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Global continuity of solutions to quasilinear equations with Morrey data

La continuité globale de solutions d'équations quasi-linéaires avec des données de Morrey

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A R T I C L E I N F O

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A B S T R A C T

We announce some recent results on boundedness and Hölder continuity up to the boundary for the weak solutions to coercive quasilinear equations with data belonging to Morrey spaces.

R É S U M É

Nous annonçons quelques résultats récents sur la régularité höldérienne globale pour les solutions faibles d'équations coercitives quasi linéaires avec des données appartenant à des espaces de Morrey.

1. Introduction

The note deals with global boundedness and Hölder continuity for the weak solutions to the Dirichlet problem for quasilinear operators whose prototype is the *m-Laplacean.* Precisely, we consider the problem

$$\begin{cases} u \in W_0^{1,m}(\Omega) \\ \operatorname{div}\big(\mathbf{a}(x,u,Du)\big) = b(x,u,Du) \quad \text{weakly in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, $m \in (1,n]$ and $\mathbf{a}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory maps, the x-behavior of which is controlled in terms of Morrey functional scales. Setting m^* for the Sobolev conjugate of m, we assume *controlled growths* of $\mathbf{a}$ and b with respect to u and $|Du|$. That is, there exist a constant $\Lambda > 0$ and non-negative measurable functions φ and ψ such that

$$|\mathbf{a}(x,z,\xi)| \leq \Lambda \left(\varphi(x) + |z|^{\frac{m^*(m-1)}{m}} + |\xi|^{m-1} \right),$$

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$$|b(x,z,\xi)| \le \Lambda \left(\psi(x) + |z|^{m^*-1} + |\xi|^{\frac{m(m^*-1)}{m^*}} \right) \tag{2}$$

for almost all (a.a.) $x \in \Omega$ and all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Let us note that (2), together with $\varphi \in L^{\frac{m}{m-1}}(\Omega)$ and $\psi \in L^{\frac{nm}{nm+m-n}}(\Omega)$, are the *minimal* hypotheses giving sense to the concept of $W_0^{1,m}(\Omega)$-weak solution to (1). However, these integrability requirements on φ and ψ are far from being sufficient to ensure the continuity and even the boundedness of the solution. That is why φ and ψ will be taken hereafter as belonging to suitable *Morrey spaces* (cf. [12] for their definitions and basic embedding properties). Namely,

$$\begin{aligned} &\varphi \in L^{p,\lambda}(\Omega) \ \text{ with } p > \frac{m}{m-1}, \qquad \lambda \in (0,n) \text{ and } (m-1)p + \lambda > n \\ &\psi \in L^{q,\mu}(\Omega) \ \text{ with } q > \frac{mn}{mn+m-n}, \quad \mu \in (0,n) \text{ and } mq + \mu > n. \end{aligned} \tag{3}$$

We will assume also the *coercivity* of the differential operator considered,

$$\mathbf{a}(x,z,\xi) \cdot \xi \ge \gamma |\xi|^m - \Lambda |z|^{m^*} - \Lambda \varphi(x)^{\frac{m}{m-1}} \tag{4}$$

for a.a. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, with a constant $\gamma > 0$.

As for the generally non-smooth boundary of Ω, we suppose that it satisfies a *density condition* given in terms of variational m-capacity, which requires the complement $\mathbb{R}^n \setminus \Omega$ to be *uniformly m-thick: there exist positive constants A_Ω and r_0 such that*

$$\mathrm{Cap}_m\left((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(x), B_{2r}(x) \right) \ge A_\Omega \, \mathrm{Cap}_m\left(\overline{B}_r(x), B_{2r}(x) \right) \tag{5}$$

for all $x \in \mathbb{R}^n \setminus \Omega$ and all $r \in (0, r_0)$. Here $B_r(x)$ stands for the ball of radius r and centered at x, and the right-hand side behaves as r^{n-m}. Replacing the capacity above with the Lebesgue measure, (5) reduces to the well-known *measure density condition* that holds, for instance, when Ω supports the uniform *exterior cone* property. Thus, each domain with at least Lipschitz continuous boundary has an m-thick complement, while the vice versa is not true in general. Moreover, if a given set satisfies the measure density condition, then it is uniformly P-thick for each $P > 1$, whereas each nonempty set is uniformly P-thick if $P > n$. Further on, a uniformly Q-thick set is also uniformly P-thick for all $P > Q$ and, as proved in [8], the uniformly P-thick sets have a deep self-improving property to be uniformly Q-thick for some $Q < P$, depending on n, P and the constant of the P-thickness. Yet another example of P-thick set for all $P > 1$ is given by those satisfying the uniform *corkscrew* condition: a set E is uniformly corkscrew if there exist constants $C > 0$ and $r_0 > 0$ such that for any $x \in E$ and any $r \in (0, r_0)$ there is a point $y \in B_r(x) \setminus E$ with the property that $B_{r/C}(y) \subset \mathbb{R}^n \setminus E$.

