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Functional analysis

Function spaces on quantum tori

Espaces de fonctions sur les tores quantiques

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A R T I C L E   I N F O

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A B S T R A C T

We study Sobolev, Besov and Triebel–Lizorkin spaces on quantum tori. These spaces share many properties with their classical counterparts. The results announced include: Besov and Sobolev embedding theorems; Littlewood–Paley-type characterizations of Besov and Triebel–Lizorkin spaces; an explicit description of the K-functional of $(L_p(\mathbb{T}^d_\theta),\ W^k_p(\mathbb{T}^d_\theta))$; descriptions of completely bounded Fourier multipliers on these spaces.

R É S U M É

On considère les espaces de Sobolev, Besov et Triebel–Lizorkin sur un tore quantique $\mathbb{T}^d_\theta$ de d générateurs. Les principaux résultats comprennent : le plongement de Besov et Sobolev ; des caractérisations à la Littlewood–Paley pour les espaces de Besov et Triebel–Lizorkin ; une formule explicite de la K-fonctionnelle de $(L_p(\mathbb{T}^d_\theta),\ W^k_p(\mathbb{T}^d_\theta))$; l'indépendance en θ des multiplicateurs de Fourier complètement bornés sur ces espaces.

Version française abrégée

L'objectif de cette note est d'annoncer les principaux résultats de [16], qui étudie des espaces de fonctions sur les tores quantiques. Soient $d \geq 2$ et $\theta = (\theta_{kj})$ une matrice carrée d'ordre d réelle et anti-symétrique. Le tore non commutatif de d générateurs est l'algèbre universelle $\mathcal{A}_\theta$ engendrée par d opérateurs unitaires $U_1, \dots, U_d$ vérifiant :

$$U_k U_j = e^{2\pi \mathrm{i}\theta_{kj}} U_j U_k, \quad 1 \leq j, k \leq d.$$

On pose $U^m = U_1^{m_1} \cdots U_d^{m_d}$ pour $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$. Un polynôme est une somme finie de la forme suivante :

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m, \quad \alpha_m \in \mathbb{C}.$$

La forme linéaire sur la famille des polynômes définie par $x \mapsto \tau(x) = \alpha_0$ s'étend alors à un état tracial fidèle sur $\mathcal{A}_\theta$. Soit $\mathbb{T}^d_\theta$ l'algèbre de von Neumann associée à la représentation GNS de τ. Elle est le tore quantique de d générateurs. Si $\theta = 0$,

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$\mathbb{T}_\theta^d = L_\infty(\mathbb{T}^d)$, où $\mathbb{T}^d$ est le d-tore usuel, muni de la mesure de Haar normalisée. Pour $1 \le p \le \infty$, $L_p(\mathbb{T}_\theta^d)$ désigne l'espace L_p non commutatif construit sur $(\mathbb{T}_\theta^d, \tau)$ (voir [9] pour les espaces L_p non commutatifs). La transformée de Fourier d'un élément $x \in L_1(\mathbb{T}_\theta^d)$ est définie par

$$\widehat{x}(m) = \tau\big((U^m)^* x\big), \quad m \in \mathbb{Z}^d.$$

Soit

$$\mathcal{S}(\mathbb{T}_\theta^d) = \Big\{ \sum_{m\in\mathbb{Z}^d} a_m U^m : \{a_m\}_{m\in\mathbb{Z}^d} \text{ rapidement décroissant}\Big\}.$$

C'est une sous-algèbre dense de $\mathcal{A}_\theta$. Comme dans le cas commutatif, $\mathcal{S}(\mathbb{T}_\theta^d)$ porte une topologie naturelle localement convexe. Son dual $\mathcal{S}'(\mathbb{T}_\theta^d)$ est alors l'espace des distributions sur $\mathbb{T}_\theta^d$. Les dérivations partielles sur $\mathcal{S}(\mathbb{T}_\theta^d)$ sont déterminées par

$$\partial_j(U_j) = 2\pi \mathrm{i}\, U_j \text{ et } \partial_j(U_k) = 0, \quad k \ne j,\ 1 \le j,k \le d.$$

Pour $m = (m_1, \dots, m_d) \in \mathbb{N}^d$, on pose $D^m = \partial_1^{m_1} \cdots \partial_d^{m_d}$ et $|m|_1 = m_1 + \cdots + m_d$. Comme d'habitude, les dérivations et la transformée de Fourier se définissent sur $\mathcal{S}'(\mathbb{T}_\theta^d)$ aussi.

