Differential geometry

Stability of holomorphically parallelizable manifolds

Stabilité des variétés holomorphiquement parallélisables

Daniele Angella [a,1,2], Adriano Tomassini [b,3]

[a] *Istituto Nazionale di Alta Matematica, Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy*
[b] *Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy*

A R T I C L E   I N F O

A B S T R A C T

We prove a stability theorem for families of holomorphically parallelizable manifolds in the category of Hermitian manifolds.

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R É S U M É

Nous montrons un théorème de stabilité pour les familles de variétés holomorphiquement parallélisables, dans la catégorie des variétés hermitiennes.

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1. Introduction

By a classical theorem by K. Kodaira and D.C. Spencer, [13, Theorem 15], small deformations of compact complex manifolds admitting a Kähler metric still admit Kähler metrics. This is a consequence of the harmonicity property of Kähler metrics and of Hodge theory on compact Kähler manifolds. On the other side, H. Hironaka provided in [12] an example of a complex-analytic family of compact complex manifolds being Kähler except for the central fibre, which is only Moǐshezon. In other words, Kählerness is not a closed property under deformations. It is expected that limits of projective manifolds are Moǐshezon, and limits of Kähler manifolds are in class $\mathcal{C}$ of Fujiki, see [9,16]. Note that $\partial\bar{\partial}$-Lemma is an invariant property under images of holomorphic birational maps, [8, Theorem 5.22]. In particular, compact complex manifolds being Moǐshezon or belonging to class $\mathcal{C}$ of Fujiki satisfy the $\partial\bar{\partial}$-Lemma, [8, Corollary 5.23]. Hence, in view of the above conjectures, in [3,5], the behaviour of the $\partial\bar{\partial}$-Lemma property under deformation is investigated. In particular, [5, Corollary 2.7] provides another argument for proving the stability of $\partial\bar{\partial}$-Lemma under small deformations; see the references therein for different proofs, while an example showing that $\partial\bar{\partial}$-Lemma is not stable under limits is provided in [3, §4, Corollary 6.1]. Note in fact that

the structures on the holomorphically parallelizable Nakamura manifold studied in [3, §4] are not in class $\mathcal{C}$ of Fujiki. In any case, nilmanifolds and solvmanifolds (that is, compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups) provide a possibly useful class of examples for investigating the above questions. In fact, Kählerness for nilmanifolds is characterized by $\partial\bar\partial$-Lemma, see [11, Theorem 1, Corollary], which is in turn characterized in terms of Bott–Chern cohomology, [5, Theorem B]. On the other hand, several results concerning the computation of Bott–Chern cohomology for nilmanifolds and solvmanifolds are known, see, e.g., [4,2] and the references therein. When restricting to the class of nilmanifolds, Kählerness is a closed properties under deformations. This follows by a theorem by A. Andreotti, H. Grauert, and W. Stoll in [1]. More precisely, they proved a stability result for complex-analytic families of complex tori.

In this short note, we prove a result similar to that by A. Andreotti, H. Grauert, and W. Stoll for the class of holomorphically parallelizable manifolds, in the category of compact Hermitian manifolds. As pointed out by the Referee, it remains an open question whether the result may be stated in the category of compact complex manifolds.

2. Main results

A compact complex manifold is called *holomorphically parallelizable* if its holomorphic tangent bundle is holomorphically trivial, see [19, page 771]. A structure theorem for holomorphically parallelizable manifolds was proven by H.-C. Wang. More precisely, holomorphically parallelizable manifolds have a complex Lie group as universal covering.

Theorem 2.1. *(See [19, Theorem 1].) Let X be a holomorphically parallelizable manifold. Then X is (biholomorphic to) a quotient G/D where G is a connected simply-connected complex Lie group and D is a discrete subgroup.*

Holomorphically parallelizable manifolds having a complex solvable Lie group as universal covering were studied by I. Nakamura in [14]. He initiated a classification of holomorphically parallelizable solvmanifolds up to complex dimension 5 in [14, §6], then completed by D. Guan in [10]. Moreover, by explicitly constructing the Kuranishi family of deformations of some holomorphically parallelizable solvmanifolds of complex dimension 3, in [14, §3], it was proved that being holomorphically parallelizable is not a stable property under small deformations of the complex structure, [14, page 86]. A detailed study of holomorphically parallelizable nilmanifolds, and of their Kuranishi space and stability was done by S. Rollenske in [17].

