Harmonic analysis/Functional analysis

Triple operator integrals in Schatten–von Neumann norms and functions of perturbed noncommuting operators

Intégrales triples opératorielles en normes de Schatten–von Neumann et fonctions d'opérateurs perturbés ne commutant pas

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ABSTRACT

We study perturbations of functions $f(A, B)$ of noncommuting self-adjoint operators A and B that can be defined in terms of double operator integrals. We prove that if f belongs to the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$, then we have the following Lipschitz-type estimate in the Schatten–von Neumann norm $\boldsymbol{S}_p$, $1 \le p \le 2$: $\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le \text{const}(\|A_1 - A_2\|_{\boldsymbol{S}_p} + \|B_1 - B_2\|_{\boldsymbol{S}_p})$. However, the condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ does not imply the Lipschitz-type estimate in $\boldsymbol{S}_p$ with $p > 2$. The main tool is Schatten–von Neumann norm estimates for triple operator integrals.

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RÉSUMÉ

Nous examinons les perturbations de fonctions $f(A, B)$ d'opérateurs auto-adjoints A et B qui ne commutent pas. De telles fonctions peuvent être définies en termes d'intégrales doubles opératorielles. Pour f dans l'espace de Besov $B^1_{\infty,1}(\mathbb{R}^2)$, nous obtenons l'estimation lipschitzienne en norme de Schatten–von Neumann $\boldsymbol{S}_p$, $1 \le p \le 2$: $\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le \text{const}(\|A_1 - A_2\|_{\boldsymbol{S}_p} + \|B_1 - B_2\|_{\boldsymbol{S}_p})$. Par ailleurs, la condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ n'implique pas l'estimation lipschitzienne en norme de $\boldsymbol{S}_p$ pour $p > 2$. L'outil principal consiste en l'estimation d'intégrales triples opératorielles dans les normes de $\boldsymbol{S}_p$.

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Nous continuons d'examiner les propriétés de fonctions d'opérateurs auto-adjoints perturbés qui ne commutent pas. Dans [3], nous avons étudié des estimations de type lipschitzien pour les fonctions d'opérateurs auto-adjoints qui ne com-

mutent pas. Si A et B sont des opérateurs auto-adjoints qui ne commutent pas forcément, on définit la fonction $f(A, B)$ comme l'intégrale double opératorielle

$$f(A, B) \stackrel{\text{def}}{=} \iint f(x_1, x_2)\, \mathrm{d}E_A(x_1)\, \mathrm{d}E_B(x_2)$$

si f est un multiplicateur de Schur (voir [5,10] et [1] pour des informations sur les multiplicateurs de Schur et sur les intégrales doubles opératorielles). Ici E_A et E_B sont les mesures spectrales de A et B.

Nous avons démontré dans [3] que, si f est une fonction de la classe de Besov $B^1_{\infty,1}(\mathbb{R}^2)$ et si A_1, B_1, A_2, B_2 sont des opérateurs auto-adjoints tels que $A_2 - A_1 \in \boldsymbol{S}_1$ (classe trace) et $B_2 - B_1 \in \boldsymbol{S}_1$, alors

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_1} \le \text{const}\, \|f\|_{B^1_{\infty,1}} \max\{\|A_1 - A_2\|_{\boldsymbol{S}_1}, \|B_1 - B_2\|_{\boldsymbol{S}_1}\}.$$

Par ailleurs, nous avons établi dans [3] que la condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ n'implique pas l'estimation lipschitzienne en norme opératorielle.