Fixons une fonction φ de Schwartz sur $\mathbb{R}^d$ vérifiant la condition usuelle de Littlewood–Paley :

$$\operatorname{supp}\varphi \subset \{\xi : 2^{-1} \le |\xi| \le 2\} \text{ et } \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1,\ \xi \ne 0.$$

Pour $k \ge 0$, soit φ_k la fonction dont la transformée de Fourier est égale à $\varphi(2^{-k}\cdot)$. Définissons, pour toute distribution x sur $\mathbb{T}_\theta^d$,

$$\widetilde{\varphi}_k * x = \sum_{m\in\mathbb{Z}^d} \varphi(2^{-k}m)\widehat{x}(m) U^m .$$

Soient $1 \le p, q \le \infty$, $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, et $J^\alpha = (1 - (2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$, où $\Delta = \partial_1^2 + \cdots + \partial_d^2$. Les quatre familles d'espaces étudiés sont définies comme suit :

- *espaces de Sobolev* :

$$W_p^k(\mathbb{T}_\theta^d) = \big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : D^m x \in L_p(\mathbb{T}_\theta^d) \text{ pour tout } m \in \mathbb{N}_0^d \text{ avec } |m|_1 \le k \big\}.$$

- *espaces de Sobolev fractionnels* :

$$H_p^\alpha(\mathbb{T}_\theta^d) = \big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : J^\alpha x \in L_p(\mathbb{T}_\theta^d) \big\}.$$

- *espaces de Besov* :

$$B_{p,q}^\alpha(\mathbb{T}_\theta^d) = \Big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : \big(|\widehat{x}(0)|^q + \sum_{k\ge 0} 2^{qk\alpha} \|\widetilde{\varphi}_k * x\|_{L_p(\mathbb{T}_\theta^d)}^q \big)^{\frac{1}{q}} < \infty \Big\}.$$

- *espaces de Triebel–Lizorkin* $(p < \infty)$:

$$F_p^{\alpha,c}(\mathbb{T}_\theta^d) = \Big\{ x \in \mathcal{S}'(\mathbb{T}_\theta^d) : \big\| \big(|\widehat{x}(0)|^2 + \sum_{k\ge 0} 2^{2k\alpha} |\widetilde{\varphi}_k * x|^2 \big)^{\frac{1}{2}} \big\|_{L_p(\mathbb{T}_\theta^d)} < \infty \Big\}.$$

Munis de leur norme naturelle, ils sont tous des espaces de Banach.

Les résultats obtenus peuvent se classer en cinq familles : (i) propriétés fondamentales ; (ii) caractérisations ; (iii) inégalités de plongement ; (iv) interpolation ; (v) multiplicateurs de Fourier. Nos arguments se basent, de façon cruciale, sur des multiplicateurs de Fourier.

1. Introduction

The aim of this note is to announce the main results of [16], which is a continuation of our previous work [2] on harmonic analysis on quantum tori. This second part is devoted to the study of Sobolev, Besov and Triebel–Lizorkin spaces. These spaces have never been investigated so far in the quantum setting, except two special cases to our knowledge. Firstly, Sobolev spaces with the L_2-norm were studied by Spera [10] in view of applications to the Yang–Mills theory for quantum tori [11]. On the other hand, inspired by noncommutative metric spaces in noncommutative geometry, Weaver [14,15] developed the Lipschitz classes of order α for $0 < \alpha \le 1$ on quantum tori. This situation is in strong contrast with the geometry of quantum tori on which there exist a considerably long list of publications. Presumably, this deficiency is due to numerous difficulties one might encounter when dealing with noncommutative L_p-spaces, since these spaces come up unavoidably if one wishes to do analysis.

Among these difficulties, a specific one is to be emphasized here: it is the lack of a noncommutative analogue of the usual pointwise maximal function. However, maximal function techniques play a paramount role in the classical theory of Besov and Triebel–Lizorkin spaces. They are no longer available in the quantum setting, which forces us to invent new techniques.

