C. R. Acad. Sci. Paris, Ser. I 353 (2015) 677–682

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory

A simple proof of the mean value of $|K_2(\mathcal{O})|$ in function fields

Une démonstration simple de la valeur moyenne de $|K_2(\mathcal{O})|$ dans des corps de fonctions

Julio Andrade [a,b]

[a] *Mathematical Institute, University of Oxford, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK*
[b] *Depto. Matematica, PUC-Rio, Rio De Janeiro, RJ, Brazil*

ARTICLE INFO

Article history:
Received 19 January 2015
Accepted after revision 22 April 2015
Available online 14 May 2015

Presented by Jean-Pierre Serre

ABSTRACT

Let F be a finite field of odd cardinality q, $A = F[T]$ the polynomial ring over F, $k = F(T)$ the rational function field over F and $\mathcal{H}$ the set of square-free monic polynomials in A of degree odd. If $D \in \mathcal{H}$, we denote by $\mathcal{O}_D$ the integral closure of A in $k(\sqrt{D})$. In this Note, we give a simple proof for the average value of the size of the groups $K_2(\mathcal{O}_D)$ as D varies over the ensemble $\mathcal{H}$ and q is kept fixed. The proof is based on character sums estimates and on the use of the Riemann hypothesis for curves over finite fields.

RÉSUMÉ

Soit F un corps fini de cardinalité impaire q, $A = F[T]$ l'anneau de polynômes sur F, $k = F(T)$ le corps des fonctions rationnelles sur F et $\mathcal{H}$ l'ensemble des polynômes unitaires et sans facteur carré en A de degré impair. Si $D \in \mathcal{H}$, on note par $\mathcal{O}_D$ la clôture intégrale de A en $k(\sqrt{D})$. Dans cette Note, nous donnons une preuve simple de la valeur moyenne de la taille des groupes $K_2(\mathcal{O}_D)$ quand D varie dans l'ensemble $\mathcal{H}$ et quand q est maintenu fixe. La preuve est basée sur des estimations des sommes de caractères et sur l'utilisation de l'hypothèse de Riemann pour les courbes sur les corps finis.

1. Introduction

In [10], the author established average value results for the size of the algebraic K groups $K_2(\mathcal{O})$ over function fields. His proof crucially depends on the mean value of quadratic Dirichlet L-functions over function fields, which in turn was first obtained with the help of functions defined on the metaplectic two-fold cover of $GL(2, k_\infty)$, where k_∞ is the completion of $k = \mathbb{F}_q(T)$ at the prime at infinity.

In this paper, we provide a simple proof for the mean value of the size of the groups $K_2(\mathcal{O})$ over the rational function field $\mathbb{F}_q(T)$. Our proof is simpler in the sense that we avoid the Eisenstein series construction involved in the proof given

E-mail address: buenodeandra@maths.ox.ac.uk.

http://dx.doi.org/10.1016/j.crma.2015.04.018
1631-073X/

by Hoffstein and Rosen [6], and we do this by computing the mean value of the required quadratic Dirichlet L-function in $\mathbb{F}_q(T)$ through character sum estimates.

We start by setting the notation. Let $F = \mathbb{F}_q$ be a finite field with q elements (q odd), $A = \mathbb{F}_q[T]$, and $k = \mathbb{F}_q(T)$. For $f \in A$, we define $|f| = q^{\deg(f)}$ if $f \neq 0$ and $|0| = 0$. If $D \in A$ is square-free, then $\mathcal{O}_D$ is the integral closure of A in the quadratic function field $K_D = k(\sqrt{D})$.

The zeta function associated with A is defined by

$$\zeta_A(s) = \sum_{\substack{f \in A \\ \text{monic}}} |f|^{-s} = \prod_{\substack{P \in A \\ \text{monic} \\ \text{irreducible}}} (1 - |P|^{-s})^{-1}. \tag{1.1}$$

