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Brody curves in complicated sets

Courbes de Brody dans quelques ensembles compliqués

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ABSTRACT

For a hyperbolic generalized Hénon mapping (in the sense of [3]), J^+, the boundary of the set of points with bounded orbit is known as a complicated set and also known to admit a lamination by biholomorphic images of $\mathbb{C}$ (see [3,6]). We prove that there exists a leaf, which is an injective Brody curve in $\mathbb{P}^2$, in the lamination of J^+ for certain generalized Hénon mappings (for Brody curves and injective Brody curves, see Subsection 2.2).

RÉSUMÉ

L'ensemble J^+ des points d'orbite bornée est connu, pour une application de Hénon généralisée hyperbolique (dans le sens de [3]), comme étant un ensemble compliqué admettant une lamination par images biholomorphes de $\mathbb{C}$ (voir [3,6]). Nous montrons que, pour certaines applications de Hénon généralisées hyperboliques, une feuille de cette lamination J^+ est une courbe de Brody injective dans $\mathbb{P}^2$ (voir la sous-section 2.2 pour les notions de courbes de Brody et courbes de Brody injectives).

1. Introduction

A generalized Hénon mapping f is defined simply by a polynomial diffeomorphism $f(z,w) = (p(z) - aw, z)$ of $\mathbb{C}^2$, where p is a monic polynomial of one complex variable and a is a non-zero constant. Then, $f^{-1}(z,w) = (w, (p(w) - z)/a)$. Define

$$K^{\pm} = \{p \in \mathbb{C}^2 : \{f^{\pm n}(p)\} \text{ is a bounded sequence of } n\},$$

and $J^{\pm} = \partial K^{\pm}$, $K = K^+ \cap K^-$, $J = J^+ \cap J^-$ and $U^{\pm} = \mathbb{C}^2 \setminus K^{\pm}$. Let $g^+ : \mathbb{C}^2 \to \mathbb{R}$ denote the Green function associated with f. Then $U^+ = \{g^+ > 0\}$ and $K^+ = \{g^+ = 0\}$, and U^+ is open and K^+ is closed.

In [7], it was proved that the level set $\{g^+ = c\}$ for $c > 0$ is foliated by biholomorphic images of $\mathbb{C}$ and that each leaf is dense in $\{g^+ = c\}$. In [1] and [2], it was proved that every leaf is actually an injective Brody curve.

In this note, we study the same or a similar property for J^+. In [3,4] and [6], the lamination structure of J^+ was studied. In particular, in [3], Bedford and Smillie proved that J^+ admits a lamination $\mathcal{F}^+$ by biholomorphic images of $\mathbb{C}$ for

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a hyperbolic generalized Hénon mapping f. Then, one might ask "Is J^+ foliated by injective Brody curves as U^+ is?" Since U^+ and J^+ have completely different dynamical natures, we cannot apply the method of [1] and [2]. So, we rather consider a preceding question in this note: "Is there any leaf of J^+ which is injective Brody?"

The purpose of this note is to prove the following:

Theorem 1.1. *Let $f(z,w) = (p(z) - aw, z)$ where p is a monic polynomial of one complex variable and a is a non-zero constant. Assume that f is hyperbolic (in the sense of [3]) and $|a| \le 1$. Then, in the natural lamination $\mathcal{F}^+$ of J^+, there exists a leaf that is an injective Brody curve of $\mathbb{P}^2$.*

Remark 1. If we restrict Theorem 1.1 to the Hénon mappings in [6], then J^+ has fractional Hausdorff dimension and, as in [1] and [2], the injective Brody curve leaf is dense in J^+.

Remark 2. Since J^+ is closed in $\mathbb{C}^2$, due to Theorem 2.1, it is not difficult to find a Brody curve in J^+ for an arbitrary generalized Hénon mapping by applying the Brody reparametrization lemma with Theorem 2.1. However, it is not clear whether this Brody curve stays inside a single leaf, whether the Brody curve is injective, and whether Remark 1 is true for this Brody curve. The main point of Theorem 1.1 is that we have these properties.

The main ingredients for Theorem 1.1 are the hyperbolicity of f and flow-boxes of the lamination of J^+ and they are quite different from those for [1] and [2].

