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Generalized Joseph's decompositions ☆

Décompositions de Joseph généralisées

Arkady Berenstein [a], Jacob Greenstein [b]

[a] Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
[b] Department of Mathematics, University of California Riverside, Riverside, CA 92521, USA

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ABSTRACT

We generalize the decomposition of $U_q(\mathfrak{g})$ introduced by A. Joseph in [5] and link it, for $\mathfrak{g}$ semisimple, to the celebrated computation of central elements due to V. Drinfeld [2]. In that case, we construct a natural basis in the center of $U_q(\mathfrak{g})$ whose elements behave as Schur polynomials and thus explicitly identify the center with the ring of symmetric functions.

RÉSUMÉ

Nous généralisons la décomposition de $U_q(\mathfrak{g})$ introduite par A. Joseph [5] et la relions, pour $\mathfrak{g}$ semi-simple, au calcul bien connu d'éléments centraux dû à V. Drinfeld [2]. Dans ce cas, nous construisons une base naturelle dans le centre de $U_q(\mathfrak{g})$, dont les éléments se conduisent comme des polynômes de Schur, et nous identifions donc explicitement le centre avec l'anneau de fonctions symétriques.

1. Introduction and main results

1.1. Let H be an associative algebra with unity over a field $\Bbbk$ and let $\mathscr{C}$ be a full abelian subcategory closed under submodules of the category $H-\mathrm{Mod}$ of left H-modules. Suppose that we have a "finite duality" functor $^\star : \mathscr{C} \to \mathrm{Mod}-H$ with $V^\star \subseteq V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ (with equality if and only if V is finite dimensional) with its natural right H-module structure, such that the restriction of the evaluation pairing $\langle\cdot,\cdot\rangle_V : V \otimes V^* \to \Bbbk$ to $V \otimes V^\star$ is non-degenerate for all objects V in $\mathscr{C}$ (see Section 2.1 for details). Following [4], we define $\beta_V : V \otimes_{D(V)} V^\star \to H^*$ where $D(V) = \mathrm{End}_H V^\star = (\mathrm{End}_H V)^{\mathrm{op}}$ by

$$\beta_V(v \otimes f)(h) = \langle h \triangleright v, f\rangle_V = \langle v, f \triangleleft h\rangle_V, \qquad v \in V,\ f \in V^\star,\ h \in H,$$

where $\triangleright$ (respectively, $\triangleleft$) denotes the left (respectively, right) H-action. It is easy to see that β_V is well-defined. Set $H^*_V = \mathrm{Im}\beta_V$. Recall that $V \otimes V^\star$ and H^* are naturally H-bimodules. The following is essentially proved in [4, §3.1] and [3, Corollary 1.16].

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E-mail addresses: arkadiy@math.uoregon.edu (A. Berenstein), jacob.greenstein@ucr.edu (J. Greenstein).

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Proposition 1.1.

(a) *For all $V \in \mathscr{C}$, β_V is a homomorphism of H-bimodules and H^*_V depends only on the isomorphism class of V. Moreover, if $V, V' \in \mathscr{C}$ are simple and $H^*_V = H^*_{V'}$ then $V \cong V'$;*
(b) *$H^*_{V\oplus V'} = H^*_V + H^*_{V'}$ for all $V, V' \in \mathscr{C}$. In particular, $H^*_{V^{\oplus n}} = H^*_V$ for all $n \in \mathbb{N}$.*
(c) *If $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule then β_V is injective.*
(d) *If V is simple finite dimensional, then $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule and hence β_V is injective.*

It is natural to call H^*_V a *generalized Peter–Weyl component*. Denote $H^*_{\mathscr{C}} = \sum_{[V]\in \mathrm{Iso}\,\mathscr{C}} H^*_V$ and $\underline{H}^*_{\mathscr{C}} = \bigoplus_{[V]\in \mathrm{Iso}^\circ\mathscr{C}} H^*_V$, where $\mathrm{Iso}\,\mathscr{C}$ (respectively, $\mathrm{Iso}^\circ\mathscr{C}$) is the set of isomorphism classes of objects (respectively, simple objects) in $\mathscr{C}$. By definition, there is a natural homomorphism of H-bimodules $\underline{H}^*_{\mathscr{C}} \to H^*_{\mathscr{C}}$. Clearly, under the assumptions of Proposition 1.1(c), it is injective. Note that $H^*_{\mathscr{C}} = \sum_{[V]\in A} H^*_V$ for any subset A of $\mathrm{Iso}\,\mathscr{C}$, which generates it as an additive monoid. The following refinement of [4, Theorem 3.10] establishes the generalized Peter–Weyl decomposition.

Theorem 1.2. *Suppose that all objects in $\mathscr{C}$ have finite length. Then*

(a) *if $H^*_{\mathscr{C}} = \underline{H}^*_{\mathscr{C}}$ then $\mathscr{C}$ is semisimple;*
(b) *if $\mathscr{C}$ is semisimple and $V \otimes_{D(V)} V^\star$ is simple for every $V \in \mathscr{C}$ simple then $H^*_{\mathscr{C}} = \underline{H}^*_{\mathscr{C}}$.*

1.2. Henceforth we denote by $\mathscr{C}^{\mathrm{fin}}$ the full subcategory of $\mathscr{C}$ consisting of all finite-dimensional objects. Clearly $V \otimes V^\star$, $V \in \mathscr{C}^{\mathrm{fin}}$, is a unital algebra with unity 1_V; set $z_V := \beta_V(1_V) \in H^*_V$. For example, if $H = \Bbbk G$ for a finite group G, then for any finite-dimensional H-module V, we have $z_V(g) = tr_V(g)$, $g \in G$, where tr_V denotes the trace of a linear endomorphism of V.