The structure theorem by H.-C. Wang allows us to generalize and simplify a stability result by A. Andreotti, H. Grauert, and W. Stoll, [1, Theorem 8]. In the proof below, the classical and well-known Montel theorem (also called generalized Vitali theorem) is used.

Theorem 2.2 (Montel theorem). *(See, e.g., [15, Proposition 6].) Let $\mathcal{F} = \{f\}$ be a family of holomorphic functions on an open set $\Omega \subseteq \mathbb{C}^n$ such that, for any compact set $K \subseteq \Omega$, there exists a positive constant M_K such that, for any $z \in K$, for any $f \in \mathcal{F}$, it holds $|f(z)| < M_K$. Then any sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence that converges uniformly on compact subsets of Ω.*

We can now state and prove the main result of this note.

Theorem 2.3. *Let $\{(X_t, g_t)\}_{t \in (-\varepsilon, 1)}$ be a smooth family of compact Hermitian manifolds, with $\varepsilon > 0$ small enough. Suppose that X_t is holomorphically parallelizable for any $t \in (0, 1)$, with a pointwise g_t-orthonormal co-frame $\{\varphi^j(t)\}_j$ of holomorphic 1-forms. Then X_0 is holomorphically parallelizable.*

Proof. First of all, by the Ehresmann theorem, for any $t \in (-\varepsilon, 1)$, we see $X_t = (X, J_t)$ where $\{J_t\}_{t \in (-\varepsilon, 1)}$ is a family of complex structures on the differentiable manifold X varying smoothly in t.

By definition, the holomorphic tangent bundle $T^{1,0}X_t$ of X_t is holomorphically-trivial for any $t \in (0, 1)$. Equivalently, the holomorphic co-tangent bundle $(T^{1,0}X_t)^*$ of X_t is holomorphically-trivial for any $t \in (0, 1)$. Hence, by the assumptions, we choose $\{\varphi^1(t), \ldots, \varphi^n(t)\}$ global pointwise g_t-orthonormal co-frame of holomorphic 1-forms on X_t depending smoothly on t, where n denotes the complex dimension of X_t.

Denote by $(\cdot, \cdot\cdot)_t$ the induced L^2-Hermitian product on 1-forms, defined as $(\varphi, \psi)_t := \int_X \varphi \wedge *_{g_t} \bar\psi$, where $*_{g_t}$ denotes the Hodge-$*$-operator associated with g_t.

For any fixed $z_0 \in X_0$, consider a local holomorphic coordinate chart

$$\left(U \times (-\delta, \delta), \left(z^1 = x^1 + \mathrm{i}x^2, \ldots, z^n = x^{2n-1} + \mathrm{i}x^{2n}, t \right) \right)$$

centred at $(z_0, 0)$ on $\{X_t\}_{t \in (-\varepsilon, 1)}$. Locally on $U \times (0, \delta)$, for $j \in \{1, \ldots, n\}$,

$$\varphi^j(z, t) \overset{\mathrm{loc}}{=} \sum_{\alpha=1}^n \varphi^j_\alpha(z^1, \ldots, z^n, t)\, \mathrm{d}z^\alpha \qquad \text{in } U \times (0, \delta),$$

where $\left\{\varphi^j_\alpha(z^1, \ldots, z^n, t)\right\}_\alpha$ are smooth in $(z^1, \ldots, z^n, t)$ and holomorphic in $(z^1, \ldots, z^n)$.