Notation. We use $\Delta(a,r)$ for the disc in $\mathbb{C}$ centered at $a \in \mathbb{C}$ and of radius $r > 0$ and Δ for the standard unit disc in $\mathbb{C}$. We denote by $\|\cdot\|$ the standard Euclidean metric of $\mathbb{C}^2$ and by $ds(P,V)$ the standard Fubini–Study metric on $\mathbb{P}^2$ of $V \in T_P\mathbb{P}^2$ at $P \in \mathbb{P}^2$. In this note, we are interested in the Fubini–Study metric on $\mathbb{C}^2 \subset \mathbb{P}^2$. With respect to the affine coordinate chart $(z,w) \in \mathbb{C}^2 \subset \mathbb{P}^2$, the standard Fubini–Study metric is defined by $\mathrm{d}s((z,w),(z',w')) = (|z'|^2 + |w'|^2 + |zw' - z'w|^2)/(1+|z|^2+|w|^2)^2)$ for $(z',w') \in T_{(z,w)}\mathbb{P}^2$. For a holomorphic curve $\gamma : U \to \mathbb{P}^2$ and for $\theta' \in U$, $\|\gamma\|_{FS,\theta'}$ denotes $\mathrm{d}s(\gamma(\theta'), \mathrm{d}\gamma|_{\theta=\theta'}(\frac{\mathrm{d}}{\mathrm{d}\theta}))$, where U is an open set in $\mathbb{C}$.

2. Preliminaries

2.1. Generalized Hénon mappings

Let $\mathbb{P}^2$ be the 2-dimensional complex projective space. We denote by $I_+ := [0:1:0]$ in the homogeneous coordinate system of $\mathbb{P}^2$. Then, f has the natural extension to $\widetilde{f} : \mathbb{P}^2 \setminus \{I_+\} \to \mathbb{P}^2 \setminus \{I_+\}$ by

$$\widetilde{f}([z:w:t]) = \left[t^d p(\frac{z}{t}) - awt^{d-1} : zt^{d-1} : t^d\right].$$

The following proposition and theorem describe the behavior of J^+.

Proposition 2.1. *(See [9].) $\overline{K^+} = K^+ \cup I_+$ in $\mathbb{P}^2$.*

Theorem 2.1. *(See Theorem 1.3 in [2].) There is no non-trivial holomorphic curve, which passes through I_+, and is supported in $\overline{K^+} \subseteq \mathbb{P}^2$.*

We recall hyperbolicity for generalized Hénon mappings in [3] (see [8] and also [6]). Recall that J is an invariant set for f. If a generalized Hénon mapping f is hyperbolic, there are continuous subbundles E_u and E_s such that $T\mathbb{C}^2_J = E^s \oplus E^u$, and $Df(E^s) = E^s$ and $Df(E^u) = E^u$, and there exist constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s}\| < c\lambda^n, n \ge 0 \quad \text{and} \quad \|Df^{-n}|_{E^u}\| < c\lambda^n, n \ge 0.$$

The Stable Manifold Theorem and Theorem 5.4 in [3] imply that, for every $x \in J$, there exists a leaf $\mathcal{L}_x$ in $\mathcal{F}^+$ such that $x \in \mathcal{L}_x$ and $T_x\mathcal{L}_x = E^s_x$ where $\mathcal{F}^+$ is the natural lamination of J^+.

2.2. Brody curves

In this subsection, we briefly introduce *Brody curves* and *injective Brody curves*.

Definition 2.2 *(Brody curve).* Let M be a compact complex manifold with a smooth metric $\mathrm{d}s_M$. Let $\psi : \mathbb{C} \to M$ be a non-constant holomorphic map.

The map ψ is said to be *Brody* if $\sup_{\theta' \in \mathbb{C}} \mathrm{d}s_M(\psi(\theta'), \mathrm{d}\psi|_{\theta=\theta'}(\frac{\partial}{\partial\theta})) < C_\psi$ for some constant $C_\psi > 0$. We call the image $\psi(\mathbb{C})$ a *Brody curve* in M. The curve $\psi(\mathbb{C})$ is said to be *injective Brody* if the parameterization ψ is injective.

Remark 3. Note that since M is assumed to be compact, Brodyness is independent of the choice of the metric $\mathrm{d}s_M$. For the purpose of simpler computations, in the remainder of the note, we will consider the Fubini–Study metric $\mathrm{d}s$ on $\mathbb{P}^2$.

Below, we consider some trivial examples. The proofs are all straightforward and so, we omit them.

Proposition 2.3. *Let α be a complex constant and p, q polynomials of one complex variable z. Then, all curves of the form $[z : p(z) : 1]$ and of the form $[p(z)\exp(z) : q(z)\exp(\alpha z) : 1]$ are Brody.*

However, not all holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^2$ are Brody. The mapping $z \to [\exp(z) : \exp(\mathrm{i}z^2) : 1]$ is not Brody. The following gives us some examples of injective but non-Brody curves.