One powerful tool used in [2] is the transference method. It consists in transferring problems on quantum tori to the corresponding ones in the case of operator-valued functions on the usual tori, in order to use existing results in the latter case or to adapt classical arguments. This method is efficient for several problems studied in [2]. It is still useful for some parts of the present work; for instance, Besov spaces can be investigated through the classical vector-valued Besov spaces by means of transference. However, it becomes inefficient for others. For example, the Sobolev or Besov embedding inequalities cannot be proved by transference. On the other hand, if one wishes to study Triebel–Lizorkin spaces on quantum tori via transference, one should first develop the theory of operator-valued Triebel–Lizorkin spaces on the classical tori. The latter is as hard as the former. Contrary to [2], the transference method plays a very limited role here. Instead, we use Fourier multipliers in a crucial way, this approach is of interest in its own right. We thus develop an intrinsic differential analysis on quantum tori, without frequently referring to the usual tori via transference as in [2]. This is a major advantage of the present methods over [2].

Given a function $\phi : \mathbb{Z}^d \to \mathbb{C}$, let M_ϕ denote the associated Fourier multiplier on $\mathbb{T}^d_\theta$, namely, $\widehat{M_\phi x}(m) = \phi(m)\widehat{x}(m)$ for any polynomial x. We call ϕ a multiplier on $L_p(\mathbb{T}^d_\theta)$ if M_ϕ extends to a bounded map on $L_p(\mathbb{T}^d_\theta)$. Similarly, if X is a Banach space of distributions on $\mathbb{T}^d_\theta$, we define bounded Fourier multipliers on X and denote by $\mathsf{M}(X)$ the space of all such multipliers, equipped with the natural norm. If X is further equipped with an operator space structure, $\mathsf{M}_{\mathrm{cb}}(X)$ is the space of all completely bounded Fourier multipliers on X. All spaces considered in this work are equipped with their natural operator space structure in Pisier's sense [8].

Most Fourier multipliers used here are the restrictions to $\mathbb{Z}^d$ of functions on $\mathbb{R}^d$. Given $\phi : \mathbb{R}^d \to \mathbb{C}$, the associated multiplier on $\mathbb{T}^d_\theta$ is denoted by $x \mapsto \widetilde{\phi} * x$.

2. The main results

We keep the notation introduced in the French version. The spaces defined previously share many properties with their classical counterparts. We classify our results into five families. We would like to emphasize that some of them improve the classical ones even in the commutative setting.

Basic properties. A common basic property of fractional Sobolev, Besov and Triebel–Lizorkin spaces is a reduction theorem by the Bessel potential. For example, J^β is an isomorphism from $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ onto $B^{\alpha-\beta}_{p,q}(\mathbb{T}^d_\theta)$ for all $1 \le p, q \le \infty$ and $\alpha, \beta \in \mathbb{R}$. Below is another type of reduction theorem.

Theorem 1. *Let $1 \le p, q \le \infty$, $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. Then $x \in F^{\alpha,c}_p(\mathbb{T}^d_\theta)$ (resp. $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$) iff all its partial derivatives of order k belong to $F^{\alpha-k,c}_p(\mathbb{T}^d_\theta)$ (resp. $B^{\alpha-k}_{p,q}(\mathbb{T}^d_\theta)$).*

Concerning Sobolev spaces, we obtain a Poincaré-type inequality:

Theorem 2. *Let $1 \le p \le \infty$. Then for any $x \in W^1_p(\mathbb{T}^d_\theta)$,*

$$\|x - \widehat{x}(0)\|_p \lesssim \sum_{j=1}^{d} \|\partial_j x\|_p .$$

Our proof of this inequality differs with standard arguments for such results in the commutative case.