Dans cette note, nous considérons le même problème dans la norme de Schatten–von Neumann $\boldsymbol{S}_p$. On trouve que, si $1 \le p \le 2$, l'inégalité suivante est vraie :

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le \text{const}\, \|f\|_{B^1_{\infty,1}} \max\{\|A_1 - A_2\|_{\boldsymbol{S}_p}, \|B_1 - B_2\|_{\boldsymbol{S}_p}\}.$$

Par ailleurs, nous établissons dans cette note que, si $p > 2$, il n'y a pas de nombre positif M pour lequel on ait

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le M \|f\|_{B^1_{\infty,1}} \max\{\|A_1 - A_2\|_{\boldsymbol{S}_p}, \|B_1 - B_2\|_{\boldsymbol{S}_p}\}.$$

Pour démontrer les résultats ci-dessus, nous utilisons la formule :

$$\begin{aligned}
f(A_1, B_1) - f(A_2, B_2) &= \iiint \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}\, \mathrm{d}E_{A_1}(x_1)(A_1 - A_2)\, \mathrm{d}E_{A_2}(x_2)\, \mathrm{d}E_{B_1}(y) \\
&\quad + \iiint \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\, \mathrm{d}E_{A_2}(x)\, \mathrm{d}E_{B_1}(y_1)(B_1 - B_2)\, \mathrm{d}E_{B_2}(y_2)
\end{aligned} \tag{1}$$

qui était établie dans [3] (voir la version en anglais de cette note pour la définition d'intégrales triples opératorielles) et nous obtenons les propriétés suivantes d'intégrales triples opératorielles.

Théorème. *Supposons que Ψ appartient au produit tensoriel de Haagerup $L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty$ et*

$$W = \iiint \Psi(x_1, x_2, x_3)\, \mathrm{d}E_1(x_1)\, T\, \mathrm{d}E_2(x_2)\, R\, \mathrm{d}E_3(x_3)$$

Alors on a :

(i) *si $p \ge 2$, $T \in \mathcal{B}(\mathcal{H})$ et $R \in \boldsymbol{S}_p$, alors $W \in \boldsymbol{S}_p$ et*

$$\|W\|_{\boldsymbol{S}_p} \le \|\Psi\|_{L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty} \|T\| \cdot \|R\|_{\boldsymbol{S}_p};$$

(ii) *si $p \ge 2$, $T \in \boldsymbol{S}_p$ et $R \in \mathcal{B}(\mathcal{H})$, alors $W \in \boldsymbol{S}_p$ et*

$$\|W\|_{\boldsymbol{S}_p} \le \|\Psi\|_{L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty} \|T\|_{\boldsymbol{S}_p} \|R\|;$$

(iii) *si $1/p + 1/q \le 1/2$, $T \in \boldsymbol{S}_p$ et $R \in \boldsymbol{S}_q$, alors $W \in \boldsymbol{S}_r$ avec $1/r = 1/p + 1/q$ et*

$$\|W\|_{\boldsymbol{S}_r} \le \|\Psi\|_{L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty} \|T\|_{\boldsymbol{S}_p} \|R\|_{\boldsymbol{S}_q}.$$

Par ailleurs, nous démontrons que si $p < 2$, alors (i) et (ii) sont faux.

Le produit tensoriel de Haagerup est défini dans la version en anglais de cette note. Remarquons que les différences divisées dans la formule (1) ne doivent pas appartenir au produit tensoriel de Haagerup $L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty$ pour toutes les fonctions $f \in B^1_{\infty,1}(\mathbb{R}^2)$. Cependant, elles appartiennent aux produits du type de Haagerup $L^\infty \otimes_{\mathrm{h}} L^\infty \otimes^{\mathrm{h}} L^\infty$ et $L^\infty \otimes^{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty$ (voir la version en anglais pour les définitions).

1. Introduction

In this note, we continue studying functions of noncommuting self-adjoint operators under perturbation. In [3], we studied Lipschitz-type estimates for functions of noncommuting pairs of self-adjoint operators. Recall that for (not necessarily commuting) self-adjoint operators A and B, we considered in [3] the functional calculus $f \mapsto f(A, B)$ defined as follows. For the class of functions f that are defined at least on the Cartesian product $\sigma(A) \times \sigma(B)$ of the spectra of the operators and such that f is a Schur multiplier with respect to the spectral measures E_A and E_B of A and B, the operator $f(A, B)$ is defined by $f(A, B) \stackrel{\text{def}}{=} \iint f(x, y) \, dE_A(x_1) \, dE_B(x_2)$. We refer the reader to [10] and [1] for the definition of Schur multipliers and double operator integrals; note also that the theory of double operator integrals was developed by Birman and Solomyak [5].