Given an H-bimodule B, define the subspace B^H of H-invariants in B by $B^H = \{b \in B : h \triangleright b = b \triangleleft h,\ \forall h \in H\}$ (B^H is sometimes referred to as the center of B). Clearly, $z_V \in (H^*_V)^H$, $z_V(1_H) = \dim_\Bbbk V \neq 0$ and $(H^*_V)^H = \Bbbk z_V$ if $\mathrm{End}_H V = \Bbbk\,\mathrm{id}_V$. Set $\mathcal{Z}_{\mathscr{C}} = \sum_{[V]\in\mathrm{Iso}\,\mathscr{C}} \mathbb{Z} z_V$. Given $V \in \mathscr{C}$, denote $|V|$ its image in the Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$. The following result contrasts sharply with Proposition 1.1 and Theorem 1.2 for non-semisimple $\mathscr{C}$.

Theorem 1.3. *Suppose that $\mathscr{C} = \mathscr{C}^{\mathrm{fin}}$. Then the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$ given by $|V| \mapsto z_V$, $[V] \in \mathrm{Iso}\,\mathscr{C}$ is an isomorphism of abelian groups.*

1.3. To introduce a multiplication on $\mathcal{Z}_{\mathscr{C}} \subset (H^*_{\mathscr{C}})^H \subset H^*_{\mathscr{C}}$, we assume henceforth that $H = (H, m, \Delta, \varepsilon)$ is a bialgebra and that $\mathscr{C}$ is a tensor subcategory of $H-\mathrm{Mod}$. Note that H^* is an algebra in a natural way. It is easy to see (Lemma 2.4) that $(H^*)^H$ is a subalgebra of H^*. We also assume that there is a natural isomorphism $(V \otimes V')^\star \cong V'^\star \otimes V^\star$ in $\mathrm{mod}-H$ for all $V, V' \in \mathscr{C}$.

Theorem 1.4.

(a) *$H^*_V \cdot H^*_{V'} = H^*_{V\otimes V'}$ for all $V, V' \in \mathscr{C}$. In particular, $H^*_{\mathscr{C}}$ is a subalgebra of H^*;*
(b) *$z_V \cdot z_{V'} = z_{V\otimes V'}$ for all $V, V' \in \mathscr{C}^{\mathrm{fin}}$. In particular, if $\mathscr{C} = \mathscr{C}^{\mathrm{fin}}$ then $\mathcal{Z}_{\mathscr{C}}$ is a subring of $(H^*_{\mathscr{C}})^H$ and the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$ from Theorem 1.3 is an isomorphism of rings.*

Thus, it is natural to regard $\mathcal{Z}_{\mathscr{C}}$ as the character ring of $\mathscr{C}$.

1.4. It turns out that we can transfer the above structures from $H^*_{\mathscr{C}}$ to H if $H = (H, m, \Delta, \varepsilon, S)$ is a Hopf algebra. For an H-bimodule B, define left H-actions ad and $\diamond$ on B via $(\mathrm{ad}\, h)(b) = h_{(1)} \triangleright b \triangleleft S(h_{(2)})$ and $h \diamond b = S^2(h_{(2)}) \triangleright b \triangleleft S(h_{(1)})$, $h \in H$, $b \in B$, where $\Delta(b) = b_{(1)} \otimes b_{(2)}$ in Sweedler's notation.

Fix a categorical completion $H \widehat{\otimes} H$ of $H \otimes H$ such that $(f \otimes 1)(H \widehat{\otimes} H) \subset H$ for all $f \in H^*_{\mathscr{C}}$. Equivalently, $\Phi_P : H^*_{\mathscr{C}} \to H$, $f \mapsto (f \otimes 1)(P)$ is a well-defined linear map. Denote $\mathscr{A}(H)$ the set of all $P \in H \widehat{\otimes} H$ such that $P \cdot (S^2 \otimes 1)(\Delta(h)) = \Delta(h) \cdot P$ for all $h \in H$. Clearly, $\mathscr{A}(H)$ is a subalgebra of $H \widehat{\otimes} H$. Elements of $\mathscr{A}(H)$ are analogous to M-matrices (see, e.g., [12]). For $V \in \mathscr{C}^{\mathrm{fin}}$, set $c_V = c_{V,P} := \Phi_P(z_V) \in \Phi_P((H^*_{\mathscr{C}})^H)$. Let $Z(H)$ be the center of H.