We also show that $W^k_\infty(\mathbb{T}^d_\theta)$ is the analogue for $\mathbb{T}^d_\theta$ of the classical Lipschitz class of order k. For $u \in \mathbb{R}^d$, define $\Delta_u x = \pi_z(x) - x$, where $z = (\mathrm{e}^{2\pi \mathrm{i} u_1}, \cdots, \mathrm{e}^{2\pi \mathrm{i} u_d})$ and π_z is the automorphism of $\mathbb{T}^d_\theta$ determined by $U_j \mapsto z_j U_j$ for $1 \le j \le d$. Then for a positive integer k, Δ^k_u is the kth difference operator on $\mathbb{T}^d_\theta$ associated with u. The kth order modulus of L_p-smoothness of an $x \in L_p(\mathbb{T}^d_\theta)$ is defined by

$$\omega^k_p(x, \varepsilon) = \sup_{0<|u|\le\varepsilon} \left\| \Delta^k_u x \right\|_p .$$

Theorem 3. *Let $1 \le p \le \infty$ and $k \in \mathbb{N}$. Then for any $x \in W^k_p(\mathbb{T}^d_\theta)$*

$$\sup_{\varepsilon>0} \frac{\omega^k_p(x, \varepsilon)}{\varepsilon^k} \approx \sum_{m\in\mathbb{N}^d_0,\ |m|_1=k} \|D^m x\|_p .$$

In particular, we recover Weaver's results [14,15] on the Lipschitz class on $\mathbb{T}^d_\theta$ when $p=\infty$ and $k=1$.

Characterizations. The second family of results are various characterizations of Besov and Triebel–Lizorkin spaces. This is the most difficult and technical part of the work. In the classical case, all existing proofs of these characterizations that we know use maximal function techniques in a crucial way. As pointed out earlier, these techniques are unavailable now. Instead, we use frequently Fourier multipliers. We would like to emphasize that our results are better than those in the literature even in the commutative case. In the following, we state our characterizations only for Besov spaces. We start with a general characterization. Let h be a Schwartz function such that

$$\operatorname{supp} h \subset \{\xi \in \mathbb{R}^d : |\xi| \le 4\} \quad \text{and} \quad h = 1 \text{ on } \{\xi \in \mathbb{R}^d : |\xi| \le 2\},$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^d$. Given $\alpha \in \mathbb{R}$ let $I_\alpha(\xi) = |\xi|^\alpha$ for $\xi \in \mathbb{R}^d \setminus \{0\}$. Let $\alpha_0, \alpha_1 \in \mathbb{R}$, and ψ be an infinitely differentiable function on $\mathbb{R}^d \setminus \{0\}$ such that

$$|\psi| > 0 \text{ on } \{\xi : 2^{-1} \le |\xi| \le 2\},\ \ \mathcal{F}^{-1}(\psi h I_{-\alpha_1}) \in L_1(\mathbb{R}^d),\ \ \sup_{j\in\mathbb{N}_0} 2^{-\alpha_0 j}\left\|\mathcal{F}^{-1}(\psi(2^j\cdot)\varphi)\right\|_1 < \infty.$$

Let ψ_k be the inverse Fourier transform of $\psi(2^{-k}\cdot)$. It is to note that compared with [12, Theorem 2.5.1], we need not require $\alpha_1 > 0$ in the following theorem.

Theorem 4. *Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$. Assume $\alpha_0 < \alpha < \alpha_1$. Let ψ satisfy the above assumption. Then a distribution x on $\mathbb{T}^d_\theta$ belongs to $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ iff*

$$\Big(\sum_{k\ge 0}\big(2^{k\alpha}\|\widetilde{\psi}_k * x\|_p\big)^q\Big)^{\frac{1}{q}} < \infty.$$

Specifying ψ to the Poisson or heat kernel, we get two concrete characterizations. Let us state the corresponding one by the circular Poisson semigroup on $\mathbb{T}^d_\theta$. Given a distribution x on $\mathbb{T}^d_\theta$ and $k \in \mathbb{Z}$, let

$$\mathbb{P}_r(x) = \sum_{m\in\mathbb{Z}^d} \widehat{x}(m) r^{|m|} U^m$$

and

$$\mathcal{J}^k_r\,\mathbb{P}_r(x) = \sum_{m\in\mathbb{Z}^d} C_{m,k}\widehat{x}(m) r^{|m|-k} U^m, \quad 0 \le r < 1\,,$$

where

$$C_{m,k} = |m|\cdots(|m|-k+1) \ \text{ if } k \ge 0 \quad \text{and} \quad C_{m,k} = \frac{1}{(|m|+1)\cdots(|m|-k)} \ \text{ if } k < 0.$$

Note that $\mathcal{J}^k_r$ is the kth derivation operator relative to r if $k \ge 0$, and the $(-k)$th integration operator if $k < 0$.