It was explained in [3] that if f is a function in the Besov space $B^1_{\infty,1}(\mathbb{R}^2)$, then f is a Schur multiplier with respect to E_A and E_B for arbitrary bounded self-adjoint operators A and B (we refer the reader to [9,12] and [2] for an introduction to Besov spaces).

In [3] we established the following Lipschitz-type estimate in trace norm for functions f in $B^1_{\infty,1}(\mathbb{R}^2)$:

$$\| f(A_1, B_1) - f(A_2, B_2) \|_{\boldsymbol{S}_1} \leq \text{const} \, \|f\|_{B^1_{\infty,1}} \max\{\|A_1 - A_2\|_{\boldsymbol{S}_1}, \|B_1 - B_2\|_{\boldsymbol{S}_1}\}.$$

On the other hand, it was shown in [3] that there is no such Lipschitz-type estimate for functions in $B^1_{\infty,1}(\mathbb{R}^2)$ in the operator norm.

Note that earlier it was shown in [10] and [11] that functions f on the real line $\mathbb{R}$ of class $B^1_{\infty,1}(\mathbb{R})$ are *operator Lipschitz*, i.e.,

$$\| f(A) - f(B) \| \leq \text{const} \, \|A - B\|$$

for arbitrary self-adjoint operators on Hilbert space and such Lipschitz-type estimates also hold in the trace norm (as well as in all Schatten–von Neumann norms). Recall that not all Lipschitz functions are operator Lipschitz; this was first proved by Farforovskaya in [6].

However, it turned out that the situation with Hölder functions is quite different. It was shown in [1] that if f is a Hölder function of order α, $0 < \alpha < 1$, then it is *operator Hölder of order α*, i.e., $\| f(A) - f(B) \| \leq \text{const} \, \|A - B\|^{\alpha}$ for self-adjoint operators A and B.

The results of [10,11], and [1] were extended in [2] to functions of normal operators (in other words, to functions of commuting pairs of self-adjoint operators) and in [8] to functions on n-tuples of commuting self-adjoint operators.

In this paper, we study Lipschitz-type estimates for functions of noncommuting self-adjoint operators in Schatten–von Neumann norms. We show that for functions in $B^1_{\infty,1}(\mathbb{R}^2)$, Lipschitz-type estimates hold in the Schatten–von Neumann norm of $\boldsymbol{S}_p$ for $p \in [1, 2]$. However, there are no Lipschitz-type estimates for $p > 2$.

To obtain Lipschitz-type estimates, we represent the difference $f(A_1, B_1) - f(A_2, B_2)$ in terms of triple operator integrals. In Section 2 we study Schatten–von Neumann properties of triple operator integrals. In Section 3, we state Lipschitz-type estimates in the Schatten–von Neumann norm $\boldsymbol{S}_p$ for $p \in [1, 2]$, while in Section 4 we show that such Lipschitz-type estimates do not hold for $p > 2$.

2. Triple operator integrals in Schatten–von Neumann norms

Let E_1, E_2, and E_3 be spectral measures on measurable spaces $(\mathcal{X}_1, \mathfrak{M}_1)$, $(\mathcal{X}_2, \mathfrak{M}_2)$, and $(\mathcal{X}_3, \mathfrak{M}_3)$ on a Hilbert space $\mathcal{H}$. In [13] triple operator integrals

$$W = \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \int_{\mathcal{X}_3} \Psi(x_1, x_2, x_3) \, dE_1(x_1) \, T \, dE_2(x_2) \, R \, dE_3(x_3) \tag{2}$$