A straightforward calculation shows that $\zeta_A(s) = (1 - q^{1-s})^{-1}$. If $D \in A$ is square-free, we set $\chi_D(f) = (D/f)$, where (D/f) is the Kronecker symbol in A, and we form the quadratic Dirichlet L-function $L(s, \chi_D) = \sum_f \chi_D(f)|f|^{-s}$. Lastly, the zeta function of the ring $\mathcal{O}_D$ is defined by $\zeta_{\mathcal{O}_D}(s) = \sum_{\mathfrak{a}} N\mathfrak{a}^{-s}$, where $\mathfrak{a}$ runs through the nonzero ideals of $\mathcal{O}_D = A[\sqrt{D}]$ and $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$, i.e., the number of elements in $\mathcal{O}_D/\mathfrak{a}$. In a similar fashion to the case of number fields [11, Proposition 17.7], one has the relation

$$\zeta_{\mathcal{O}_D}(s) = \zeta_A(s)L(s, \chi_D). \tag{1.2}$$

Making use of (1.2) together with the results of Quillen [9] and Tate [12], Rosen [10] was able to relate the number $L(2, \chi_D)$ to the size of the group $K_2(\mathcal{O}_D)$. We will use such a relationship to prove our main result.

2. The algebraic K groups $K_2(\mathcal{O}_D)$ and a theorem of Rosen

Let $D \in A$ be a monic and square-free polynomial. For ease of discussion, we only consider the case where the degree of D is odd since the case with an even degree of D is similar and there are no important differences. Let $F = \mathbb{F}_q$ and K/F be a function field in one variable with a finite constant field $\mathbb{F}_q$. The primes in K are denoted by v, and $\mathcal{O}_v$ is the valuation ring at v. We denote by $\mathcal{P}_v$ the maximal ideal of $\mathcal{O}_v$ and by $\bar{F}_v$ the residue class field at v. The tame symbol $(*, *)_v$ is a mapping from $K^* \times K^*$ to $\bar{F}_v^*$ defined by

$$(a, b)_v = (-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)} \text{ modulo } \mathcal{P}_v. \tag{2.1}$$

Let $a \in K^*$ such that $a \neq 0, 1$ so the group $K_2(K)$ is defined to be $K^* \otimes K^*$ modulo the subgroup generated by the elements $a \otimes (1 - a)$. Moore (see [12] for more details) proved that the following sequence is exact

$$(0) \longrightarrow \ker(\lambda) \longrightarrow K_2(K) \xrightarrow{\lambda} \bigoplus_v \bar{F}_v^* \xrightarrow{\mu} F^* \longrightarrow (0), \tag{2.2}$$

where $\lambda : K_2(K) \to \bigoplus_v \bar{F}_v^*$ is the sum of the tame symbol maps, and $\mu : \bigoplus_v \bar{F}_v^* \to F^*$ is the map given by $\mu(\ldots, a_v, \ldots) = \prod_v a_v^{m_v/m}$ where $m_v = N\mathcal{P}_v - 1$ and $m = |F^*| = q - 1$.

By making use of the above discussion with Tate's proof [12] of the Birch–Tate conjecture concerning the size of $\ker(\lambda)$, i.e.,

$$|\ker(\lambda)| = (q - 1)(q^2 - 1)\zeta_K(-1), \tag{2.3}$$

where $\zeta_K(s) = \prod_v (1 - N\mathcal{P}_v^{-s})^{-1}$, the product being over all the primes v of the function field K, Rosen [10, Proposition 2] established that

$$\#K_2(\mathcal{O}_D) = q^{(3/2)\deg(D)} q^{-3/2} L(2, \chi_D). \tag{2.4}$$

With this in hand Rosen [10, Proposition 2(a)] proves the following theorem.

Theorem 2.1 *(Rosen). Let m be a square-free polynomial of degree M, with M odd, and $\varepsilon > 0$ given. Then*

$$(q - 1)^{-1}(q^M - q^{M-1})^{-1} \sum_{\substack{m \in A \\ m \text{ square-free}}} |K_2(\mathcal{O}_m)| = \zeta_A(2)\zeta_A(4)c(2)q^{-3/2}q^{3(M/2)} + O(q^{M(1+\varepsilon)}), \tag{2.5}$$

where

$$c(2) = \prod_P \left(1 - |P|^{-2} - |P|^{-5} + |P|^{-6}\right), \tag{2.6}$$

the product is taken over all monic irreducible polynomials in A.

3. The main result

Without further postponements, we present below the main result of this note.

