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Un raffinement du théorème de Gauss–Lucas (d'après W.P. Thurston)

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ABSTRACT

We present a refinement of the Gauss–Lucas theorem, following an idea of W.P. Thurston.

RÉSUMÉ

Nous présentons un raffinement du théorème de Gauss–Lucas, d'après une idée de W.P. Thurston.

1. Introduction

This article, although self contained, can be considered as a completion of the article [1], where the statements and ideas of proofs have been exposed.

Let f be a polynomial with complex coefficients of degree $m \geq 1$. By the fundamental theorem of algebra, f has exactly m complex roots counting multiplicities. The derivative f' is a polynomial of degree $m-1$ and its roots of are called the *critical points* of f.

The following theorem by Gauss and rediscovered by Lucas describes a beautiful relationship between the roots of a polynomial and the roots of its derivative.

Theorem 1.1 *(Classical version of the Gauss–Lucas theorem). If f is a polynomial of degree at least two, then the convex hull of the roots of f contains the roots of f'.*

The proof of this theorem is fairly elementary and can be found in many text books. W. Thurston gave a purely geometric proof of the theorem eliminating all the calculations except the fact that a polynomial is a product of linear factors. We present his proof in Section 2. He also showed another version of this theorem, which goes as follows.

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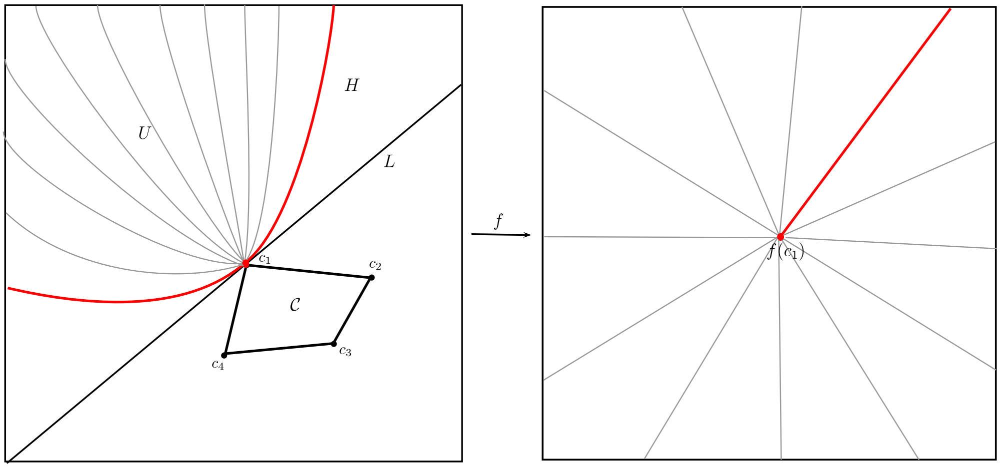

Fig. 1. (Color online.) Bijective region of f.

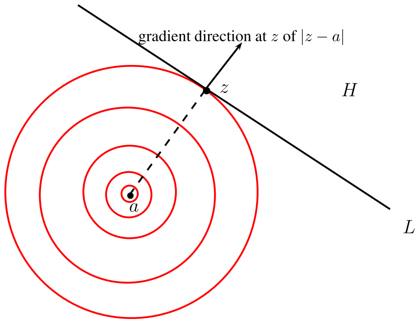

Fig. 2. (Color online.) Gradient direction of $z \mapsto |z-a|$.

Proposition 1.1 *(A surjective version of the Gauss–Lucas theorem). Let f be any polynomial of degree at least two. Denote by $\mathcal{C}$ the convex hull of the critical points of f. Then $f : E \to \mathbb{C}$ is surjective for any closed half-plane E intersecting $\mathcal{C}$.*

This is an equivalent statement to the Gauss–Lucas theorem. In fact, if the convex hull of the roots of f does not contain $\mathcal{C}$, there would exist a closed half-plane E intersecting $\mathcal{C}$, but avoiding all roots of f. Hence $f|_E$ would be non-surjective and hence Proposition 1.1 implies Theorem 1.1. Conversely, one sees that Theorem 1.1 implies Proposition 1.1 by considering the convex hull of roots of $f - w$ for all complex constants w.

By this proposition, if we want to cover the whole plane with a half-plane containing the least possible points, we should take the one just touching $\mathcal{C}$. To achieve this covering more efficiently, one may look for a connected region D such that f is surjective on D and injective on the interior of D. Thurston, in private communications, indicated to us the following result, which is more powerful than the classical Gauss–Lucas theorem.