were defined for bounded linear operators T and R and for functions Ψ in the projective tensor product $L^\infty(E_1) \hat{\otimes} L^\infty(E_2) \hat{\otimes} L^\infty(E_3)$. For such functions Ψ the following holds:

$$T \in \mathcal{B}(\mathcal{H}), \quad R \in \boldsymbol{S}_p, \quad p \geq 1 \quad \Longrightarrow \quad W \in \boldsymbol{S}_p, \tag{3}$$

$$T \in \boldsymbol{S}_p, \quad R \in \boldsymbol{S}_q, \quad \frac{1}{p} + \frac{1}{q} \leq 1 \quad \Longrightarrow \quad W \in \boldsymbol{S}_r, \quad \text{where} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{4}$$

Later in [7], the definition of triple operator integrals of the form (2) was extended to functions Ψ in the *Haagerup tensor product* $L^\infty(E_1) \otimes_{\mathrm{h}} L^\infty(E_2) \otimes_{\mathrm{h}} L^\infty(E_3)$ of the spaces $L^\infty(E_j)$, $j = 1, 2, 3$. It consists of functions Ψ that admit a representation

$$\Psi(x_1, x_2, x_3) = \sum_{j,k \geq 0} \alpha_j(x_1) \beta_{jk}(x_2) \gamma_k(x_3), \tag{5}$$

where $\{\alpha_j\}_{j\geq 0}$, $\{\gamma_k\}_{k\geq 0} \in L^\infty(\ell^2)$, and $\{\beta_{jk}\}_{j,k\geq 0} \in L^\infty(\mathcal{B})$. Here $\mathcal{B}$ is the space of infinite matrices that induce bounded linear operators on ℓ^2; $\mathcal{B}$ is endowed with the operator norm. We refer the reader to [14] for Haagerup tensor products. Moreover, the following inequality holds:

$$\|W\| \leq \|\Psi\|_{L^\infty\otimes_h L^\infty\otimes_h L^\infty}\|T\|\cdot\|R\|,$$

where

$$\|\Psi\|_{L^\infty\otimes_h L^\infty\otimes_h L^\infty} \overset{\text{def}}{=} \inf\left\{\|\{\alpha_j\}_{j\geq 0}\|_{L^\infty(\ell^2)}\|\{\beta_{jk}\}_{j,k\geq 0}\|_{L^\infty(\mathcal{B})}\|\{\gamma_k\}_{k\geq 0}\|_{L^\infty(\ell^2)}\right\},$$

the infimum being taken over all representations of Ψ in the form (5).

However, unlike in the case $\Psi \in L^\infty(E_1)\hat{\otimes}L^\infty(E_2)\hat{\otimes}L^\infty(E_3)$, for functions Ψ in the Haagerup tensor product $L^\infty(E_1) \otimes_h L^\infty(E_2) \otimes_h L^\infty(E_3)$, the situation with implications (3) and (4) is more complicated. We proved in [3] that there exist a function Ψ in $L^\infty(E_1) \otimes_h L^\infty(E_2) \otimes_h L^\infty(E_3)$, a bounded linear operator T, and an operator R of trace class such that the triple operator integral (2) does not belong to trace class $\boldsymbol{S}_1$.

Nevertheless, it turns out that implications in (3) and (4) hold under certain assumptions on p and q for an arbitrary function Ψ in $L^\infty(E_1) \otimes_h L^\infty(E_2) \otimes_h L^\infty(E_3)$.