We claim that, for any $\alpha \in \{1, \ldots, n\}$, the set $\left\{\varphi_\alpha^j(z^1, \ldots, z^n, t)\right\}_{t \in \left(0, \frac{\delta}{2}\right]}$ is a uniformly-bounded family of holomorphic functions on compact subsets. More precisely, this follows from the following two observations. Fix an open relatively compact V in U and consider t varying in $\left(0, \frac{\delta}{2}\right]$. First, there exists a positive constant $\tilde{C}$ such that, for every $(z, t) \in \overline{V} \times \left(0, \frac{\delta}{2}\right]$, there hold

$$\sum_{\alpha, \beta=1}^n g_t^{\alpha, \bar{\beta}}(z) u_\alpha \bar{u}_\beta \geq \tilde{C} \sum_{\gamma=1}^n |u_\gamma|^2 \qquad \text{and} \qquad \sqrt{\det\left(g_{t, \alpha, \beta}(z)\right)_{\alpha, \beta}} \geq \tilde{C}$$

where $(u_\alpha)_{\alpha \in \{1, \ldots, n\}}$ is a vector in $\mathbb{C}^n$. Second, for any $j \in \{1, \ldots, n\}$, the 1-form $\varphi^j(t)$ has uniformly bounded norm with respect to the L^2-Hermitian product induced by g_t. Therefore, we have

$$\begin{aligned}
\max_{t \in \left[0, \frac{\delta}{2}\right]} &\text{Vol}(X, g_t) \\
&\geq \left(\varphi^j(t), \varphi^j(t)\right)_t \\
&= \int_X \sum_{\alpha, \beta} g_t^{\alpha\bar{\beta}}(z) \varphi_\alpha^j(z, t) \bar{\varphi}_\beta^j(z, t) \sqrt{\det\left(g_{t, \alpha, \beta}(z)\right)_{\alpha, \beta}} \, dx^1 \wedge \cdots \wedge dx^{2n} \\
&\geq \tilde{C}^2 \int_V \sum_{\gamma=1}^n \left|\varphi_\gamma^j(z, t)\right|^2 dx^1 \wedge \cdots \wedge dx^{2n}
\end{aligned}$$

from which we get

$$C^{-2} \geq \int_V \left|\varphi_\gamma^j(z, t)\right|^2 dx^1 \wedge \cdots \wedge dx^{2n} .$$

Classical estimates now imply that, given a compact subset K in V, there exists a constant $A_K > 0$ such that $\left|\varphi_\gamma^j(z, t)\right| \leq A_k$ for any $(z, t) \in K \times \left(0, \frac{\delta}{2}\right]$. Then we apply the Montel theorem. In fact, for fixed j and γ, the family $\left\{\varphi_\gamma^j(z^1, \ldots, z^n, t)\right\}_{t \in \left(0, \frac{\delta}{2}\right]}$ is a family of holomorphic functions on V uniformly bounded on compact subsets K in V. Then, up to pick-out a subsequence, $\varphi_\alpha^j(z^1, \ldots, z^n, t)$ converges uniformly on compact subsets, as $t \to 0$, to a holomorphic function $\varphi_\alpha^j(z^1, \ldots, z^n, 0)$ on V.

Now, we take another local holomorphic coordinate chart intersecting the first one. Then we extract another subsequence in such a way that the convergence holds for both charts. By continuing in this way, we can construct a set $\left\{\varphi^1(0), \ldots, \varphi^n(0)\right\}$ of holomorphic 1-forms on X_0.

We claim that this set is orthogonal with respect to the L^2-Hermitian product associated with g_0. Indeed, consider a smooth partition of unity $\{\rho_\ell\}_\ell$ associated with a covering $\{U_\ell\}_\ell$ with coordinate holomorphic charts as in notations above. We have