Theorem 1.5. *Let $P \in \mathscr{A}(H)$. Then $\Phi_P : H^*_{\mathscr{C}} \to H$ is a homomorphism of left H-modules, where H acts on $H^*_{\mathscr{C}}$ and H via $\diamond$ and* ad*, respectively. Moreover, $\Phi_P((H^*_{\mathscr{C}})^H) \subset Z(H)$ and the assignment $|V| \mapsto c_V$, $[V] \in \mathrm{Iso}\,\mathscr{C}^{\mathrm{fin}}$ defines a homomorphism of abelian groups* $\mathrm{ch}_{\mathscr{C}} : K_0(\mathscr{C}^{\mathrm{fin}}) \to Z(H)$.

Surprisingly, Φ_P is often close to be an algebra homomorphism. To make this more precise, we generalize the notion of an algebra homomorphism as follows. Let A, B be $\Bbbk$-algebras and let $\mathscr{F}$ be a collection of subspaces in A. We say that a $\Bbbk$-linear map $\Phi : A \to B$ is an *$\mathscr{F}$-homomorphism* if $\Phi(U) \cdot \Phi(U') \subset \Phi(U \cdot U')$ for all $U, U' \in \mathscr{F}$. We say that $\mathscr{F}$ is multiplicative if $U \cdot U' \in \mathscr{F}$ for all $U, U' \in \mathscr{F}$. It is easy to see that $|\mathscr{F}| := \sum_{U \in \mathscr{F}} U$ is a subalgebra of A and $\Phi(|\mathscr{F}|)$ is a subalgebra of B for any multiplicative family $\mathscr{F}$.

In what follows, we denote by $\mathscr{F}_{\mathscr{C}}$ the collection of all subspaces of H^* of the form H^*_V where $V \in \mathscr{C}$. By Theorem 1.4, $\mathscr{F}_{\mathscr{C}}$ is multiplicative.

Example 1.6. Let $H = \Bbbk G$, where G is a finite group and $\mathscr{C}$ is the category of its finite-dimensional representations. Then the assignment $\delta_g \mapsto g^{-1}$ where $\delta_g(h) = \delta_{g,h}$, $g, h \in G$ defines an isomorphism of H-bimodules $\Phi : H^* \to H$. Let $\mathscr{F}_G = \{H^*_V : [V] \in \mathrm{Iso}^\circ \mathscr{C}, \mathrm{Hom}_G(V, V \otimes V) \neq 0\} \subset \mathscr{F}_{\mathscr{C}}$. If $|G| \in \Bbbk^\times$, then Φ is an $\mathscr{F}_G$-homomorphism since $\Phi(H^*_V) \cdot \Phi(H^*_{V'}) = 0$ if $[V] \neq [V'] \in \mathrm{Iso}^\circ \mathscr{C}$ and $\Phi(H^*_V) \cdot \Phi(H^*_V) = \Phi(H^*_V)$.

Denote by $\mathscr{M}(H)$ the set of all $P \in H \widehat{\otimes} H$ such that Φ_P is an $\mathscr{F}_{\mathscr{C}}$-homomorphism and by $\mathscr{M}_0(H)$ the set of all $P \in \mathscr{M}(H)$ such that Φ_P restricts to a homomorphism of algebras $(H^*_{\mathscr{C}})^H \to Z(H)$. We abbreviate $H_{V,P} := \Phi_P(H^*_V)$ and $H_{\mathscr{C},P} := \Phi_P(H^*_{\mathscr{C}}) = \sum_{[V] \in \mathrm{Iso}\, \mathscr{C}} H_{V,P}$. Since $\mathscr{F}_{\mathscr{C}}$ is multiplicative, $H_{\mathscr{C},P}$ is a subalgebra of H for $P \in \mathscr{M}(H)$. The following is immediate.

Proposition 1.7. *Suppose that $P \in \mathscr{A}(H) \cap \mathscr{M}(H)$ and Φ_P is injective. Then:*

(a) *if $V \otimes_{D(V)} V^\star$ is a simple H-bimodule then it is isomorphic to $H_{V,P}$ as a left H-module;*
(b) *$H_{\mathscr{C},P} = \bigoplus_{[V] \in \mathrm{Iso}^\circ \mathscr{C}} H_{V,P}$ if $\mathscr{C}$ is semisimple and $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule for each $V \in \mathscr{C}$ simple;*
(c) *if $P \in \mathscr{M}_0(H)$ then $\mathrm{ch}_{\mathscr{C}} : K_0(\mathscr{C}^{\mathrm{fin}}) \to Z(H)$ is injective.*

The following theorem provides a sufficiently large subclass of $\mathscr{A}(H) \cap \mathscr{M}(H)$ and $\mathscr{A}(H) \cap \mathscr{M}_0(H)$.

Theorem 1.8. *Suppose that $P \in \mathscr{A}(H)$ such that $(\Delta \otimes 1)(P) = (m \otimes m \otimes 1)((T \otimes 1)P_{15}P_{35})$ for some $T \in H \widehat{\otimes} H \widehat{\otimes} H \widehat{\otimes} H$. Then $P \in \mathscr{M}(H)$. Moreover, if $(m^{\mathrm{op}} \otimes m^{\mathrm{op}})(T) = 1 \otimes 1$ then $P \in \mathscr{M}_0(H)$.*

It should be noted that $\mathscr{M}(H)$ and $\mathscr{M}_0(H)$ are not exhausted by the above condition.