Proposition 2.4. *The map $f_n : z \to (z, \exp(z^n))$ is not Brody in $\mathbb{C}^2 \subset \mathbb{P}^2$ for $n \geq 3$. In particular, not all holomorphic images of $\mathbb{C}$ in $\mathbb{P}^2$ are Brody.*

We close this section by pointing out a property of the injective Brody curves. Since the proof is straightforward, we omit it.

Proposition 2.5. *For an injective Brody curve $\mathcal{C}$ in $\mathbb{P}^2$, every parameterization of $\mathcal{C}$ has uniformly bounded Fubini–Study metrics. In short, the injective Brodyness property does not depend on the choice of the parameterization.*

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We first define a family of analytic discs. From Corollary 6.13 in [3], periodic points are dense in J. Pick a periodic point $P \in J$ and say N its period. Let $\mathcal{L}_P$ be a leaf in the lamination $\mathcal{F}^+$ of J^+ passing through P as discussed in Section 2. Fix an analytic disc $\psi : \Delta \to \mathcal{L}_P$ such that $\psi(0) = P$ and $\|\psi\|_{FS,0} > 0$. Then we consider a family of analytic discs as follows:

$$\varphi_n := f^{-Nn} \circ \psi : \Delta \to \mathcal{L}_P.$$

Then, since $\mathcal{L}_P$ is a stable manifold of P, from the hyperbolicity of f, $\|\varphi_n\|_{FS,0} \to \infty$ as $n \to \infty$.

Now we apply the Brody reparameterization lemma as in [5]. Note that φ_n's are holomorphic in a slightly larger disc. Define $H_n : \Delta \to \mathbb{R}^+$ by $H_n(\theta) := \|\varphi_n\|_{FS,\theta}(1 - |\theta|^2)$. Then, there exists $\theta_n \in \Delta$ with $H_n(\theta_n) = \max_{\theta\in\Delta} H_n(\theta)$. For each n, define a Möbius transformation $\mu_n(\zeta) := (\zeta + \theta_n)/(1 + \overline{\theta_n}\zeta)$ mapping 0 to θ_n. Let $g_n := \varphi_n \circ \mu_n$. Then

$$\|g_n\|_{FS,\zeta}(1 - |\zeta|^2) = \|\varphi_n\|_{FS,\theta}|\mu_n'(\zeta)|(1 - |\zeta|^2) = \|\varphi_n\|_{FS,\theta}(1 - |\theta|^2).$$

So, $\|g_n\|_{FS,\zeta} \leq \|g_n\|_{FS,0}/(1 - |\zeta|^2)$. Let $R_n = \|g_n\|_{FS,0}$ and define $k_n(\theta) = g_n(\theta/R_n)$. Then,

$$\|k_n\|_{FS,\theta} = \frac{\|g_n\|_{FS,\theta/R_n}}{R_n} \leq \frac{\|g_n\|_{FS,0}}{R_n(1 - |\theta/R_n|^2)} \leq 2,$$

on $\Delta(0, R_n/2)$. Note that $\|k_n\|_{FS,0} = 1$ and that from the hyperbolicity of f, we see that $R_n \to \infty$ as $n \to \infty$. Hence, from a normal family argument applied to $\{k_n\}$ and the compactness of $\mathbb{P}^2$, there exists a holomorphic map $\Phi : \mathbb{C} \to \overline{\mathcal{L}_P} \subseteq \overline{J^+} \subset \mathbb{P}^2$ with $\|\Phi\|_{FS,0} = 1$ and a subsequence $\{k_{n_j}\}$ locally uniformly converging toward Φ. In particular, Φ is a Brody map. From Proposition 2.1, we have $\overline{J^+} = J^+ \cup \{I_+\}$. However, Theorem 2.1 implies that $\Phi(\mathbb{C}) \subset J^+$.

