\right)^2 \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}$$
$$= \frac{\sqrt{-1}\,\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}}{4|\lambda|^2(\log|\lambda|)^2}. \qquad \square$$

3. Further remarks

On the one hand, the above computational proof seems difficult to be generalized. On the other hand, for compact Riemann surfaces with higher genus, there might be a non-computational approach working for this problem. Based on the result of this paper, the author continues the study of the relative Bergman kernel metric for a family of elliptic curves and obtains explicitly a two-term asymptotic expansion formula in the limiting case. It turns out that the second term in the asymptotic expansion contains more "logarithmic" information. For the details, please see [7].

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References

[1] L.V. Ahlfors, Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable, 3rd ed., McGraw-Hill, New York, 1979.
[2] B. Berndtsson, Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains, Ann. Inst. Fourier (Grenoble) 56 (6) (2006) 1633–1662.
[3] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Ann. Math. (2) 169 (2) (2009) 531–560.
[4] B. Berndtsson, L. Lempert, A proof of the Ohsawa–Takegoshi theorem with sharp estimates, arXiv:1407.4946 [math.CV], 2014.
[5] B. Berndtsson, M. Păun, Bergman kernels and the pseudoeffectivity of relative canonical bundles, Duke Math. J. 145 (2) (2008) 341–378.

[6] S.K. Donaldson, S. Sun, Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry, Acta Math. 213 (1) (2014) 63–106.
[7] R.X. Dong, Boundary asymptotics of the relative Bergman kernel metric for elliptic curves II, submitted for publication, 2015.
[8] Q.-A. Guan, X.-Y. Zhou, A solution of an L^2 extension problem with optimal estimate and applications, Ann. Math. (2) 181 (3) (2015) 1139–1208.
[9] F. Maitani, H. Yamaguchi, Variation of Bergman metrics on Riemann surfaces, Math. Ann. 330 (3) (2004) 477–489.
[10] T. Ohsawa, K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. 195 (1987) 197–204.
[11] G. Tian, On Calabi conjecture for complex surfaces with positive first Chern class, Invent. Math. 101 (1) (1990) 101–172.
[12] H. Tsuji, Curvature semipositivity of relative pluricanonical systems, preprint, arXiv:math/0703729, 2007.