Example 1.9. Let $G = S_3$. Suppose that $\mathrm{char}\, \Bbbk \neq 2, 3$ and let $P_{\lambda,\mu} = \frac{1}{6} \sum_{\sigma \in S_3} 1 \otimes \sigma + \frac{1}{36} \big[s_1 \otimes (1 + (2\mu - 1)s_1 - (\mu + 1)(s_2 + s_1 s_2 s_1) + s_1 s_2 + s_2 s_1) \big]_{S_3} + \frac{1}{18} \big[s_1 s_2 \otimes (2 + (\lambda - 1)s_1 s_2 - (\lambda + 1)s_2 s_1) \big]_{S_3}$, where $\lambda, \mu \in \Bbbk$, $s_i = (i, i+1)$ and we abbreviate $[x]_G := \sum_{g \in G} (g \otimes g)x(g^{-1} \otimes g^{-1})$ for $x \in \Bbbk G \otimes \Bbbk G$. Then one can show that $P_{\lambda,\mu} \in \mathscr{A}(H) \cap \mathscr{M}_0(H)$ and that Φ_P is an isomorphism if and only if $(\lambda, \mu) \in (\Bbbk^\times)^2$. However, there is no $T \in H^{\otimes 4}$ such that the condition of Theorem 1.8 holds.

It turns out that $P \in \mathscr{A}(\Bbbk G) \cap \mathscr{M}_0(\Bbbk G)$ with Φ_P injective does not always exist for a given finite group G (for instance, it does not exist for dihedral groups different from $S_2 \times S_2$ and S_3) and thus it would be interesting to classify all finite groups G that admit such a P. Its existence provides a decomposition of $\Bbbk G$ into a direct sum of adjoint G-modules $H_{V,P}$ over all simple $\Bbbk G$-modules V (a mock Peter–Weyl decomposition), which is an alternative to the well-known Maschke decomposition into the direct sum of matrix algebras. As a further example, we constructed an 8-parameter family of such P for $G = S_4$. The answer is rather cumbersome (it involves 34 terms of the form $[g \otimes x]_{S_4}$, $g \in S_4$, $x \in \Bbbk S_4$ and is available at https://ishare.ucr.edu/jacobg/jdec-example.pdf).

Specializing Proposition 1.7 and Theorem 1.8 to quantized universal enveloping algebras, we can recover Joseph's decomposition [5]. Namely, let $H = U_q(\mathfrak{g})$ for a Kac–Moody algebra $\mathfrak{g}$ and $\mathscr{C}_{\mathfrak{g}}$ be the (semisimple) category of highest weight integrable $U_q(\mathfrak{g})$-modules (of type **1**, see e.g. [1]); then $V^\star$ is the graded dual of V. Let Λ^+ be the monoid of dominant weights for $\mathfrak{g}$ and denote $V(\lambda)$ a highest weight simple integrable module of highest weight $\lambda \in \Lambda^+$. We construct $P = P_{\mathfrak{g}}$ with $\Phi_{P_{\mathfrak{g}}}$ injective in Lemma 2.9 and obtain the following theorem, which refines the results of [5].

Theorem 1.10.

(a) *For $\lambda \in \Lambda^+$, $H_{V(\lambda),P} = \mathrm{ad}\, U_q(\mathfrak{g})(K_{2\lambda}) \cong V(\lambda) \otimes V(\lambda)^\star$.*
(b) *The sum $\sum_{\lambda \in \Lambda^+} \mathrm{ad}\, U_q(\mathfrak{g})(K_{2\lambda})$ is direct and is a subalgebra of $U_q(\mathfrak{g})$.*

Furthermore, part (c) of Proposition 1.7, which generalizes a classic result of Drinfeld [2], yields the following theorem.

Theorem 1.11. *Let $\mathfrak{g}$ be semisimple. Then the assignment $|V| \mapsto c_V$ defines an isomorphism of algebras $\mathbb{Q}(q) \otimes_{\mathbb{Z}} K_0(\mathfrak{g} - \mathrm{mod}) \to Z(U_q(\mathfrak{g}))$.*

This provides the following refinements of classic results of Duflo, Harish-Chandra and Rosso [10].

Corollary 1.12. *For $\mathfrak{g}$ semisimple, $Z(U_q(\mathfrak{g}))$ is freely generated by the $c_{V(\omega)}$ where the ω are fundamental weights of $\mathfrak{g}$, and $c_{V(\lambda)}c_{V(\mu)} = \sum_{\nu\in\Lambda^+}[V(\lambda)\otimes V(\mu) : V(\nu)]c_{V(\nu)}$ for any $\lambda, \mu \in \Lambda^+$.*

2. Notation and proofs

Recall that, given an H-bimodule B, B^* is naturally an H-bimodule via $(h \triangleright f \triangleleft h')(b) = f(h' \triangleright b \triangleleft h)$, $f \in B^*$, $h, h' \in H$, $b \in B$. In particular, H^* is an H-bimodule.

2.1. Proof of Theorem 1.3

The following are immediate.

Lemma 2.1. $\langle V, W^\star\rangle_{V\oplus W} = 0 = \langle W, V^\star\rangle_{V\oplus W}$.