$$\begin{aligned}
\left(\varphi^j(0), \varphi^k(0)\right)_0 \\
&= \sum_\ell \int_{U_\ell} \rho_\ell \sum_{\alpha, \beta=1}^n g_{\ell, 0}^{\alpha\bar{\beta}}(z) \varphi_{\ell, \alpha}^j(z, 0) \overline{\varphi_{\ell, \beta}^k(z, 0)} \sqrt{\det\left(g_{t, \alpha, \beta}(z)\right)_{\alpha, \beta}} \, dx^1 \wedge \cdots \wedge dx^{2n} \\
&= \sum_\ell \int_{U_\ell} \rho_\ell \sum_{\alpha, \beta=1}^n \lim_{t \to 0} g_{\ell, t}^{\alpha\bar{\beta}}(z) \varphi_{\ell, \alpha}^j(z, t) \overline{\varphi_{\ell\beta}^k(z, t)} \sqrt{\det\left(g_{t, \alpha, \beta}(z)\right)_{\alpha, \beta}} \, dx^1 \wedge \cdots \wedge dx^{2n} \\
&= \lim_{t \to 0} \sum_\ell \int_{U_\ell} \rho_\ell \sum_{\alpha, \beta=1}^n g_{\ell, t}^{\alpha\bar{\beta}}(z) \varphi_{\ell, \alpha}^j(z, t) \overline{\varphi_{\ell\beta}^k(z, t)} \sqrt{\det\left(g_{t, \alpha, \beta}(z)\right)_{\alpha, \beta}} \, dx^1 \wedge \cdots \wedge dx^{2n} \\
&= \lim_{t \to 0} \left(\varphi^j(t), \varphi^k(t)\right)_t = \lim_{t \to 0} \delta^{jk} = \delta^{jk} .
\end{aligned}$$

Now, we claim that $\left\{\varphi^j(0)\right\}_{j \in \{1, \ldots, n\}}$ are in fact linearly independent at every point. Indeed, consider a coordinate holomorphic chart as in notations above. Then

$$\varphi^1(z, t) \wedge \cdots \wedge \varphi^n(z, t) = \det\left(\varphi_\alpha^j(z, t)\right)_{j, \alpha} dz^1 \wedge \cdots \wedge dz^n .$$

The holomorphic functions $\det\left(\varphi_\alpha^j(z,t)\right)_{j,\alpha}$ converge to $\det\left(\varphi_\alpha^j(z,0)\right)_{j,\alpha}$ for $t \to 0$ uniformly on compact subsets. We show that $\det\left(\varphi_\alpha^j(z,0)\right)_{j,\alpha}$ is not identically zero. Indeed, $\{\varphi^1\lfloor_p(t), \ldots, \varphi^n\lfloor_p(t)\}$ being orthonormal with respect to $g_t\lfloor_p$ at any point $p \in X$, we have:

$$
\int_X \left| \det\left(\varphi_\alpha^j(0)\right)_{j,\alpha} \right|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n
$$
$$
= \lim_{t \to 0} \int_X \left| \det\left(\varphi_\alpha^j(t)\right)_{j,\alpha} \right|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n
$$
$$
= \lim_{t \to 0} c(n) \cdot \mathrm{Vol}(X, g_t)
$$
$$
= c(n) \cdot \mathrm{Vol}(X, g_0) > 0,
$$

where $c(n)$ is a constant depending just on the complex dimension. It follows that $\left| \det\left(\varphi_\alpha^j(z,0)\right)_{j,\alpha} \right|^2$ is not identically zero. Therefore, up to pick-out a subsequence, by the Bochner theorem, [7, Theorem VIII.8], we obtain that $\det\left(\varphi_\alpha^j(z,0)\right)_{j,\alpha}$ is nowhere vanishing, proving the claim.

Therefore $\left\{\varphi^j(0)\right\}_{j \in \{1, \ldots, n\}}$ provides a global co-frame of holomorphic 1-forms for X_0, and hence X_0 is holomorphically parallelizable. □

Note that if $d\varphi^j(t) = 0$, then also $d\varphi^j(0) = 0$, for any $j \in \{1, \ldots, n\}$. In particular, by [19, Theorem 1], one recovers the stability result for differentiable families of complex tori by A. Andreotti, H. Grauert, and W. Stoll.