The question of Hölder continuity of the weak solutions to (1) has been a long-standing problem in the PDEs theory, related to the Hilbert 19th Problem. It has been brilliantly solved by De Giorgi in [4] for $W_0^{1,2}$-weak solutions to *linear* equations over Lipschitz continuous domains when $m = 2$, $\varphi \in L^p$, with $p > n$ and $\psi \in L^q$ with $2q > n$, and this provided the initial breakthrough in the modern theory of nonlinear equations in more than two independent variables. The De Giorgi result was extended to *general linear* operators within L^p-framework by Stampacchia in [14], and in the *non-L^p* settings (i.e., when a sort of (3) holds) by Morrey in [11] and Lewy and Stampacchia in [9] to equations with measures at the right-hand side, assuming $\varphi \in L^{2,\lambda}$, $\psi \in L^{1,\mu}$ with $\lambda, \mu > n-2$. Moving to the quasilinear equation (1), we dispose of the seminal L^p-result of Serrin [13], which provides *interior* boundedness and Hölder continuity of the $W_0^{1,m}$-weak solutions to (1) in the *sub-controlled* case when the nonlinearities grow as $|u|^{m-1} + |Du|^{m-1}$, and the behavior with respect to x of $\mathbf{a}(x,u,Du)$ and $b(x,u,Du)$ is controlled in terms of φ and ψ, respectively, that satisfy

$$\begin{aligned} &\varphi \in L^p(\Omega) \ \text{ with } p > \frac{m}{m-1}, \qquad (m-1)\,p > n \\ &\psi \in L^q(\Omega) \ \text{ with } q > \frac{mn}{mn+m-n}, \quad mq > n. \end{aligned} \tag{6}$$

Global boundedness of the $W_0^{1,m}$-weak solutions to (1) with general nonlinearities of *controlled* growths has been obtained by Ladyzhenskaya and Ural'tseva (cf. [7]) under the hypotheses (6) and for domains with exterior cone property. In the case $m = 2$, the paper [2] extends their result to the problem (1) with Morrey data satisfying (3). Assuming *natural growths* of the data (that is, $\mathbf{a}(x,u,Du) = \mathcal{O}(\varphi(x) + |Du|^{m-1})$ and $b(x,u,Du) = \mathcal{O}(\psi(x) + |Du|^m)$) and (6), Ladyzhenskaya and Ural'tseva proved also Hölder continuity up to the boundary for the *bounded* weak solutions of (1), and Gariepy and Ziemer extended in [5] that result in domains with *m-thick complements*. It was Trudinger [15] the first to get *global* Hölder continuity of the *bounded* solutions in the *non-L^p* settings under the *natural structure hypotheses* of Ladyzhenskaya and Ural'tseva with $\varphi \in L^{n/(m-1),\varepsilon}$, $\psi \in L^{n/m,\varepsilon}$ for a small $\varepsilon > 0$, while Lieberman derived in [10] a very general result on *interior* Hölder continuity when φ and ψ are suitable measures.

We announce here our recent results from [3] regarding *global boundedness* (Theorem 2.1) and *Hölder continuity up to the boundary* (Theorem 3.1) for *each* $W_0^{1,m}(\Omega)$-weak solution to the *coercive* Dirichlet problem (1) over domains with *m-thick* complements, assuming *controlled growths* of the nonlinearities and Morrey data φ and ψ satisfying (3). Apart from the more general class of domains considered, we extend this way the classical L^p*-results* of Ladyzhenskaya and Ural'tseva to the *non-*L^p*-settings* by weakening the hypotheses on φ and ψ to the scales of *Morrey type*. Comparing (3) with (6), it is clear that the decrease of the degrees p and q of Lebesgue integrability of the data φ and ψ is at the expense of increase of the Morrey exponents λ and μ, and the range of these variations is always controlled by the relations $(m-1)p+\lambda>n$ and $mq+\mu>n$. Indeed, in the particular case $\lambda=\mu=0$ and domains with exterior cone property, our results reduce to these of Ladyzhenskaya and Ural'tseva [7]. However, our Theorems 2.1 and 3.1 generalize substantially the results of [7] because, even if $(m-1)p\le n$ and $mq\le n$, there exist functions $\varphi\in L^{p,\lambda}$ with $(m-1)p+\lambda>n$ and $\psi\in L^{q,\mu}$ with $mq+\mu>n$ for which (3) hold, but $\varphi\notin L^{p'}$ $\forall p'>n/(m-1)$ and $\psi\notin L^{q'}$ $\forall q'>n/m$ and therefore (6) fail. Moreover, the *controlled* growths and the restrictions (3) on the Lebesgue–Morrey exponents turn out to be *optimal* for the global boundedness and the subsequent Hölder continuity of the weak solutions to (1).