Lemma 2.2. *Let V, W be left H-modules and let $\rho : H \otimes_{\Bbbk} W \to V$ be a $\Bbbk$-linear map. Then:*

(a) *the assignment $h \triangleright_\rho (v, w) = (h \triangleright v + \rho(h \otimes w), h \triangleright w)$, $h \in H$, $v \in V$, $w \in W$, defines a left H-module structure $V \oplus_\rho W$ on $V \oplus W$ if and only if*

$$\rho(hh' \otimes w) = \rho(h \otimes h' \triangleright w) + h \triangleright \rho(h' \otimes w), \qquad h, h' \in H,\ w \in W. \tag{1}$$

In that case, V is an H-submodule of $V \oplus_\rho W$ and $W = (V \oplus_\rho W)/V$.

(b) *A short exact sequence of H-modules $0 \to V \to U \to W \to 0$ is equivalent to $0 \to V \to V \oplus_\rho W \to W \to 0$ for some ρ satisfying (1).*

Thus, given $V \subset U$ in $\mathscr{C}$, we can replace the natural short exact sequence $0 \to V \to U \to U/V \to 0$ by the one from Lemma 2.2.

Lemma 2.3. *Let V, W be left H-modules and let ρ be as in Lemma 2.2. Then $\beta_{V\oplus_\rho W}(x + y) = \beta_V(x) + \beta_V(y)$ for any $x \in V \otimes V^\star$, $y \in W \otimes W^\star$.*

Proof. It suffices to verify the assertion for $x = v \otimes f$ and $y = w \otimes g$, $v \in V$, $w \in W$, $f \in V^\star$, $g \in W^\star$. We have, by Lemmata 2.1, 2.2(a):

$$\begin{aligned}\beta_{V\oplus_\rho W}(v \otimes f + w \otimes g)(h) &= \langle h \triangleright_\rho v \otimes f + h \triangleright_\rho w \otimes g\rangle_{V\oplus W}\\ &= \langle h \triangleright v, f\rangle_V + \langle \rho(h \otimes w), f\rangle_{V\oplus W} + \langle h \triangleright w, g\rangle_W\\ &= \beta_V(v \otimes f)(h) + \beta_W(w \otimes g)(h). \qquad \square\end{aligned}$$

Since $1_{V\oplus_\rho W} = 1_V + 1_W$ where $1_V \in V \otimes V^\star$, $1_W \in W \otimes W^\star$, it follows from Lemma 2.3 that $z_{V\oplus_\rho W} = z_V + z_W$ and the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$, $|V| \mapsto z_V$ is a well-defined surjective homomorphism of abelian groups. Also, $z_V \in \sum_{[S]\in \mathrm{Iso}^\circ\mathscr{C}} \mathbb{Z}z_S$ for each $V \in \mathscr{C} = \mathscr{C}^{\mathrm{fin}}$ because it has finite length. Since the set $\{z_V\}_{[V]\in\mathrm{Iso}^\circ\mathscr{C}} \subset \underline{H}^*_{\mathscr{C}}$ is $\Bbbk$-linearly independent by Proposition 1.1(d), the injectivity follows. $\square$

2.2. Algebra structure on $H^*_{\mathscr{C}}$

Henceforth we assume that $H = (H, m, \Delta, \varepsilon)$ is a bialgebra. Then H^* is a unital algebra with the multiplication defined by $(\phi \cdot \xi)(h) = \phi(h_{(1)})\xi(h_{(2)})$, $h \in H$, $\phi, \xi \in H^*$, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ in Sweedler notation and with the unity being ε.

Lemma 2.4. *$(H^*)^H$ is a subalgebra of H^*.*

Proof. Observe that $\phi \in (H^*)^H$ if and only if $\phi(hh') = \phi(h'h)$ for all $h, h' \in H$. Then, given $h, h' \in H$ and $\xi, \xi' \in (H^*)^H$, we have:

$$(\xi \cdot \xi')(hh') = \xi(h_{(1)}h'_{(1)})\xi'(h_{(2)}h'_{(2)}) = \xi(h'_{(1)}h_{(1)})\xi'(h'_{(2)}h_{(2)}) = (\xi \cdot \xi')(h'h). \qquad \square$$

Proof of Theorem 1.4. Note that in the category of $\Bbbk$-vector spaces there is a natural isomorphism $\kappa : (V \otimes V^\star) \otimes (V' \otimes V'^\star) \to (V \otimes V') \otimes (V \otimes V')^\star$, $\kappa(v \otimes f \otimes v' \otimes f') = v \otimes v' \otimes f' \otimes f$, $v \in V$, $v' \in V'$, $f \in V^\star$, $f' \in V'^\star$. Then, clearly, $\langle\cdot,\cdot\rangle_{V\otimes V'} \circ \kappa = \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_{V'}$, which immediately implies that $\tilde\beta_V \otimes \tilde\beta_{V'} = \tilde\beta_{V\otimes V'} \circ \kappa$ where $\tilde\beta_U := \beta_U \circ \pi_U$ and $\pi_U : U \otimes_{\Bbbk} U^\star \to U \otimes_{D(U)} U^\star$ is the natural projection. This proves the first assertion and also the second once we observe that $1_{V\otimes V'} = \kappa(1_V \otimes 1_{V'})$. $\square$

2.3. The Hopf algebra case

Suppose now that $H=(H,m,\Delta,\varepsilon,S)$ is a Hopf algebra. Since H is naturally an H-bimodule, $\mathrm{ad}: H\to \mathrm{End}_{\Bbbk} H$ is a homomorphism of algebras. We also define $\mathrm{ad}^*: H^{\mathrm{op}}\to \mathrm{End}_{\Bbbk} H$ by $(\mathrm{ad}^* h)(h')=S(h_{(1)})h'S^2(h_{(2)})$, which is a homomorphism of algebras. Henceforth, given $a\in H^{\otimes n}$ we write it in Sweedler-like notation as $a=a_1\otimes\cdots\otimes a_n$ with summation understood.