Theorem 2.1. *Let* $\Psi \in L^\infty(E_1) \otimes_h L^\infty(E_2) \otimes_h L^\infty(E_3)$. *Then the following holds:*

(i) *if* $p \geq 2$, $T \in \mathcal{B}(\mathcal{H})$, *and* $R \in \boldsymbol{S}_p$, *then the triple operator integral in* (2) *belongs to* $\boldsymbol{S}_p$ *and*

$$\|W\|_{\boldsymbol{S}_p} \leq \|\Psi\|_{L^\infty\otimes_h L^\infty\otimes_h L^\infty}\|T\|\cdot\|R\|_{\boldsymbol{S}_p}; \tag{6}$$

(ii) *if* $p \geq 2$, $T \in \boldsymbol{S}_p$, *and* $R \in \mathcal{B}(\mathcal{H})$, *then the triple operator integral in* (2) *belongs to* $\boldsymbol{S}_p$ *and*

$$\|W\|_{\boldsymbol{S}_p} \leq \|\Psi\|_{L^\infty\otimes_h L^\infty\otimes_h L^\infty}\|T\|_{\boldsymbol{S}_p}\|R\|;$$

(iii) *if* $1/p + 1/q \leq 1/2$, $T \in \boldsymbol{S}_p$, *and* $R \in \boldsymbol{S}_q$, *then the triple operator integral in* (2) *belongs to* $\boldsymbol{S}_r$ *with* $1/r = 1/p + 1/q$ *and*

$$\|W\|_{\boldsymbol{S}_r} \leq \|\Psi\|_{L^\infty\otimes_h L^\infty\otimes_h L^\infty}\|T\|_{\boldsymbol{S}_p}\|R\|_{\boldsymbol{S}_q}.$$

We will see in Section 4 that statements (i) and (ii) of Theorem 2.1 do not hold for $p \in [1, 2)$.

To prove Theorem 2.1, we first prove statements (i) and (ii) and then use complex interpolation of bilinear operators, see Theorem 4.4.1 in [4].

In [3] we established the following formula for $f(A_1, B_1) - f(A_2, B_2)$ in the case when $f \in B^1_{\infty,1}(\mathbb{R}^2)$ and the pair (A_2, B_2) is a trace class perturbation of the pair (A_1, B_1):

$$f(A_1, B_1) - f(A_2, B_2) = \iiint \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}\,dE_{A_1}(x_1)(A_1 - A_2)\,dE_{A_2}(x_2)\,dE_{B_1}(y)$$
$$+ \iiint \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\,dE_{A_2}(x)\,dE_{B_1}(y_1)(B_1 - B_2)\,dE_{B_2}(y_2). \tag{7}$$

However, the divided differences

$$(x_1, x_2, y) \mapsto \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2} \quad\text{and}\quad (x, y_1, y_2) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

do not have to belong to the Haagerup tensor product $L^\infty \otimes_h L^\infty \otimes_h L^\infty$ (this follows from Theorem 3.1 of [3]). Nevertheless, we defined in [3] Haagerup like tensor products $L^\infty \otimes_h L^\infty \otimes^h L^\infty$ and $L^\infty \otimes^h L^\infty \otimes_h L^\infty$, defined triple operator integrals for such Haagerup like tensor products, and proved that the first divided difference belongs to $L^\infty \otimes_h L^\infty \otimes^h L^\infty$, while the second divided difference belongs to $L^\infty \otimes^h L^\infty \otimes_h L^\infty$.

We are going to use the above integral representation in the case when the pair (A_2, B_2) is an $\boldsymbol{S}_p$ perturbation of the pair (A_1, B_1) for $p \in [1, 2]$.

Definition. A function Ψ is said to belong to the Haagerup-like tensor product $L^\infty \otimes_h L^\infty \otimes^h L^\infty$ of the first kind if it admits a representation

$$\Psi(x_1, x_2, x_3) = \sum_{j,k\geq 0} \alpha_j(x_1)\beta_k(x_2)\gamma_{jk}(x_3) \tag{8}$$

with $\{\alpha_j\}_{j\geq 0}$, $\{\beta_k\}_{k\geq 0} \in L^\infty(\ell^2)$, and $\{\gamma_{jk}\}_{j,k\geq 0} \in L^\infty(\mathcal{B})$. For a bounded linear operator R and for an operator T of class $\boldsymbol{S}_p$, $1 \leq p \leq 2$, we define the triple operator integral

$$W = \iiint \Psi(x_1, x_2, x_3)\, dE_1(x_1)\, T\, dE_2(x_2)\, R\, dE_3(x_3) \tag{9}$$

as the following continuous linear functional on the Schatten–von Neumann class $\boldsymbol{S}_{p'}$, $1/p' = 1 - 1/p$, (on the class of compact operators if $p = 1$):

$$Q \mapsto \operatorname{trace}\left(\left(\iiint \Psi(x_1, x_2, x_3)\, dE_2(x_2)\, R\, dE_3(x_3)\, Q\, dE_1(x_1)\right) T\right). \tag{10}$$