Theorem 5. *Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$ with $k > \alpha$. Then for any distribution x on $\mathbb{T}^d_\theta$,*

$$\|x\|_{B^\alpha_{p,q}} \approx \Big(\max_{|m|<k} |\widehat{x}(m)|^q + \int_0^1 (1-r)^{(k-\alpha)q}\left\|\mathcal{J}^k_r\,\mathbb{P}_r(x_k)\right\|_p^q \frac{dr}{1-r}\Big)^{\frac{1}{q}},$$

where $x_k = x - \sum_{|m|<k} \widehat{x}(m) U^m$.

The use of integration operators (corresponding to negative k) in the above statement seems completely new even in the case $\theta = 0$. On the other hand, for Triebel–Lizorkin spaces, another improvement of the above result over the classical one lies on the assumption on k: in the classical case, k is required to be greater than $d + \max(\alpha, 0)$, while we only need to assume $k > \alpha$.

The classical characterization of Besov spaces by differences also extends to the quantum setting.

Theorem 6. *For $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ with $0 < \alpha < k$. Then $x \in B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ iff*

$$\|x\|_{B^{\alpha,\omega}_{p,q}} = \Big(\int_0^1 \varepsilon^{-\alpha q}\omega^k_p(x,\varepsilon)^q \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{q}} < \infty.$$

This implies that $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d_\theta)$ is a quantum Zygmund class. In particular, for $0<\alpha<1$, $B^{\alpha}_{\infty,\infty}(\mathbb{T}^d_\theta)$ is the Hölder class of order α, already studied by Weaver [15].

In the commutative case, the limit behavior of $\|x\|_{B^{\alpha,\omega}_{p,q}}$ as $\alpha\to k$ or $\alpha\to 0$ is the subject of several recent publications (see [1] and [6]). Here, we obtain the following analogue for $\mathbb{T}^d_\theta$: For $1\le p\le\infty$, $1\le q<\infty$ and $0<\alpha<k$ with $k\in\mathbb{N}$,

$$\lim_{\alpha\to k}(k-\alpha)^{\frac1q}\|x\|_{B^{\alpha,\omega}_{p,q}}\approx q^{-\frac1q}\sum_{m\in\mathbb{N}^d_0,\ |m|_1=k}\|D^m x\|_p,$$

$$\lim_{\alpha\to 0}\alpha^{\frac1q}\|x\|_{B^{\alpha,\omega}_{p,q}}\approx q^{-\frac1q}\|x\|_p.$$

Embedding. The third family of results deal with embedding. Below is the embedding theorem of Besov spaces:

Theorem 7. *Assume that $1\le p<p_1\le\infty$, $1\le q\le q_1\le\infty$ and $\alpha,\alpha_1\in\mathbb{R}$ such that $\alpha-\frac{d}{p}=\alpha_1-\frac{d}{p_1}$. Then $B^{\alpha}_{p,q}(\mathbb{T}^d_\theta)\subset B^{\alpha_1}_{p_1,q_1}(\mathbb{T}^d_\theta)$.*

Using interpolation, we then deduce the following.

Theorem 8. *Let $\alpha,\alpha_1\in\mathbb{R}$ with $\alpha>\alpha_1$.*

(i) *If $1<p<p_1<\infty$ are such that $\alpha-\frac{d}{p}=\alpha_1-\frac{d}{p_1}$, then $H^{\alpha}_p(\mathbb{T}^d_\theta)\subset H^{\alpha_1}_{p_1}(\mathbb{T}^d_\theta)$.*
(ii) *If $1\le p<\infty$ is such that $p(\alpha-\alpha_1)>d$ and $\alpha_1=\alpha-\frac{d}{p}$, then $H^{\alpha}_p(\mathbb{T}^d_\theta)\subset B^{\alpha_1}_{\infty,\infty}(\mathbb{T}^d_\theta)$.*

As a corollary, we get

Corollary 9. *If $1<p<q<\infty$ such that $\frac1q=\frac1p-\frac{k}{d}$, then $W^k_p(\mathbb{T}^d_\theta)\subset L_q(\mathbb{T}^d_\theta)$.*

Our proofs of these embedding inequalities are based on Varopolous' semigroup approach [13] to the Littlewood–Sobolev theory, which has been transferred to the noncommutative setting by Junge and Mei [5]. Thus the characterization of Besov spaces by Poisson semigroup is essential in our argument.