Proof of Theorem 1.5. We need the following equivalent descriptions of $\mathscr{A}(H)$.

Lemma 2.5. *Let $P=P_1\otimes P_2\in H\widehat{\otimes}H$. The following are equivalent:*

(a) $P\cdot(S^2\otimes 1)\circ\Delta(h)=\Delta(h)\cdot P$;
(b) $(1\otimes h)\cdot P=(\mathrm{ad}^* h_{(1)})(P_1)\otimes P_2h_{(2)}$;
(c) $(\mathrm{ad}^* h\otimes 1)(P)=(1\otimes \mathrm{ad}\, h)(P)$.

Proof. By (a) we have $h_{(1)}\otimes P_1S^2(h_{(2)})\otimes P_2h_{(3)}\otimes h_{(4)}=h_{(1)}\otimes h_{(2)}P_1\otimes h_{(3)}P_2\otimes h_{(4)}$ for all $h\in H$. Then (b) and (c) follow by applying $m(S\otimes 1)\otimes 1\otimes\varepsilon$ and $m(S\otimes 1)\otimes m(1\otimes S)$, respectively, to both sides. Part (b) implies (a) since $h_{(1)}(\mathrm{ad}^* h_{(2)})(h')=h'S^2(h)$. Finally, (c) implies (b) since $(\mathrm{ad}^* h_{(1)})(P_1)\otimes P_2h_{(2)}=P_1\otimes \mathrm{ad}\, h_{(1)}(P_2)h_{(2)}=P_1\otimes hP_2$. $\square$

Lemma 2.6. *Let B be an H-bimodule and set $B^{\diamond H}:=\{b\in B : h\diamond b=\varepsilon(h)b,\ h\in H\}$. Then $B^H\subset B^{\diamond H}\subset B^{S(H)}$ with the equality if S is invertible.*

Proof. Let $h\in H$. Then for all $b\in B^H$ we have $h\diamond b=S^2(h_{(2)})\triangleright b\triangleleft S(h_{(1)})=S^2(h_{(2)})S(h_{(1)})\triangleright b=S(h_{(1)}S(h_{(2)}))\triangleright b=\varepsilon(h)b$. On the other hand, for all $b\in H^{\diamond H}$, $S(h)\triangleright b=\varepsilon(h_{(1)})S(h_{(2)})\triangleright m=S(h_{(3)})S^2(h_{(2)})\triangleright m\triangleleft S(h_{(1)})=S(S(h_{(2)})h_{(3)})\triangleright m\triangleleft S(h_{(1)})=m\triangleleft S(h)$. $\square$

The following lemma is well known and can be proved similarly.

Lemma 2.7. $Z(H)=H^H=H^{\mathrm{ad}\,H}:=\{h'\in H : (\mathrm{ad}\, h)(h')=\varepsilon(h)h',\ h\in H\}$. $\square$

By Lemma 2.5(c) we have, for all $h\in H$, $\xi\in H^*_{\mathscr{C}}$

$$\Phi_P(h\diamond\xi)=(S^2(h_{(2)})\triangleright\xi\triangleleft S(h_{(1)}))(P_1)P_2=\xi((\mathrm{ad}^* h)P_1)P_2=\xi(P_1)(\mathrm{ad}\, h)(P_2)=(\mathrm{ad}\, h)\Phi_P(\xi).$$

Furthermore, if $\xi\in(H^*_{\mathscr{C}})^H$ then $\Phi_P(h\diamond\xi)=\varepsilon(h)\Phi_P(\xi)=(\mathrm{ad}\, h)\Phi_P(\xi)$, whence $\Phi_P(\xi)\in Z(H)$. $\square$

Proof of Theorem 1.8. Suppose that P satisfies $(\Delta\otimes 1)(P)=t_1P_1t_2\otimes t_3P'_1t_4\otimes P_2P'_2$, for some $T=t_1\otimes t_2\otimes t_3\otimes t_4\in H^{\widehat{\otimes}4}$ where $P=P_1\otimes P_2=P'_1\otimes P'_2$. Then for any $\xi,\xi'\in H^*_{\mathscr{C}}$

$$\begin{aligned}\Phi_P(\xi\cdot\xi')&=(\xi\cdot\xi')(P_1)P_2=\xi(t_1P_1t_2)\xi'(t_3P'_1t_4)P_2P'_2=(t_2\triangleright\xi\triangleleft t_1)(P_1)(t_4\triangleright\xi'\triangleleft t_3)(P'_1)P_2P'_2\\&=\Phi_P(t_2\triangleright\xi\triangleleft t_1)\cdot\Phi_P(t_4\triangleright\xi'\triangleleft t_3).\end{aligned}\tag{2}$$