The fact that the linear functional (10) is continuous is a consequence of inequality (6), which also implies the following estimate:

$$\|W\|_{\boldsymbol{S}_p} \le \|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty} \|T\|_{\boldsymbol{S}_p} \|R\|,$$

where $\|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty}$ is the infimum of

$$\|\{\alpha_j\}_{j \ge 0}\|_{L^\infty(\ell^2)} \|\{\beta_k\}_{k \ge 0}\|_{L^\infty(\ell^2)} \|\{\gamma_{jk}\}_{j,k \ge 0}\|_{L^\infty(\mathcal{B})}$$

over all representations in (8).

Similarly, suppose that Ψ belongs to the *Haagerup-like tensor product* $L^\infty \otimes^{\mathrm h} L^\infty \otimes_{\mathrm h} L^\infty$ *of the second kind*, i.e., Ψ admits a representation

$$\Psi(x_1, x_2, x_3) = \sum_{j,k \ge 0} \alpha_{jk}(x_1)\beta_j(x_2)\gamma_k(x_3),$$

where $\{\beta_j\}_{j \ge 0}$, $\{\gamma_k\}_{k \ge 0} \in L^\infty(\ell^2)$, $\{\alpha_{jk}\}_{j,k \ge 0} \in L^\infty(\mathcal{B})$, T is a bounded linear operator, and $R \in \boldsymbol{S}_p$, $1 \le p \le 2$. Then the continuous linear functional

$$Q \mapsto \operatorname{trace}\left(\left(\iiint \Psi(x_1, x_2, x_3)\, dE_3(x_3)\, Q\, dE_1(x_1)\, T\, dE_2(x_2)\right) R\right)$$

on the class $\boldsymbol{S}_{p'}$ determines an operator

$$W \stackrel{\mathrm{def}}{=} \iiint \Psi(x_1, x_2, x_3)\, dE_1(x_1)\, T\, dE_2(x_2)\, R\, dE_3(x_3)$$

of class $\boldsymbol{S}_p$. Moreover,

$$\|W\|_{\boldsymbol{S}_p} \le \|\Psi\|_{L^\infty \otimes^{\mathrm h} L^\infty \otimes_{\mathrm h} L^\infty} \|T\| \cdot \|R\|_{\boldsymbol{S}_p}.$$

The following result can be deduced from Theorem 2.1.

Theorem 2.2. *Let* $\Psi \in L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty$. *Suppose that* $T \in \boldsymbol{S}_p$ *and* $R \in \boldsymbol{S}_q$, *where* $1 \le p \le 2$, *and* $1/p + 1/q \le 1$. *Then the operator* W *in* (9) *belongs to* $\boldsymbol{S}_r$, $1/r = 1/p + 1/q$, *and*

$$\|W\|_{\boldsymbol{S}_r} \le \|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty} \|T\|_{\boldsymbol{S}_p} \|R\|_{\boldsymbol{S}_q}.$$

If T *is a bounded linear operator and* $R \in \boldsymbol{S}_p$, $1 \le p \le 2$, *then* $W \in \boldsymbol{S}_p$ *and*

$$\|W\|_{\boldsymbol{S}_p} \le \|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty} \|T\|_{\boldsymbol{S}_p} \|R\|.$$

A similar result holds for triple operator integrals defined above for functions Ψ in $L^\infty \otimes^{\mathrm h} L^\infty \otimes_{\mathrm h} L^\infty$.