We prove that the Brody curve $\Phi(\mathbb{C})$ sits inside a single leaf of the lamination $\mathcal{F}^+$ of J^+. Suppose the contrary. Then, there exist two points $\alpha, \beta \in \mathbb{C}$ such that $\Phi(\alpha)$ and $\Phi(\beta)$ live in two different leaves and that α, β are sufficiently close so that some small piece of the complex curve $\Phi(\mathbb{C})$ connecting $\Phi(\alpha)$ and $\Phi(\beta)$ sits in a single flow-box of the lamination of J^+. Let $\gamma \subset \Phi(\mathbb{C})$ denote the piece of the complex curve $\Phi(\mathbb{C})$ connecting $\Phi(\alpha)$ and $\Phi(\beta)$. Then, there exists a constant $\epsilon > 0$ such that for any plaque T in the flow-box, $\sup_{(z,w)\in\gamma} \mathrm{dist}((z, w), T) > \epsilon$ where $\mathrm{dist}(\cdot,\cdot)$ is with respect to the standard Euclidean distance of $\mathbb{C}^2$. This is a contradiction to the local uniform convergence of $\{k_{n_j}\}$ to Φ, since the image of each reparameterized analytic disc sits inside a single leaf of the lamination $\mathcal{F}^+$ of J^+. This proves that $\Phi(\mathbb{C})$ sits in a single leaf of the lamination $\mathcal{F}^+$ of J^+.

We show that Φ is one-to-one. Suppose on the contrary that Φ is not one-to-one. Then, there are $\alpha, \beta \in \mathbb{C}$ and $q \in \Phi(\mathbb{C})$ such that $\alpha \neq \beta$ and $\Phi(\alpha) = \Phi(\beta) = q$. Consider a sufficiently large $R_q > 1$ such that $\alpha, \beta \in \Delta(0, R_q)$. Let F be a compact set of $\mathbb{C}^2$ such that its interior contains J. Consider a finite covering of $J^+ \cap F$ consisting of flow-boxes of $\mathcal{F}^+$. Note that since $|a| \leq 1$, Theorem 5.9 in [3] says that for any leaf $\mathcal{L}$ in J^+, there exists a point $x \in J$ such that $\mathcal{L}$ is a stable manifold of the point x. Since $\Phi(\Delta(0, 2R_q))$ lives in a single leaf, there exists sufficiently large $N_q \in \mathbb{N}$ such that the analytic disc $f^{N_q}(\Phi(\Delta(0, 2R_q)))$ is entirely contained in a flow-box in the finite covering. By passing to a subsequence, we may assume that k_n converges to Φ locally uniformly and $f^{N_q}(k_n(\Delta(0, 2R_q)))$ is entirely contained in the same flow-box. Let π denote the projection onto the base in the flow-box. Then, since the image of each reparameterized analytic disc sits inside a single leaf of the lamination $\mathcal{F}^+$ of J^+, $\pi \circ f^{N_q} \circ k_n$'s are injective. Since the convergence of $\{k_n\}$ to Φ is locally uniform and Φ is not a constant map, the Hurwitz theorem of one complex variable implies that $\pi \circ f^{N_q} \circ \Phi$ is injective. So, we have $\pi \circ f^{N_q} \circ \Phi(\alpha) \neq \pi \circ f^{N_q} \circ \Phi(\beta)$, which contradicts $\Phi(\alpha) = \Phi(\beta) = q$. Hence, we just proved that Φ is injective.

Note that each leaf in the lamination $\mathcal{F}^+$ of J^+ is biholomorphic to $\mathbb{C}$ (see [3]). Since there is no proper biholomorphic image of $\mathbb{C}$ inside $\mathbb{C}$, the leaf containing the injective Brody curve itself should be an injective Brody curve. This proves our theorem. $\square$

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References

[1] T. Ahn, Foliation structure for generalized Hénon mappings, PhD thesis, University of Michigan, USA, 2012.
[2] T. Ahn, Foliation structure for generalized Hénon mappings, preprint, arXiv:1410.6576.
[3] E. Bedford, J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity, Invent. Math. 103 (1) (1991) 69–99.
[4] E. Bedford, M. Lyubich, J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$: IV. The measure of maximal entropy and laminar currents, Invent. Math. 112 (1) (1993) 77–125.
[5] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math. Soc. 235 (1978) 213–219.
[6] J.E. Fornæss, N. Sibony, Complex Hénon mappings in $\mathbb{C}^2$ and Fatou–Bieberbach domains, Duke Math. J. 65 (1992) 345–380.
[7] J.H. Hubbard, R. Oberste-Vorth, Hénon mappings in the complex domain I: the global topology of dynamical space, Publ. Math. Inst. Hautes Études Sci. 79 (1994) 5–46.
[8] M. Shub, Global Stability of Dynamical Systems, Springer-Verlag, 1987.
[9] N. Sibony, Dynamique des applications rationnelles de $\mathbb{P}^k$, Panor. Synth. 8 (1999) 97–185.