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Differential geometry

Uniqueness of asymptotic cones of complete noncompact shrinking gradient Ricci solitons with Ricci curvature decay

Unicité des cônes asymptotiques des solitons gradients de Ricci contractants complets non compacts avec courbure de Ricci décroissante

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A R T I C L E   I N F O

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A B S T R A C T

We show that any complete noncompact shrinking gradient Ricci soliton with (1) $|\mathrm{Rc}| \to 0$ at infinity or (2) $R \to 0$ at infinity, $|\mathrm{Rm}|$ bounded, and κ-noncollapsed has a unique asymptotic cone.

R É S U M É

Nous montrons que tout soliton gradient de Ricci contractant complet non compact vérifiant la propriété (1) $|\mathrm{Rc}| \to 0$ à l'infini ou (2) $R \to 0$ à l'infini, avec $|\mathrm{Rm}|$ bornée et κ-non-effrondée, possède un cône asymptotique unique.

Let $(\mathcal{M}^n, \bar{g}, \bar{f})$ be a complete noncompact shrinking gradient Ricci soliton (GRS). The study of its geometry near infinity is of crucial importance. A basic question is the volume growth of $\bar{g}$. By Cao and Zhou [1], with an observation of Munteanu using Chen's result in [5] that $R \geq 0$, the volume growth is at most Euclidean. Moreover, in [1] a strong lower bound for the potential function $\bar{f}$ was obtained. This may be interpreted as generally indicating rigidity for shrinking GRS. Further evidence reflecting the aforementioned rigidity are the classification and nonexistence results starting with Hamilton [8,9] and Perelman [18] and then Cao, Chen and Zhu [2], Ni and Wallach [16], Naber [15], and Petersen and Wylie [19]. The most general and recent result in this direction is by Munteanu and Wang [14]. Regarding curvature estimates, notable are the works [12] and [13] of Munteanu and Wang.

Invariants reflecting the geometry near infinity of a complete noncompact Riemannian manifold $(\mathcal{N}^n, h)$ are its asymptotic cones, where an asymptotic cone is defined as the pointed Gromov–Hausdorff limit of $\left(\mathcal{N}, \lambda_i^{-1} d_h, p\right)$ for some sequence $\lambda_i \to \infty$ and $p \in \mathcal{N}$ (this limit is independent of p). By the compactness theorem of Gromov [7], if $\mathrm{Rc}_h \geq 0$, then there exists such a metric space limit. This leads to the questions of the uniqueness and regularity of such limits.

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That is, are the limits independent of the sequence $\lambda_i \to \infty$? Is an asymptotic cone necessarily a regular asymptotic cone (i.e., the cross section is a smooth $(n-1)$-dimensional manifold)? Even in the presence of positive Ricci curvature, the limit may be nonunique by the examples of Perelman [17]. For the study of complete noncompact Ricci flat manifolds, there are the deep works of Cheeger and Colding (e.g., [3]), Cheeger and Tian [4], and Colding and Minicozzi (e.g., [6]), where in the last paper it is proved that if there exists a regular asymptotic cone, then the asymptotic cone is unique.

Regarding the geometry near infinity of GRS, Kotschwar and Wang [11] proved that for a simply-connected complete noncompact shrinking GRS $(\mathcal{M}_1^n, g_1, f_1)$, if a topological end is geometrically asymptotic near infinity to a unique regular asymptotic cone, then any other simply-connected shrinking GRS $(\mathcal{M}_2^n, g_2, f_2)$ with a topological end geometrically asymptotic to the same cone must be isometric to $(\mathcal{M}_1^n, g_1)$. This holographic principle was based in part on their earlier separate works on backward uniqueness of the Ricci flow (see Kotschwar [10]) and the holographic principle for shrinking self-similar solutions to the mean curvature flow (see Wang [20]).

Let $(\mathcal{M}^n, \bar{g}, \bar{f})$ be a complete noncompact shrinking GRS with $\bar{f}$ normalized, so that $\mathrm{Rc}_{\bar{g}} + \nabla^2_{\bar{g}} \bar{f} = \frac{1}{2}\bar{g}$ and $R_{\bar{g}} + |\nabla \bar{f}|^2 = \bar{f}$. We assume that $|\mathrm{Rc}_{\bar{g}}|(x) \to 0$ as $x \to \infty$. Munteanu and Wang [13] proved that, fixing $p \in \mathcal{M}$, there exists a constant $C < \infty$ such that $|\mathrm{Rm}_{\bar{g}}|(x) \le C \bar{f}(x)^{-1} \le C \left(d_{\bar{g}}(x, p) + 1\right)^{-2}$ for $x \in \mathcal{M}$. This implies that asymptotic cones are regular. In this paper, based mostly on §2 of [11], we consider the uniqueness issue.