Interpolation. The fourth family of results are related to interpolation. Like in the classical case, the interpolation of Besov spaces is quite simple, and that of Triebel–Lizorkin spaces can be easily reduced to the corresponding problem of Hardy spaces. Thus the really hard task here concerns the interpolation of Sobolev spaces for which we have obtained only partial results. The most interesting couple is $\big(W^k_1(\mathbb{T}^d_\theta),\ W^k_\infty(\mathbb{T}^d_\theta)\big)$. Recall that the complex interpolation of this couple remains always unsolved even in the commutative case, while its real interpolation spaces were completely determined by DeVore and Scherer [3]. We do not know, unfortunately, how to prove the quantum analogue of DeVore and Scherer's theorem.

However, we are able to extend to the present setting the K-functional formula of the couple $\big(L_p(\mathbb{R}^d),\ W^k_p(\mathbb{R}^d)\big)$ obtained by Johnen and Scherer [4].

Theorem 10. *Let $1\le p\le\infty$ and $k\in\mathbb{N}$. Then*

$$K(x,\varepsilon^k;\ L_p(\mathbb{T}^d_\theta),W^k_p(\mathbb{T}^d_\theta))\approx\varepsilon^k|\widehat{x}(0)|+\omega^k_p(x,\varepsilon),\quad 0<\varepsilon\le 1,\ x\in L_p(\mathbb{T}^d_\theta).$$

The real interpolation of $\big(L_p(\mathbb{R}^d),\ W^k_p(\mathbb{R}^d)\big)$ is closely related to the limit behavior of Besov norms described previously. We show that it implies the optimal order (relative to α) of the best constant in the embedding of $B^{\alpha}_{p,p}(\mathbb{T}^d_\theta)$ into $L_q(\mathbb{T}^d_\theta)$ for $\frac1q=\frac1p-\frac{\alpha}{d}$ and $0<\alpha<1$, which is the quantum analogue of a result of Bourgain, Brézis, and Mironescu [1]. On the other hand, the latter result is equivalent to the Sobolev embedding $W^1_p(\mathbb{T}^d_\theta)\subset L_q(\mathbb{T}^d_\theta)$ for $\frac1q=\frac1p-\frac1d$.

Multipliers. The last family of results describe Fourier multipliers on the preceding spaces. We are mainly concerned with completely bounded multipliers. The following is the Sobolev analogue of the L_p result of [2]. The main tool is the transference theorem of [7].

Theorem 11. *Let $1\le p\le\infty$ and $k\in\mathbb{N}$. Then*

$$\mathsf{M}_{\rm cb}(W^k_p(\mathbb{T}^d_\theta))=\mathsf{M}_{\rm cb}(W^k_p(\mathbb{T}^d))\ \ \textit{with equal norms}.$$

Similar statements hold for the other spaces too. Moreover, the situation for Besov spaces is very satisfactory since it is well known that Fourier multipliers behave much better on Besov spaces than on L_p-spaces (in the commutative case).

Theorem 12. *Let $\alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$. Let $\phi : \mathbb{Z}^d \to \mathbb{C}$. Then ϕ is a Fourier multiplier on $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ iff all $\phi\varphi(2^k\cdot)$'s are Fourier multipliers on $L_p(\mathbb{T}^d_\theta)$ uniformly in $k \ge 0$. A similar completely bounded version holds too.*

Consequently, the completely bounded multipliers on $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ depend solely on p. In the case of $p = 1$, a multiplier is bounded on $B^\alpha_{1,q}(\mathbb{T}^d_\theta)$ iff it is completely bounded iff it is the Fourier transform of an element of $B^0_{1,\infty}(\mathbb{T}^d)$. We also show that there exists ϕ that is a completely bounded Fourier multiplier on $B^\alpha_{p,q}(\mathbb{T}^d_\theta)$ for all p, but is unbounded on $L_p(\mathbb{T}^d_\theta)$ for any $p \ne 2$.

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