Theorem 1.2 *(The Gauss–Lucas–Thurston theorem). Let f be any polynomial of degree at least two, and denote by $\mathcal{C}$ the convex hull of the critical points of f. Let L be a straight line intersecting $\mathcal{C}$ and that bounds an open half plane H disjoint from $\mathcal{C}$. If $c \in L$ is a critical point of f, then there exists a set of geodesics for the metric $|f'(z)| \cdot |\mathrm{d}z|$ starting at c and with directions in H that forms an open subset U of H for which $f(\overline{U}) = \mathbb{C}$. Furthermore, f maps U bijectively onto $\mathbb{C}$ minus a radial line from $f(c)$ to ∞ (see Fig. 1).*

We had never seen such statement elsewhere. We have completed a proof of it following the guidelines of Thurston. The proof will be presented in Section 3.

2. A geometric proof of the Gauss–Lucas theorem, due to Thurston

Proof. Let $a \in \mathbb{C}$. Consider the distance function $z \mapsto |z-a|$. The level curves of this function are concentric circles with center a, so the gradient direction of the function at any point $z \neq a$ is the radial direction from a to z (see Fig. 2). Let L be the line through z and perpendicular to the segment $[a, z]$. Denote by H the open half-plane determined by L disjoint from a. Then the function $z \mapsto |z-a|$ has a strictly positive directional derivation at z in any direction pointing into H.

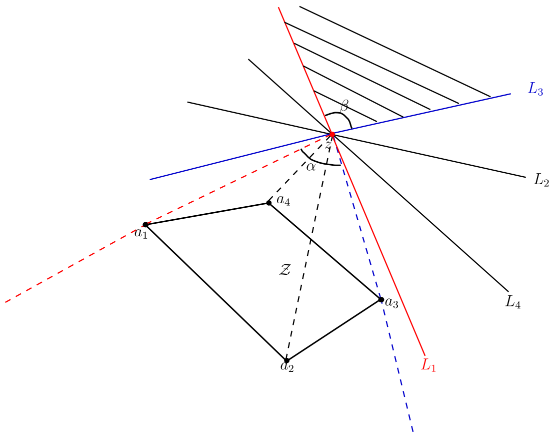

Fig. 3. (Color online.) The intersection of all H_j.

Let f be a polynomial of degree at least two. Denote by $\mathcal{Z}$ the convex hull of the roots of f and let z be a point outside $\mathcal{Z}$. For each root a_j of f, we can draw a line L_j through z and perpendicular to the segment $[z, a_j]$. Denote by H_j the open half-planes determined by L_j disjoint from a_j (see Fig. 3). By the argument above, each distance function $z \mapsto |z - a_j|$ has a strictly positive directional derivation at z in any direction pointing into H_j.

The fact that $z \notin \mathcal{Z}$ is equivalent to say that the point z admits a cone of angle $\alpha < \pi$ containing all roots of f. Then the intersection of all H_j is a cone based at z with angle $\beta = \pi - \alpha > 0$ (the shadow region in Fig. 3). Therefore, there exists a direction in which the directional derivative of $z \mapsto |z - a_j|$ is strictly positive for each root a_j of f. We see that, in this direction, the directional derivative of the function $z \mapsto |f(z)|$ is still strictly positive, hence non-zero. Then we conclude that $f'(z) \neq 0$. Otherwise, if $f'(z) = 0$, the directional derivative of f in each direction at z is 0, so is the one of $|f|$, which is a contradiction to the argument above. $\square$

3. The proof of Theorem 1.2

Proof. We firstly treat the case of simple critical points, i.e., $f''(c) \neq 0$. In this case, we will show that the region U in Theorem 1.2 consists of all geodesics starting at c and with directions in H.

We may assume that $c = 0$, $f(0) = f'(0) = 0$, $H = \mathbb{H}_+$, the upper half-plane, and $f''(0) = 1$. Note that the conformal metric $|f'(z)||\mathrm{d}z|$, which is a Riemannian metric with the singularities at critical points of f, is obtained by pulling-back the Euclidean metric by f, so that the geodesics in this metric are pullback by f of straight lines. Then we just need to prove that there exist two branches of $f^{-1}(\mathbb{R}^+)$ emanating from 0, belonging to $\overline{\mathbb{H}_+} := \mathbb{H}_+ \cup \mathbb{R}$, and such that the open region $U \subset \mathbb{H}_+$ bounded by these two curves contains no curves of $f^{-1}(\mathbb{R}^+)$. If so, the map $f : U \to \mathbb{C} \setminus \mathbb{R}^+$ is a covering, and hence must be bijective by the Riemann–Hurwitz formula.