Theorem 3.1. *Let* $\mathcal{H} = \{D \in A \text{ monic}, \textit{square-free and } \deg(D) = 2g+1\}$ *and* $\varepsilon > 0$. *Then*

$$\frac{1}{\#\mathcal{H}} \sum_{D \in \mathcal{H}} \#K_2(\mathcal{O}_D) = q^{\frac{3}{2}(2g+1)} q^{-3/2} \zeta_A(4) P(4) + O(q^{(2g+1)(1+\varepsilon)}), \tag{3.1}$$

where

$$P(s) = \prod_{\substack{P \in A \\ \text{monic} \\ \text{irreducible}}} \left(1 - \frac{1}{(|P|+1)|P|^s}\right). \tag{3.2}$$

A theorem similar to this was previously studied by Rosen. There are essentially two differences between Rosen's result (Theorem 2.1) and the main result in this paper. The first one is that the average value of $|K_2(\mathcal{O}_D)|$, as presented by Rosen, is an average taken over *all* square-free D and in our result we only consider the *monic* and square-free D, i.e., positive and fundamental discriminants over function fields. Comparing Eqs. (2.5) and (3.1), we observe that the constants multiplying the main term are close, but not equal. This is due to the fact that, in our result, we are summing over monic and square-free while in Rosen's result he is summing over all square-free, and in this sense Rosen's result is more general. This phenomenon is not new and it has appeared when one compares the main theorem of [2] with [6, Theorem 5.2], and it also appears in number fields when you compare the first moment of quadratic Dirichlet L-functions at the central point as given by [8, Theorem 1] and [5, Theorem (1)]; in both comparisons we see again that the constants multiplying the leading term in the mean values are different. The second, and most important difference, is the argument used to prove such a result. Rosen needs to invoke a mean value of $L(s, \chi_D)$ that was previously proved by himself and Hoffstein [6] through the use of the theory of the Eisenstein series and the metaplectic two-fold cover of $GL(2, k_\infty)$, whereas our method is solely based on estimating characters sums and in the use of the Riemann hypothesis for curves over finite fields.

4. Preparatory results

In this section we present a few auxiliary results that will be used in the proof of the main theorem of this note.

Lemma 4.1. *Let $f \in A$ be a fixed monic polynomial. Then for all $\varepsilon > 0$ we have that*

$$\sum_{\substack{D \in \mathcal{H} \\ \gcd(D,f)=1}} 1 = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P \mid f}} \left(\frac{|P|}{|P|+1}\right) + O\left(|D|^{\frac{1}{2}} |f|^{\varepsilon}\right). \tag{4.1}$$

Proof. See [2, Proposition 5.2]. □

We also need the following lemma.

Lemma 4.2. *We have*

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \prod_{P|f} (1+|P|^{-1})^{-1} = q^n \sum_{\substack{d \text{ monic} \\ \deg(d) \le n}} \frac{\mu(d)}{|d|} \prod_{P|d} (|P|+1)^{-1}. \tag{4.2}$$

Proof. See [2, Lemma 5.7]. □

The last result that we need before proceeding to the proof of our main theorem is given below, and it has appeared in a different form in [2–4] and its proof, as appears here, was first given in [1].

Lemma 4.3. *If $f \in A$ is not a perfect square then*

$$\sum_{\substack{D \in \mathcal{H} \\ f \neq \square}} \left(\frac{D}{f}\right) \ll |D|^{1/2} |f|^{1/4}. \tag{4.3}$$

Proof. We write

$$\begin{aligned}\sum_{D\in\mathcal{H}}\left(\frac{D}{f}\right) &= \sum_{2\alpha+\beta=2g+1}\sum_{\deg(B)=\beta}\sum_{\deg(A)=\alpha}\mu(A)\left(\frac{A^2B}{f}\right)\\ &= \sum_{0\le\alpha\le g}\sum_{\deg(A)=\alpha}\mu(A)\left(\frac{A^2}{f}\right)\sum_{\deg(B)=2g+1-2\alpha}\left(\frac{B}{f}\right)\\ &\le \sum_{0\le\alpha\le g}\sum_{\deg(A)=\alpha}\sum_{\deg(B)=2g+1-2\alpha}\left(\frac{B}{f}\right). \qquad (4.4)\end{aligned}$$