We refer the reader to [3] for the full proofs of the results here announced.

2. Global boundedness

Our first result claims essential boundedness of the weak solutions to (1).

Theorem 2.1. *Under the assumptions (2)–(5), each $W_0^{1,m}(\Omega)$-weak solution to the Dirichlet problem (1) is globally essentially bounded. That is, there exists a constant M, depending on known quantities,[1] on $\|Du\|_{L^m(\Omega)}$ and on the uniform integrability of $|Du|^m$, such that*

$$\|u\|_{L^\infty(\Omega)}\le M. \tag{7}$$

A crucial starting point of proving Theorem 2.1 is ensured by the next Gehring–Giaquinta–Modica type result, which asserts *better integrability* for the gradient of the weak solution over domains with m-thick complements.

Lemma 2.2. *(See [3, Lemma 3.8].) Assume (2), (4), (5), and let $u\in W_0^{1,m}(\Omega)$ be a weak solution to (1). Then there exist exponents $m_0>m$ and $m_1>m^*$ such that $u\in W^{1,m_0}(\Omega)\cap L^{m_1}(\Omega)$ with $\|Du\|_{L^{m_0}(\Omega)}+\|u\|_{L^{m_1}}(\Omega)$ bounded in terms of known quantities, of $\|Du\|_{L^m(\Omega)}$ and of the uniform integrability of $|Du|^m$ in Ω.*

With Lemma 2.2 at hand, our strategy relies on the De Giorgi approach to the boundedness as adapted by Ladyzhenskaya and Ural'tseva (cf. [7, Chapter IV]) to quasilinear equations, and consists of obtaining exact decay estimates for the total mass of the weak solution taken over its level sets. However, unlike the L^p-approach in [7], the mass we have to do with is taken with respect to the Radon measure

$$\mathrm{d}\mathcal{M}:=\left(\chi(x)+\varphi(x)^{\frac{m}{m-1}}+\psi(x)+|u(x)|^{\frac{m^2}{n-m}}\right)\mathrm{d}x,$$

where $\chi(x)$ is the characteristic function of Ω. (If $m=n$, then m is to be taken $n-\frac{n^2}{m^*(n+1)}$ in the above formula.) Thanks to (3) and Lemma 2.2, we have

$$\mathcal{M}(B_r)\le Kr^{n-m+\varepsilon},\quad \varepsilon>0$$

with absolute constant K; this allows us to employ very fine inequalities of trace type due to D.R. Adams [1] in order to estimate the $\mathcal{M}$-mass of u in terms of the m-energy of u. Precisely, setting $u_k(x):=\max\{u(x)-k,0\}$ and $\Omega_k:=\{x\in\Omega\colon\ u(x)>k\}$ for arbitrary $k\ge 1$, and using u_k as test function, we get

$$\int_\Omega u_k(x)\,\mathrm{d}\mathcal{M}\le C(\mathcal{M}(\Omega_k))^{1-\frac{n-m}{m(n-m+\varepsilon)}}\left(\int_{\Omega_k}|Du_k(x)|^m\,\mathrm{d}x\right)^{1/m} \tag{8}$$

as a consequence of [1]. On the other hand, rather technical applications of (2), (3), (4) and Lemma 2.2 lead to

$$\int_{\Omega_k}|Du_k(x)|^m\,\mathrm{d}x\le Ck^m\mathcal{M}(\Omega_k)\qquad\forall\,k\ge k_0$$

with large enough k_0, and this rewrites (8) into

[1] The omnibus term "*known quantities*" means the data in hypotheses (2)–(5) that include n, m, m^*, γ, Λ, p, λ, q, μ, $\|\varphi\|_{L^{p,\lambda}(\Omega)}$, $\|\psi\|_{L^{q,\mu}(\Omega)}$, $\operatorname{diam}\Omega$, A_Ω and r_0.