Proposition 1. *Let $(\mathcal{M}^n, g(t), f(t))$, $t < 1$, be the associated time-dependent canonical form of $(\mathcal{M}, \bar{g}, \bar{f})$. Then there exists a compact set $K \subset \mathcal{M}$ such that, as $t \to 1^-$, $g(t)$ converges pointwise in C^∞ on compact subsets of $\mathcal{M} - K$ to a smooth Riemannian metric g_1, where $(\mathcal{M} - K, g_1)$ is isometric to the complement of a compact set in a regular cone.*

Proof. Firstly, observe that the quadratic curvature decay implies that $|\nabla \bar{f}|^2 \ge \bar{f} - \frac{a^2}{\bar{f}}$ for some positive constant a. By the definition of the canonical form, there exist diffeomorphisms $\varphi_t : \mathcal{M} \to \mathcal{M}$, defined by $\frac{\partial}{\partial t}\varphi_t(x) = \frac{1}{1-t}\left(\nabla_{\bar{g}} \bar{f}\right)(\varphi_t(x))$, $\varphi_0 = \mathrm{id}$, such that $g(t) = (1-t)\varphi_t^* \bar{g}$ is a solution to the Ricci flow and $f(x,t) \doteqdot \bar{f}(\varphi_t(x)) > 0$ satisfies $\mathrm{Rc}_{g(t)} + \nabla^2_{g(t)} f(t) - \frac{1}{2(1-t)} g(t) = 0$ and $\frac{\partial f}{\partial t}(x,t) = \frac{1}{1-t}|\nabla_{\bar{g}} \bar{f}|^2(\varphi_t(x))$. Hence

$$\frac{\partial f}{\partial t}(x,t) \ge \frac{1}{1-t}\left(f(x,t) - \frac{a^2}{f(x,t)}\right). \tag{1}$$

Suppose that $x \in \mathcal{M}$ satisfies $\bar{f}(x) \ge \frac{a}{\sqrt{1-\varepsilon^2}}$, $\varepsilon > 0$. By $\frac{f}{f^2 - a^2}\frac{\partial f}{\partial t} \ge \frac{1}{1-t}$, we have $f(x,t)^2 - a^2 \ge (1-t)^{-2}\left(\bar{f}(x)^2 - a^2\right)$. Therefore

$$f(x,t) \ge (1-t)^{-1}\left(\bar{f}(x)^2 - a^2\right)^{1/2} \ge \varepsilon (1-t)^{-1} \bar{f}(x) \quad \text{for } t \in [0,1). \tag{2}$$

We have

$$|\mathrm{Rm}_{g(t)}|_{g(t)}(x) = (1-t)^{-1}\left|\mathrm{Rm}_{\bar{g}}\right|_{\bar{g}}(\varphi_t(x)) \le \frac{C}{(1-t) f(x,t)} \le \frac{C\sqrt{1-\varepsilon^2}}{a\varepsilon}.$$

By this uniform bound for curvature, by $\int_0^1 \left|\frac{\partial}{\partial t} g(x,t)\right|_{g(x,t)} dt \le \frac{C\sqrt{1-\varepsilon^2}}{\varepsilon a}$, and by Shi's local derivative of curvature estimates, there exists a smooth metric g_1 on $\{\bar{f} > a\}$ such that $g(t)$ converges to g_1 in C^∞ on $\{\bar{f} \ge a + \varepsilon\}$, for every $\varepsilon > 0$.

Now $\frac{\partial f}{\partial t}(x,t) \le \frac{1}{1-t} f(x,t)$ implies that $h(x,t) \doteqdot (1-t) f(x,t) \le \bar{f}(x)$. By $0 \le R_{\bar{g}}(\varphi_t(x)) \le \frac{C(1-t)}{\varepsilon \bar{f}(x)}$ and

$$\frac{\partial h}{\partial t}(x,t) = -f(x,t) + |\nabla_{\bar{g}} \bar{f}|^2(\varphi_t(x)) = -R_{\bar{g}}(\varphi_t(x)) \tag{3}$$

for $x \in \{\bar{f} \ge \frac{a}{\sqrt{1-\varepsilon^2}}\}$ and $t \in [0,1)$, we see that $h(t)$ converges in C^0 on $\{\bar{f} > a\}$ as $t \to 1$ to a function h_1. By $(1-t)\mathrm{Rc}_{g(t)} + \nabla^2_{g(t)} h(t) - \frac{1}{2} g(t) = 0$ and standard elliptic theory, the convergence is in C^∞. Taking the limit of this equation as $t \to 1$, we obtain $\nabla^2_{g_1} h_1 - \dfrac{1}{2} g_1 = 0$. Since $(1-t)^2 R_{g(t)} + |\nabla h(t)|^2_{g(t)} = h(t)$, we have $|\nabla h_1|^2_{g_1} = h_1$. Moreover, $\varepsilon \bar{f}(x) \le h_1(x) \le \bar{f}(x)$. Since $\left|\frac{\partial h}{\partial t}\right| \le \frac{C(1-t)}{\varepsilon f}$, we have $|h(x,t) - h_1(x)| \le \frac{C(1-t)^2}{\varepsilon f(x)}$ on $\{\bar{f} \ge \frac{a}{\sqrt{1-\varepsilon^2}}\} \times [0,1)$.