If $f\neq\Box$ then $\sum_{\deg(B)=2g+1-2\alpha}\left(\frac{B}{f}\right)$ is a character sum to a non-principal character modulo f. So using [7, Proposition 2.1] (which is the Pólya–Vinogradov inequality for $\mathbb{F}_q[T]$), we have that

$$\sum_{\deg(B)=2g+1-2\alpha}\left(\frac{B}{f}\right)\ll|f|^{1/2}. \qquad (4.5)$$

Further we can estimate trivially the non-principal character sum by

$$\sum_{\deg(B)=2g+1-2\alpha}\left(\frac{B}{f}\right)\ll\frac{|D|}{|A|^2}=q^{2g+1-2\alpha}. \qquad (4.6)$$

Thus, if $f\neq\Box$, we obtain that

$$\begin{aligned}\sum_{D\in\mathcal{H}}\left(\frac{D}{f}\right) &\ll \sum_{0\le\alpha\le g}\sum_{\deg(A)=\alpha}\min\left(|f|^{1/2},\frac{|D|}{|A|^2}\right)\\ &\ll |D|^{\frac{1}{2}}|f|^{\frac{1}{4}}, \qquad (4.7)\end{aligned}$$

upon using the first bound (4.5) for $\alpha\le g-\frac{\deg(f)}{4}$ and the second bound (4.6) for larger α. And this concludes the proof of the lemma. $\Box$

5. Proof of the Main Theorem

From now on, we are assuming that all the sums are being taken over monic polynomials and the products are over monic and irreducible polynomials P in $\mathbb{F}_q[T]$.

By [11, Proposition 4.3], we have:

$$\begin{aligned}\sum_{D\in\mathcal{H}}L(2,\chi_D) &= \sum_{D\in\mathcal{H}}\sum_{\deg(f)\le 2g}\chi_D(f)|f|^{-2}\\ &= \sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\le 2g\\ f=\Box}}\chi_D(f)|f|^{-2}+\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\le 2g\\ f\neq\Box}}\chi_D(f)|f|^{-2}. \qquad (5.1)\end{aligned}$$

For the sum above, where f is not a square of a polynomial, we use Lemma 4.3, which depends on the Riemann hypothesis for curves over finite fields, to obtain that

$$\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\le 2g\\ f\neq\Box}}\chi_D(f)|f|^{-2}\ll q^g. \qquad (5.2)$$

For the sum with f a square of a polynomial in (5.1), we need some extra manipulations. First we use Lemma 4.1; so we can write:

$$\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\le 2g\\ f=\Box}}\chi_D(f)|f|^{-2}=\frac{|D|}{\zeta_A(2)}\sum_{m=0}^{g}\frac{1}{q^{4m}}\sum_{\deg(l)=m}\prod_{P|l}\left(\frac{|P|}{|P|+1}\right)+O\left(|D|^{1/2}\sum_{n=0}^{2g}q^{n\epsilon-n}\right). \qquad (5.3)$$

From Lemma 4.2, we have:

$$\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\leq 2g\\ f=\square}}\chi_D(f)|f|^{-2}=\frac{|D|}{\zeta_A(2)}\sum_{\deg(d)\leq g}\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\sum_{\deg(d)\leq m\leq g}q^{-3m}$$
$$+O\left(q^{-g}\frac{(q^{2g\epsilon+\epsilon}-q^{2g+1})}{q^{\epsilon}-q}\right). \tag{5.4}$$

After some arithmetic manipulations and summing the geometric series we can rewrite (5.4) as

$$\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\leq 2g\\ f=\square}}\chi_D(f)|f|^{-2}=\zeta_A(4)\frac{|D|}{\zeta_A(2)}\left\{\sum_{d\text{ monic}}-\sum_{\deg(d)>g}\right\}\left(\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}\right)$$
$$-\frac{q^{-3g}}{q^3-1}\frac{|D|}{\zeta_A(2)}\left\{\sum_{d\text{ monic}}-\sum_{\deg(d)>g}\right\}\left(\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\right)$$
$$+O\left(q^{-g}\frac{(q^{2g\epsilon+\epsilon}-q^{2g+1})}{q^{\epsilon}-q}\right). \tag{5.5}$$