Take $\xi\in H^*_V$, $\xi'\in H^*_{V'}$. Then $\xi\cdot\xi'\in H^*_{V\otimes V'}$ by Theorem 1.4(a) and $\Phi_P(\xi\cdot\xi')\in H_{V,P}\cdot H_{V',P}$ by (2). Therefore, $P\in\mathscr{M}(H)$. Furthermore, assume that $t_2t_1\otimes t_4t_3=1\otimes 1$, and let $\xi,\xi'\in(H^*_{\mathscr{C}})^H$. Then (2) yields $\Phi_P(\xi\cdot\xi')=\Phi_P(t_2t_1\triangleright\xi)\cdot\Phi_P(t_4t_3\triangleright\xi')=\Phi_P(\xi)\cdot\Phi_P(\xi')$. This implies that $P\in\mathscr{M}_0(H)$. $\square$

2.4. Applications

Let $\mathscr{R}(H)$ be the set of pairs (R^+,R^-), $R^\pm\in H\widehat{\otimes}H$, such that $R^+_{21}R^-\cdot\Delta(h)=\Delta(h)\cdot R^+_{21}R^-$ for all $h\in H$ and $(\Delta\otimes 1)(R^\pm)=R^\pm_{13}R^\pm_{23}$, $(1\otimes\Delta)(R^+)=R^+_{13}R^+_{12}$. Clearly, $(R,R)\in\mathscr{R}(H)$ if R is an R-matrix for H.

Lemma 2.8. *Suppose that there exists $\mathbf{g}\in H$ group-like such that $\mathbf{g}S^2(h)=h\mathbf{g}$ for all $h\in H$. Let $(R^+,R^-)\in\mathscr{R}(H)$. Then $P:=R^+_{21}\cdot R^-\cdot(\mathbf{g}\otimes 1)\in\mathscr{A}(H)\cap\mathscr{M}_0(H)$.*

Proof. Write $R^\pm=r^\pm_1\otimes r^\pm_2=s^\pm_1\otimes s^\pm_2$. Since $R^+_{21}R^-\cdot\Delta(h)=\Delta(h)\cdot R^+_{21}R^-$ we have

$$P\cdot(S^2\otimes 1)(\Delta(h))=r^+_2r^-_1\mathbf{g}S^2(h_{(1)})\otimes r^+_1r^-_2h_{(2)}=r^+_2r^-_1h_{(1)}\mathbf{g}\otimes r^+_1r^-_2h_{(2)}=\Delta(h)\cdot P.$$

Thus, $P\in\mathscr{A}(H)$. Furthermore, $(\Delta\otimes 1)(P)=R^+_{32}R^+_{31}R^-_{13}R^-_{23}(\mathbf{g}\otimes\mathbf{g}\otimes 1)=P_1\otimes r^+_2r^-_1\mathbf{g}\otimes r^+_1P_2r^-_2$. Since $(\Delta\otimes 1)(R^+)=r^+_1\otimes s^+_1\otimes r^+_1s^+_1$, by Lemma 2.5(b), we obtain:

$$(\Delta\otimes 1)(P)=(\mathrm{ad}^* r_1^+)(P_1)\otimes r_2^+ s_2^+ r_1^- \mathbf{g}\otimes P_2 s_1^+ r_2^- =(\mathrm{ad}^* r_1^+)(P_1)\otimes r_2^+ P_1'\otimes P_2 P_2'$$
$$= S(r_1^+)P_1 S^2(s_1^+)\otimes r_2^+ s_2^+ P_1'\otimes P_2 P_2'.$$

Thus, $P\in\mathscr{M}(H)$ with $T=(S\otimes S^2\otimes 1\otimes 1)(R_{13}^+\cdot R_{23}^+)$. Finally, $(m^{\mathrm{op}}\otimes m^{\mathrm{op}})(T)=S^2(s_2^+)S(r_1^+)\otimes r_2^+ s_2^+ =(S\otimes 1)(R^+\cdot(S\otimes 1)(R^+))=1\otimes 1$. Thus, $P\in\mathscr{M}_0(H)$. □

If P is as in Lemma 2.8, we obtain

$$\Phi_P(\beta_V(v\otimes f))=r_1^+\langle r_2^+ r_1^- \mathbf{g}\triangleright v, f\rangle_V r_2^- = r_1^+\langle r_1^-\triangleright \mathbf{g}(v), f\triangleleft r_2^+\rangle_V r_2^-, \qquad v\in V,\ f\in V^\star. \tag{3}$$