$$\int_{\Omega_k} u_k(x)\,\mathrm{d}\mathcal{M} \le Ck\big(\mathcal{M}(\Omega_k)\big)^{1+\frac{\varepsilon}{m(n-m+\varepsilon)}} \qquad \forall\, k \ge k_0.$$

This way, applying the Cavalieri principle to the left-hand side, we get

$$\int_{\Omega_k} u_k(x)\,\mathrm{d}\mathcal{M} = \int_{\Omega_k} (u(x)-k)\,\mathrm{d}\mathcal{M} = \int_k^\infty \mathcal{M}(\Omega_t)\,\mathrm{d}t \le Ck\big(\mathcal{M}(\Omega_k)\big)^{1+\delta} \quad \forall k \ge k_0,\ \delta > 0.$$

At this point, the *Hartman–Stampacchia maximum principle* (see [6]) ensures the existence of a number $k_{\max}$, depending on known quantities and on $\|Du\|_{L^m(\Omega)}$, such that $\mathcal{M}(\Omega_k) = 0$ for all $k \ge k_{\max}$, which means

$$u(x) \le k_{\max} \qquad \text{a.e. } \Omega.$$

The same procedure, applied to $-u(x)$, leads to a bound from below for $u(x)$ and this gives (7).

3. Global Hölder continuity

Once having the result of Theorem 2.1, it follows under the same hypotheses that the weak solutions of (1) are Hölder continuous functions up to the boundary of Ω. Moreover, this holds true for the *bounded* weak solutions if the controlled growths (2) are relaxed to the *natural structure* conditions of Ladyzhenskaya and Ural'tseva. Thus, instead of (2) and (4), we will assume that there exist a non-decreasing function $\Lambda(t)$ and a non-increasing function $\gamma(t)$, both positive and continuous, such that

$$\begin{aligned}
&|\mathbf{a}(x,z,\xi)| \le \Lambda(|z|)\left(\varphi(x) + |\xi|^{m-1}\right),\\
&|b(x,z,\xi)| \le \Lambda(|z|)\left(\psi(x) + |\xi|^{m}\right),\\
&\mathbf{a}(x,z,\xi)\cdot\xi \ge \gamma(|z|)|\xi|^m - \Lambda(|z|)\varphi(x)^{\frac{m}{m-1}}
\end{aligned} \tag{9}$$

for almost all $x \in \Omega$ and all $(z,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

Theorem 3.1. *Assume (9), (3) and (5). Then each* bounded *weak solution to the Dirichlet problem (1) is Hölder continuous in $\overline{\Omega}$ with exponent $\alpha \in (0,1)$ depending on the same quantities as M in (7).*

The *interior Hölder continuity* is a direct consequence of the hypotheses (9) and the fine results obtained by Lieberman in [10]. To get the claim of Theorem 3.1 up to $\partial\Omega$, we adopt to the Morrey framework the approach of Gariepy and Ziemer from [5], which relies on the Moser iteration technique in obtaining growth estimates for the gradient of the solution. A crucial step here is ensured by the following lemma.

Lemma 3.2. *(See [3, Lemma 5.1].) Assume (9), (3) and let $u \in L^\infty(\Omega)\cap W_0^{1,m}(\Omega)$ be a weak solution to the problem (1) extended at zero outside Ω.*

Let B_ρ be a ball of radius $\rho \in (0, \operatorname{diam}\Omega)$ and centered at a point of $\partial\Omega$, and $\eta \in C_0^\infty(B_{\rho/2})$ with $|D\eta| \le c/\rho$. Define $M(\rho) = \operatorname{ess\,sup}_{B_\rho} u^+$, $u^+ = \max\{u,0\}$, $A(\rho) = \rho + \|\varphi\|_{L^{p,\lambda}(B_\rho)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\mu}(B_\rho)}^{\frac{1}{m}}$ and $w^{-1} = M(\rho) + A(\rho) - u^+$.