3. Lipschitz-type estimates in $\boldsymbol{S}_p$ with $p \le 2$

Recall that in [3] we established formula (7) for functions $f \in B^1_{\infty,1}(\mathbb{R}^2)$ and pairs of self-adjoint operators (A_1, B_1) and (A_2, B_2) such that (A_2, B_2) is a trace class perturbation of (A_1, B_1). Moreover, we proved in [3] that the first divided difference in (7) belongs to $L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty$, while the second divided difference belongs to $L^\infty \otimes^{\mathrm h} L^\infty \otimes_{\mathrm h} L^\infty$.

The following theorem shows that the same is true if we replace trace norm with the norm in $\boldsymbol{S}_p$ for $p \in [1, 2]$. It can be deduced from Theorem 2.1 and formula (7).

Theorem 3.1. *Let* $1 \le p \le 2$ *and let* $f \in B^1_{\infty,1}(\mathbb{R}^2)$. *Suppose that* (A_1, B_1) *and* (A_2, B_2) *are pairs of self-adjoint operators such that* $A_2 - A_1 \in \boldsymbol{S}_p$ *and* $B_2 - B_1 \in \boldsymbol{S}_p$. *Then*

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le \operatorname{const} \|f\|_{B^1_{\infty,1}} (\|A_1 - A_2\|_{\boldsymbol{S}_p} + \|B_1 - B_2\|_{\boldsymbol{S}_p}). \tag{11}$$

We have defined functions $f(A, B)$ for f in $B^1_{\infty,1}(\mathbb{R}^2)$ only for bounded self-adjoint operators A and B. However, as in the case of trace class perturbations (see [3]), formula (7) allows us to define the difference $f(A_1, B_1) - f(A_2, B_2)$ in the case when $f \in B^1_{\infty,1}(\mathbb{R}^2)$ and the self-adjoint operators A_1, A_2, B_1, B_2 are possibly unbounded once we know that the pair (A_2, B_2) is an $\boldsymbol{S}_p$ perturbation of the pair (A_1, B_1), $1 \le p \le 2$. Moreover, inequality (11) also holds for such operators.

4. No Lipschitz-type estimates in $\boldsymbol{S}_p$ with $p > 2$

Recall that we showed in [3] that the condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ does not imply Lipschitz-type estimates in the operator norm for functions of pairs of (not necessarily commuting) self-adjoint operators. It turns out that the same is true in the $\boldsymbol{S}_p$-norms for $p > 2$.

The main result of this section shows that unlike in the case of commuting pairs of self-adjoint operators, the condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ does not imply Lipschitz-type estimates in the norm of $\boldsymbol{S}_p$ with $p > 2$.

Theorem 4.1. *Let $p > 2$. There is no positive number M such that*

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_p} \le M\|f\|_{L^\infty(\mathbb{R}^2)}(\|A_1 - A_2\|_{\boldsymbol{S}_p} + \|B_1 - B_2\|_{\boldsymbol{S}_p})$$

for all bounded functions f on $\mathbb{R}^2$ with Fourier transform supported in $[-1, 1]^2$ and for all finite-rank self-adjoint operators A_1, A_2, B_1, B_2.

The proof of Theorem 4.1 uses a modification of the construction given in [3].
We conclude the paper with a theorem that can be deduced from Theorem 4.1.

Theorem 4.2. *Let $1 \le p < 2$. There are spectral measures E_1, E_2 and E_3 on Borel subsets of $\mathbb{R}$, a function Ψ in the Haagerup tensor product $L^\infty(E_1) \otimes_h L^\infty(E_2) \otimes_h L^\infty(E_3)$ and an operator Q in $\boldsymbol{S}_p$ such that*

$$\iiint \Psi(x_1, x_2, x_2)\, dE_1(x_1)\, dE_2(x_2) Q\, dE_3(x_3) \notin \boldsymbol{S}_p.$$

Acknowledgements

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