Let $\Bbbk=\mathbb{Q}(q)$ and let $U_q(\mathfrak{g})$ be a quantized enveloping algebra corresponding to a symmetrizable Kac–Moody algebra $\mathfrak{g}$, which is a Hopf algebra generated by E_i, F_i, $i\in I$ and K_μ, $\mu\in\Lambda$, where Λ is a weight lattice of $\mathfrak{g}$, with $\Delta(E_i)=1\otimes E_i+E_i\otimes K_{\alpha_i}$, $\Delta(F_i)=F_i\otimes 1+K_{-\alpha_i}\otimes F_i$, $\Delta(K_\mu)=K_\mu\otimes K_\mu$, $\varepsilon(E_i)=\varepsilon(F_i)=0$ and $\varepsilon(K_\mu)=1$, where α_i, $i\in I$ are simple roots of $\mathfrak{g}$. Let $\mathcal{K}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the K_μ, $\mu\in\Lambda$. After [2,8], there exists a unique R-matrix in a weight completion $U_q(\mathfrak{g})\widehat{\otimes}U_q(\mathfrak{g})$ of the form $R=R_0R_1$, where $R_1\in U_q^+(\mathfrak{g})\widehat{\otimes}U_q^-(\mathfrak{g})$ is essentially Θ^{op} in the notation of [8] and satisfies $(\varepsilon\otimes 1)(R_1)=(1\otimes\varepsilon)(R_1)=1\otimes 1$, while $R_0\in\mathcal{K}\widehat{\otimes}\mathcal{K}$ is determined by the following condition: for any $\mathcal{K}$-modules $V^\pm$ such that $K_\mu|_{V^\pm}=q^{(\mu,\mu_\pm)}\mathrm{id}_{V^\pm}$, $\mu,\mu_\pm\in\Lambda$, we have $R_0|_{V^-\otimes V^+}=q^{(\mu_-,\mu_+)}\mathrm{id}_{V^-\otimes V^+}$. Here $(\cdot,\cdot)$ is the Kac–Killing form on $\Lambda\times\Lambda$ ([6]). The following is immediate.

Lemma 2.9. *Let $R=r_1\otimes r_2$ be as above. Let $v_\lambda\in V(\lambda)$ ($f_\lambda\in V(\lambda)^\star$) be a highest (respectively, lowest) weight vector of weight λ (respectively, $-\lambda$), $\lambda\in\Lambda^+$. Then $r_1\triangleright v_\lambda\otimes r_2=v_\lambda\otimes K_\lambda$ and $r_1\otimes f_\lambda\triangleleft r_2=K_\lambda\otimes f_\lambda$.* □

Proof of Theorem 1.10. Since $V(\lambda)$ is a simple highest weight module, $D(V(\lambda))\cong\Bbbk$. Note that for any $\lambda,\mu\in\Lambda^+$, $V(\lambda)\otimes V(\mu)$ is a simple $U_q(\mathfrak{g}\oplus\mathfrak{g})=U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$-module of highest weight (λ,μ). Twisting $V(\lambda)$ with the anti-automorphism of $U_q(\mathfrak{g})$ interchanging F_i and E_i, we conclude that $V(\lambda)\otimes V(\lambda)^\star$ is a simple $U_q(\mathfrak{g})$-bimodule. Taking into account that $\mathbf{g}=K_{-2\rho}$ we obtain from Lemma 2.9 and (3) that $\Phi_P(\beta_{V(\lambda)}(v_\lambda\otimes f_\lambda))=K_\lambda\langle\mathbf{g}\triangleright v_\lambda, f_\lambda\rangle K_\lambda\in\Bbbk^\times K_{2\lambda}$. Since $V(\lambda)\otimes V(\lambda)^\star$ is cyclic on $v_\lambda\otimes f_\lambda$ as $U_q(\mathfrak{g})$-module with the $\diamond$ action, $H_{V(\lambda)}$ is cyclic on $K_{2\lambda}$ as the $\mathrm{ad}\,U_q(\mathfrak{g})$-module by the above. Since $\beta_{V(\lambda)}$ is injective by Theorem 1.1(c) and Φ_P is injective by [2] (see also [9,11]), it follows that $H_{V(\lambda)}\cong V(\lambda)\otimes V(\lambda)^\star$. This proves (a). Then the sum in (b) is direct by Proposition 1.7(b) and coincides with $H_{\mathscr{C}_\mathfrak{g},P}$, which is always a subalgebra of H. □

Proof of Theorem 1.11. Since $D(V(\lambda))\cong\Bbbk$, Theorem 1.10 implies that $Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}})=\bigoplus_{\lambda\in\Lambda^+}\Bbbk c_{V(\lambda)}$, hence the assignment $|V(\lambda)|\mapsto c_{V(\lambda)}$ is an isomorphism $\Bbbk\otimes_\mathbb{Z} K_0(\mathscr{C}_\mathfrak{g})\to\Phi_{P_\mathfrak{g}}((H^*_{\mathscr{C}_\mathfrak{g}})^H)=Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}})$ as in Proposition 1.7(c). By [7], $K_0(\mathscr{C}_\mathfrak{g})=K_0(\mathfrak{g}-\mathrm{mod})$ where $\mathfrak{g}-\mathrm{mod}$ is the category of finite dimensional $\mathfrak{g}$-modules. On the other hand, each non-zero element of $Z(U_q(\mathfrak{g}))$ is ad-invariant, hence generates a one-dimensional $\mathrm{ad}\,U_q(\mathfrak{g})$-module and thus is contained in $H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}}$ by [5]. Therefore, $Z(U_q(\mathfrak{g}))\subset H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}}$ hence $Z(U_q(\mathfrak{g}))=Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}})$. □

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