The sums over $\deg(d)>g$ in (5.5) are respectively bounded by $O(q^{-4g})$ and $O(q^{-g})$ as can be seen below:

$$\sum_{\deg(d)>g}\frac{\mu(d)}{|d|^4}\prod_{P|d}(|P|+1)^{-1}\ll\sum_{\deg(d)>g}\frac{1}{|d|^4}\prod_{P|d}\frac{1}{|P|}\ll\sum_{\deg(d)>g}\frac{1}{|d|^5}$$
$$=\sum_{n>g}q^{-4n}\ll q^{-4g} \tag{5.6}$$

and

$$\sum_{\deg(d)>g}\frac{\mu(d)}{|d|}\prod_{P|d}(|P|+1)^{-1}\ll\sum_{\deg(d)>g}\frac{1}{|d|}\prod_{P|d}\frac{1}{|P|}\ll\sum_{\deg(d)>g}\frac{1}{|d|^2}$$
$$=\sum_{n>g}q^{-n}\ll q^{-g}, \tag{5.7}$$

and therefore they do not contribute to the main term.

By expressing the sums over all monic d in (5.5) as Euler products we derive that

$$\sum_{D\in\mathcal{H}}\sum_{\substack{\deg(f)\leq 2g\\ f=\square}}\chi_D(f)|f|^{-2}=\zeta_A(4)\frac{|D|}{\zeta_A(2)}P(4)+O\left(q^{-g}\frac{(q^{2g\epsilon+\epsilon}-q^{2g+1})}{q^{\epsilon}-q}\right), \tag{5.8}$$

where $P(s)$ is given as in the statement of Theorem 3.1.

Combining (5.2) and (5.8) we get that

$$\sum_{D\in\mathcal{H}}L(2,\chi_D)=\frac{|D|}{\zeta_A(2)}\zeta_A(4)P(4)+O(q^g)+O\left(\frac{q^g+q^{g(2\epsilon-1)+\epsilon}}{q^{\epsilon}-q}\right). \tag{5.9}$$

We invoke [11, Proposition 2.3], which shows that $\#\mathcal{H}=|D|/\zeta_A(2)$, together with Eq. (2.4) and a few arithmetic maneuvers to complete the proof of the main theorem in this letter. $\square$

Acknowledgements

This research was supported by EPSRC grant EP/K021132X/1. The author is thankful to the comments of an anonymous referee, who helped us to give more clarity to the presentation of this note. The author also wishes to thank Professor Alain Connes for the several discussions related to the problem treated in this paper.

References

[1] J.C. Andrade, Rudnick and Soundararajan's theorem for function fields, preprint, 2014.
[2] J.C. Andrade, J.P. Keating, The mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble, J. Number Theory 132 (2012) 2793–2816.
[3] J.C. Andrade, J.P. Keating, Conjectures for the integral moments and ratios of L-functions over function fields, J. Number Theory 142 (2014) 102–148.
[4] D. Faifman, Z. Rudnick, Statistics of the zeros of zeta functions in families of hyperelliptic curves over a finite field, Compos. Math. 146 (2010) 81–101.
[5] D. Goldfeld, J. Hoffstein, Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series, Invent. Math. 80 (1985) 185–208.

[6] J. Hoffstein, M. Rosen, Average values of L-series in function fields, J. Reine Angew. Math. 426 (1992) 117–150.
[7] C. Hsu, Estimates for coefficients of L-functions for function fields, Finite Fields Appl. 5 (1999) 76–88.
[8] M. Jutila, On the mean value of $L(\frac{1}{2}, \chi)$ for real characters, Analysis 1 (1981) 149–161.
[9] D. Quillen, On the cohomology and K-theory of the general linear group over a finite field, Ann. Math. (2) 96 (1972) 552–586.
[10] M. Rosen, Average value of $|K_2(\mathcal{O})|$ in function fields, Finite Fields Appl. 1 (1995) 235–241.
[11] M. Rosen, Number Theory in Function Fields, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002.
[12] J. Tate, Symbols in arithmetic, in: International Congress of Mathematics, vol. 1, Gauthier-Villars, Paris, 1971, pp. 201–211.