Theorem 2.4. *(See [1, Theorem 8] by A. Andreotti, H. Grauert, and W. Stoll.) Let $\{X_t\}_{t \in (-\varepsilon, 1)}$ be a differentiable family of compact complex manifolds, with $\varepsilon > 0$ small enough. Suppose that X_t is a complex torus for any $t \in (0, 1)$. Then X_0 is a complex torus.*

Proof. We take a co-frame $\{\varphi^j(t)\}_j$ of holomorphic 1-forms on $X_t = \Gamma_t \backslash \mathbb{C}^n = (X, J_t)$ varying smoothly in t. We set $g_t := \sum_j \varphi^j(t) \odot \bar{\varphi}^j(t)$. We claim that $g_0 := \lim_{t \to 0} g_t$ is a Hermitian metric on X_0. Indeed, we have

$$
\int_X \varphi^1(0) \wedge \cdots \wedge \varphi^n(0) \wedge \bar{\varphi}^1(0) \wedge \cdots \wedge \bar{\varphi}^n(0)
$$
$$
= \lim_{t \to 0} \int_X \varphi^1(t) \wedge \cdots \wedge \varphi^n(t) \wedge \bar{\varphi}^1(t) \wedge \cdots \wedge \bar{\varphi}^n(t)
$$
$$
= \lim_{t \to 0} \det \begin{pmatrix} \Omega(t) \\ \bar{\Omega}(t) \end{pmatrix}
$$
$$
= \det \begin{pmatrix} \Omega(0) \\ \bar{\Omega}(0) \end{pmatrix} \neq 0
$$

by [1, page 341], where $\Omega(t)$ is the period matrix of X_t.

Finally, we claim that a compact complex holomorphically parallelizable manifold $X = G/D$ of complex dimension n is a torus if and only if the first Betti number is $b_1 = 2n$.

Indeed, note that $b_1 = \dim_{\mathbb{R}} G/[G,G]$ by the Sakane theorem, [18, Theorem 1], see also [19, Corollary 1]. The statement follows, since $t \mapsto b_1(X_t)$ is locally constant at 0 by the Ehresmann theorem. □

In particular, one gets the following.

Corollary 2.5. *In the class of holomorphically parallelizable manifolds, respectively nilmanifolds, the property of being Kähler is stable for both small and large deformations.*

Proof. By [19, Corollary 2], holomorphically parallelizable manifolds admit Kähler metrics if and only if they are complex tori. Respectively, by [6, Theorem A], nilmanifolds admit Kähler structures if and only if they are tori. By [1, Theorem 8], the statement follows. □

Remark 2.6. Complex solvable Lie groups up to complex dimension 5 are classified by I. Nakamura in [14, §6]. For example, the family in class (IV.5) is characterized by the structure equations

$$\mathrm{d}\varphi^1 = 0\,, \quad \mathrm{d}\varphi^2 = \varphi^1 \wedge \varphi^2\,, \quad \mathrm{d}\varphi^3 = \alpha\,\varphi^1 \wedge \varphi^3\,, \qquad \mathrm{d}\varphi^4 = -(1+\alpha)\,\varphi^1 \wedge \varphi^4\,,$$

where $\alpha \in \mathbb{C}$ is such that $\alpha(1+\alpha) \neq 0$. Note that, for $\alpha \to 0$, the above family degenerates to the class (IV.4), which is characterized by the structure equations

$$\mathrm{d}\varphi^1 = 0\,, \quad \mathrm{d}\varphi^2 = 0\,, \quad \mathrm{d}\varphi^3 = \varphi^2 \wedge \varphi^3\,, \qquad \mathrm{d}\varphi^4 = \varphi^2 \wedge \varphi^4\,.$$

Holomorphically parallelizable solvmanifolds up to complex dimension 5 are classified by D. Guan in [10, Classification Theorem, Theorem 2, Theorem 3]. In this case, we cannot find an example showing that the property of being holomorphically parallelizable with a fixed universal covering is not preserved at the limit. We wonder whether such an example can be found.

Remark 2.7. As pointed out by the Referee, we ask whether the result in Theorem 2.3 may be stated in the category of compact complex manifolds.

Acknowledgements

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