First, we deal with the case that at least one critical point is in the open lower half plane $\mathbb{H}_-$. As 0 is a simple critical point, then there are two branches R_1, R_2 in $f^{-1}(\mathbb{R}^+)$ emanating from 0, and each component of $\mathbb{C} \setminus (R_1 \cup R_2)$ has the angle $\pi/2$ at 0. We want to show that $R_1 \cup R_2 \subset \mathbb{H}_+ \cup \{0\}$.

For this we study, the behavior of the vector field $1/f'(z)$ on $L = \mathbb{R}$ near 0, because the two curves R_1, R_2 are trajectories of the vector field $\frac{1}{f'(z)} \frac{\partial}{\partial z}$.

We have $f'(z) = \frac{z}{A} \cdot \prod_j (c_j - z)$ with $A = \prod c_j$ and $\Im(c_j) \leq 0$ for any non-zero critical point c_j of f.

Fix $x \in \mathbb{R}$ close to 0. Write $c_j - x = r_j(x) \mathrm{e}^{\mathrm{i}\theta_j(x)}$ with $r_j(x) > 0$ and $0 \leq \theta_j(x) < 2\pi$. In particular, $c_j = r_j(0)\mathrm{e}^{\mathrm{i}\theta_j(0)}$. So $A = \prod r_j(0)\mathrm{e}^{\mathrm{i}\theta_j(0)}$. Then, for $x \neq 0$, there is some $r(x) > 0$ so that

$$\frac{1}{f'(x)} = \frac{A}{x}\left(\prod_j (c_j - x)\right)^{-1} = \frac{r(x)}{x} \mathrm{e}^{\mathrm{i}\sum(\theta_j(0) - \theta_j(x))}.$$

Note that for x close to 0, the quantity $\theta_j(0) - \theta_j(x)$ is either 0 ($c_j \in \mathbb{R}$) or close to 0 and has the same sign as x ($c_j \in \mathbb{H}_-$) (see Fig. 4). Since at least one critical point is in $\mathbb{H}_-$, then $\Im \frac{1}{f'(x)} > 0$ for any $x \neq 0$ close to 0. It follows that for z close to 0, the vector $1/f'(z)$ points into $\mathbb{H}_+$. Since the two curves R_1, R_2 are trajectories of the vector field $\frac{1}{f'(z)} \frac{\partial}{\partial z}$, according to Gronwall's comparison lemma, we conclude that they belong locally to $\mathbb{H}_+$.

We now show that the two trajectories stay entirely in $\mathbb{H}_+$. Let $a \in \mathbb{H}_+$ be a point of the two trajectories. Let $\gamma(t)$ be either R_1 or R_2 so that $f(\gamma(t)) = t \cdot f(a)$, $t \in \mathbb{R}^+$. Then $\gamma'(t) \cdot f'(\gamma(t)) = f(a)$ and $\gamma''(t) \cdot f'(\gamma(t)) + \gamma'(t)^2 f''(\gamma(t)) \equiv 0$. It

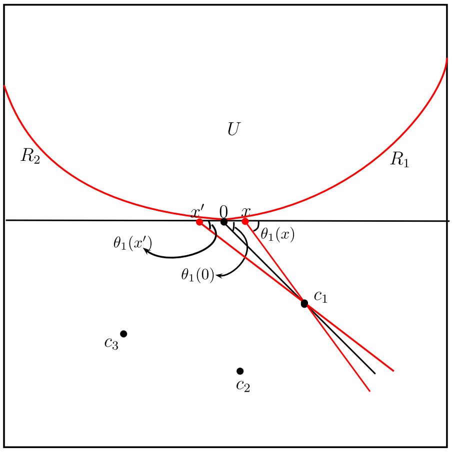

Fig. 4. (Color online.) The same sign of $\theta_1(0) - \theta_1(x)$ and x.