There exists a constant C depending on the same quantities as M in (7), such that

$$\int_{B_{\rho/2}} |D(\eta w^{-1})|^m\,\mathrm{d}x \le C\big(M(\rho)+A(\rho)\big)\left(M(\rho) - M\left(\frac{\rho}{2}\right) + A(\rho)\right)^{m-1}\rho^{n-m}.$$

With Lemma 3.2 at hand, let $x_0 \in \partial\Omega$ be arbitrary and set B_ρ for the ball of radius ρ and centered at x_0. Since $(m-1)p+\lambda > n$ and $mq+\mu > n$, there exist positive constants λ' and μ' such that $n < (m-1)\,p + \lambda' < (m-1)\,p + \lambda$, $n < mq + \mu' < mq + \mu$. It follows from [12] that $L^{p,\lambda}(B_\rho) \subset L^{p,\lambda'}(B_\rho)$, $L^{q,\mu}(B_\rho) \subset L^{q,\mu'}(B_\rho)$ and it is not hard to see that

$$\|\varphi\|^p_{L^{p,\lambda'}(B_\rho)} \le \|\varphi\|^p_{L^{p,\lambda}(B_\rho)}\rho^{\lambda-\lambda'}, \qquad \|\psi\|^q_{L^{q,\mu'}(B_\rho)} \le \|\psi\|^q_{L^{q,\mu}(B_\rho)}\rho^{\mu-\mu'}. \tag{10}$$

Take now a cutoff function $\eta \in C_0^\infty\left(B_{\rho/2}\right)$ so that $0 \le \eta \le 1$, $\eta = 1$ on $B_{\rho/4}$, and $|D\eta| \le c/\rho$. The m-thickness condition (5) ensures that $\mathbb{R}^n \setminus \Omega$ is also m-thick and, making use of $B_{\rho/4}\setminus\Omega \subset \{x \in B_{\rho/4}\colon\ u^+(x) = 0\}$, we have

$$\begin{aligned}
C\rho^{n-m} =&\ A_\Omega \operatorname{Cap}_m\left(B_{\rho/4}, B_{\rho/2}\right) \le \operatorname{Cap}_m\left(B_{\rho/4}\setminus\Omega, B_{\rho/2}\right)\\
\le&\ \operatorname{Cap}_m\Big(\left\{x \in B_{\rho/4}\colon\ u^+(x) = 0\right\}, B_{\rho/2}\Big)
\end{aligned}$$

for all $\rho \le r_0$. Further on, $\eta w^{-1} = M(\rho) + A'(\rho)$ on the set $\{x \in B_{\rho/4}\colon\ u^+(x) = 0\}$ where $A'(\rho) = \rho + \|\varphi\|_{L^{p,\lambda'}(B_\rho)}^{\frac{1}{m-1}} + \|\psi\|_{L^{q,\mu'}(B_\rho)}^{\frac{1}{m}}$, and then

$$\mathrm{Cap}_m\Big(\big\{x \in B_{\rho/4}\colon\ u^+(x) = 0\big\}, B_{\rho/2}\Big) \le \int_{B_{\rho/2}} \left| D\left(\frac{\eta w^{-1}}{M(\rho) + A'(\rho)}\right)\right|^m \mathrm{d}x.$$

Putting together all these inequalities, Lemma 3.2 gives:

$$\begin{aligned}\rho^{n-m} &\le C\big((M(\rho) + A'(\rho)\big)^{-m} \int_{B_{\rho/2}} |D(\eta w^{-1})|^m \\ &\le C\big(M(\rho) + A'(\rho)\big)^{1-m}\Big(M(\rho) - M\left(\frac{\rho}{2}\right) + A'(\rho)\Big)^{m-1}\rho^{n-m}\end{aligned}$$

whence

$$M\left(\frac{\rho}{2}\right) \le \frac{C-1}{C}\big(M(\rho) + A'(\rho)\big)$$

for all $\rho \le R$ with R depending on r_0 from (5). At this point, a known interpolation inequality applies (cf. [7, Chapter II]) which, together with (10), leads to

$$M(\rho) \le C\rho^\alpha \qquad \forall \rho \in (0, R)$$

with an exponent $\alpha \in (0, 1)$. Repeating the same procedure with $-u(x)$ instead of $u(x)$ gives finally

$$\sup_{B_\rho(x_0)} |u| \le C\rho^\alpha \qquad \forall x_0 \in \partial\Omega,\ \forall \rho \in (0, R). \tag{11}$$

Once having (11), it is a standard matter to combine it with the results of Lieberman from [10] in order to get the Hölder continuity up to the boundary as claimed in Theorem 3.1.

Remark 1. Lemma 3.2 remains valid for the $W_0^{1,m}(\Omega)$-weak solutions of (1) if one requires the *controlled growths* (2) and (4) instead of the *natural* ones (9), since the essential boundedness is *a priori* guaranteed by Theorem 2.1. As a consequence, the hypotheses (2), (3), (4) and (5) are sufficient to ensure global *Hölder continuity* for the $W_0^{1,m}(\Omega)$-weak solutions of the problem (1).

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