Define $\Omega = \{h_1 > a\} \subset \mathcal{M}$. Taking $\varepsilon = \frac{1}{\sqrt{2}}$, we obtain $\{\bar{f} > \sqrt{2} a\} \subset \Omega \subset \{\bar{f} > a\}$. The function $\rho_1 \doteqdot 2\sqrt{h_1}$ on Ω satisfies $\nabla^2_{g_1}\left(\rho_1^2\right) = 2 g_1$, $|\nabla \rho_1|^2_{g_1} = 1$, $\nabla^{g_1}(\rho_1^2)$ is a vector field generating a 1-parameter family $\left\{\varphi_t^1\right\}_{t \in [0,\infty)}$ of homotheties of (Ω, g_1) into itself, the integral curves to $\nabla^{g_1} \rho_1$ are geodesics, and there is a diffeomorphism between Ω and the product of $(2\sqrt{a}, \infty)$ and a compact manifold Σ^{n-1} such that $g_1 = d\rho_1^2 + \rho_1^2 \tilde{g}_1$, where $\tilde{g}_1$ is a C^∞ metric on Σ. This implies that (Ω, g_1) extends to a regular asymptotic cone. □

Theorem 2. *Any two asymptotic cones of a complete noncompact shrinking gradient Ricci soliton $(\mathcal{M}^n, \bar{g}, \bar{f})$ with $|\mathrm{Rc}|(x) \to 0$ as $x \to \infty$ are isometric.*

Proof. Let O be a minimum point of $\bar{f}$, so that $\varphi_t(O) = O$. Suppose that an Euclidean metric cone C is the pointed Gromov–Hausdorff limit of $\left(\mathcal{M}, \lambda_i^{-1} d_{\bar{g}}, O\right)$ for some sequence $\lambda_i \to \infty$. Since $g(1-\lambda_i^{-2}) = \lambda_i^{-2} \varphi^*_{1-\lambda_i^{-2}} \bar{g}$ converges pointwise in C^∞ on compact subsets of Ω to g_1, we have that $(\varphi_{1-\lambda_i^{-2}}(\Omega), \lambda_i^{-2}\bar{g})$ converges in the C^∞ Cheeger–Gromov sense using the diffeomorphisms $\varphi_{1-\lambda_i^{-2}}$. Since

$$
\begin{aligned}
d_{\lambda_i^{-2}\bar{g}}(\varphi_{1-\lambda_i^{-2}}(x), O) &= d_{(\varphi^{-1}_{1-\lambda_i^{-2}})^* g(1-\lambda_i^{-2})}(\varphi_{1-\lambda_i^{-2}}(x), \varphi_{1-\lambda_i^{-2}}(O)) \\
&= d_{g(1-\lambda_i^{-2})}(x, O) \le C d_{\bar{g}}(x, O),
\end{aligned}
$$

the Cheeger–Gromov convergence matches with the pointed Gromov–Hausdorff convergence. We obtain that (Ω, g_1) is isometric to the complement of a compact set in C. So C is independent of the choice of λ_i. □

Finally, we observe a sufficient condition for a shrinking GRS to satisfy $|\mathrm{Rc}| \to 0$.

Proposition 3. *If a κ-noncollapsed complete noncompact shrinking GRS $(\mathcal{M}^n, \bar{g}, \bar{f})$ satisfies $|\mathrm{Rm}| \le C$ and $R(x) \to 0$ as $x \to \infty$, then $|\mathrm{Rc}|(x) \to 0$ as $x \to \infty$.*

Proof. Suppose $|\mathrm{Rm}| \le C$ and $R(x) \to 0$ as $x \to \infty$, but $|\mathrm{Rc}|(x) \nrightarrow 0$ as $x \to \infty$. Choose a sequence of points $x_i \to \infty$ with $|\mathrm{Rc}|(x_i) \ge c > 0$. Let $(\mathcal{M}, g(t), f(t))$ be the canonical form of $(\mathcal{M}, \bar{g}, \bar{f})$. Then $(\mathcal{M}, g(t), x_i)$ will subconverge to a complete ancient solution $(\mathcal{M}^n_\infty, g_\infty(t), x_\infty)$ with bounded curvature and $|\mathrm{Rc}|(x_\infty, 0) \ge c$ and $R(x_\infty, 0) = 0$. This is a contradiction since by the strong maximum principle applied to the equation $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Rc}|^2$, a complete ancient solution to Ricci flow with $R = 0$ at some point must be Ricci flat. □

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