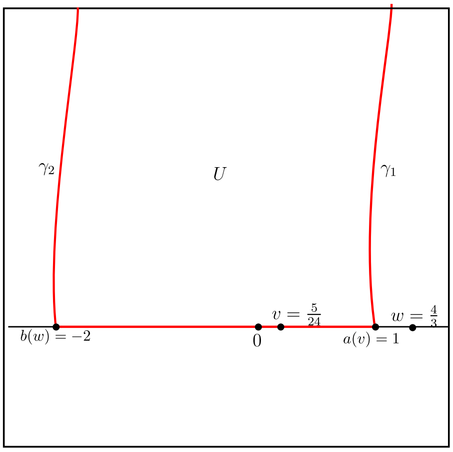

Fig. 5. (Color online.) Bijective region of $f = \frac{1}{2}z^2 - \frac{1}{6}z^3 - \frac{1}{8}z^4$.

follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Im\frac{1}{\gamma'(t)}) = \Im(-\frac{\gamma''(t)}{\gamma'(t)^2}) = \Im\frac{f''(\gamma(t))}{f'(\gamma(t))} = \Im\frac{1}{\gamma(t)} + \sum_j \Im\frac{1}{\gamma(t) - c_j} < 0.$$

This means that $\Im\frac{1}{\gamma'(t)}$ is decreasing and therefore $\Im\gamma'(t)$ is increasing. Write $\gamma(t) = x(t) + \mathrm{i}y(t)$. We know that $y'(t) > 0$ for small t and $y'(t)$ is increasing. So $y'(t) > 0$ for all $t \in \mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\}$. It follows that $y(t)$ is also increasing. But $y(t) > 0$ for t small. So $y(t) > 0$ for all $t \in \mathbb{R}^+_*$.

It remains to show that the open region $U \subset \mathbb{H}_+$ bounded by R_1, R_2 contains no branches in $f^{-1}(\mathbb{R}^+)$. As no critical points of f are in $\mathbb{H}^+$, the region $U \setminus f^{-1}(\mathbb{R}^+)$ is connected and simply-connected, and the map $f : U \setminus f^{-1}(\mathbb{R}^+) \to \mathbb{C} \setminus \mathbb{R}^+$ is bijective by the Riemann–Hurwitz formula. Assume that a branch R of $f^{-1}(\mathbb{R}^+)$ belongs to U. Since $f(R) = f(R_1) = \mathbb{R}^+$, there exists a point in a neighborhood of $\mathbb{R}^+$ having two pre-images in $U \setminus f^{-1}(\mathbb{R}^+)$ that belong to the neighborhoods of R and R_1 respectively. It is a contradiction. So we obtain that $U \cap f^{-1}(\mathbb{R}^+) = \emptyset$.

Now we assume that all critical points of f are in $\mathbb{R}$. Then f is a polynomial with real coefficients. For each $t \in \mathbb{R}^+$ close to 0, there is a unique positive (resp. negative) number $a(t)$ (resp. $b(t)$) such that $f(a(t)) = t$ (resp. $f(b(t)) = t$). The function $a(t)$ (resp. $b(t)$) is increasing (resp. decreasing) in the interval $(0, v)$ (resp. $(0, w)$), where $a(v)$ (resp. $b(w)$) is the minimal positive (resp. maximal negative) critical point of f. Suppose the multiplicity as a critical point of $a(v)$ (resp. $b(w)$) is $k \geq 1$, then there is a curve γ_1 in $f^{-1}[v, +\infty)$ (resp. γ_2 in $f^{-1}[w, +\infty)$) such that the angle between γ_1 and $[0, a(v)]$ at $a(v)$ (resp. γ_2 and $[b(w), 0]$ at $b(w)$) is $\frac{\pi}{k+1}$ and γ_1 (resp. γ_2) belongs locally to $\mathbb{H}_+$. With the same argument as in the first case, we get that γ_1 and γ_2 stay entirely in $\mathbb{H}_+$ (see Fig. 5).

Set $R_1 := [0, a(v)] \cup \gamma_1$ and $R_2 := [b(w), 0] \cup \gamma_2$. Under our construction, the open region $U \subset \mathbb{H}_+$ bounded by R_1 and R_2 is not disconnected by $f^{-1}(\mathbb{R}^+)$. So we have $U \cap f^{-1}(\mathbb{R}+) = \emptyset$ by the same reason as in the first case. Then the proof of the case of simple critical points is completed.

For the case of multiple critical points, if $c \in L$ is a critical point of f with multiplicity $k \geq 2$, we conclude that a set of geodesics from c with directions in a cone in H of angle $\frac{2\pi}{k+1}$ at c forms a region U with the properties in Theorem 1.2. The proof is very similar to the case of simple critical points, where the local part is somehow simpler and the global part is completely the same. $\square$

Remark 1. In fact, it is not difficult to find the bijective regions, and we have a lot of choices for the image of their interior by f. The interest of this theorem is that one of the bijective regions is in the half-plane H supported by $\mathcal{C}$.

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