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Probability theory

A central limit theorem for fields of martingale differences

Théorème limite centrale pour des champs de différences de martingale

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ABSTRACT

We prove a central limit theorem for stationary random fields of martingale differences $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, where $T_{\underline{i}}$ is a $\mathbb{Z}^d$ action and the martingale is given by a commuting filtration. The result has been known for Bernoulli random fields; here only ergodicity of one of commuting transformations generating the $\mathbb{Z}^d$ action is supposed.

RÉSUMÉ

Le théorème limite centrale pour un champ aléatoire $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, de différences d'une martingale est démontré. Le résultat est connu pour les champs aléatoires de Bernoulli ; ici, l'ergodicité d'un seul générateur de l'action $T_{\underline{i}}$ est supposée.

1. Introduction

In the study of the central limit theorem for dependent random variables, the case of martingale difference sequences has played an important role, cf. Hall and Heyde, [9]. Limit theorems for random fields of martingale differences were studied for example by Basu and Dorea [1], Morkvenas [14], Nahapetian [15], Poghosyan and Roelly [17], Wang and Woodroofe [21]. Limit theorems for martingale differences enable a research of much more complicated processes and random fields. The method of martingale approximations, often called Gordin's method, originated by Gordin's 1969 paper [7]. The approximation is possible for random fields as well; for most recent results, see, e.g., [21] and [18]. Remark that another approach was introduced by Dedecker in [6] (and is being used since); it applies both to sequences and to random fields.

For random fields, the martingale structure can be introduced in several different ways. Here we will deal with a stationary random field $f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, where f is a measurable function on a probability space $(\Omega, \mu, \mathcal{A})$ and $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, is a group of commuting probability preserving transformations of $(\Omega, \mu, \mathcal{A})$ (a $\mathbb{Z}^d$ action). By $e_i \in \mathbb{Z}^d$ we denote the vector $(0, \dots, 1, \dots, 0)$ having 1 on the i-th place and 0 at all other places, $1 \le i \le d$.

$\mathcal{F}_{\underline{i}}$, $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$, is an invariant commuting filtration (cf. D. Khosnevisan, [11]) if

(i) $\mathcal{F}_{\underline{i}} = T_{-\underline{i}} \mathcal{F}_{\underline{0}}$ for all $\underline{i} \in \mathbb{Z}^d$,

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(ii) $\mathcal{F}_{\underline{i}} \subset F_{\underline{j}}$ for $\underline{i} \le \underline{j}$ in the lexicographic order, and

(iii) $\mathcal{F}_{\underline{i}} \cap \mathcal{F}_{\underline{j}} = \mathcal{F}_{\underline{i} \wedge \underline{j}}$, $\underline{i}, \underline{j} \in \mathbb{Z}^d$, and $\underline{i} \wedge \underline{j} = (\min\{i_1, j_1\}, \dots, \min\{i_d, j_d\})$.

If, moreover, $E\Big(E(f|\mathcal{F}_{\underline{i}})\big|\mathcal{F}_{\underline{j}}\Big) = E(f|\mathcal{F}_{\underline{i} \wedge \underline{j}})$, for every integrable function f, we say that the filtration is *completely commuting* (cf. [8,18]).

By $\mathcal{F}_l^{(q)}$, $1 \le q \le d$, $l \in \mathbb{Z}$, we denote the σ-algebra generated by the union of all $\mathcal{F}_{\underline{i}}$ with $i_q \le l$. For $d = 2$ we by $\mathcal{F}_{\infty,j} = \mathcal{F}_j^{(2)}$ denote the σ-algebra generated by the union of all $\mathcal{F}_{i,j}$, $i \in \mathbb{Z}$, and in the same way we define $\mathcal{F}_{i,\infty}$.

We sometimes denote $f \circ T_{\underline{i}}$ by $U_{\underline{i}} f$; f will always be from $\mathcal{L}^2$; by $\mathcal{L}^2$ we understand $\mathcal{L}^2(\mu)$ and by L^2 we denote $L^2(\mu)$.

We say that $U_{\underline{i}} f$, $\underline{i} \in \mathbb{Z}^d$, is a *field of martingale differences* if f is $\mathcal{F}_{\underline{0}}$-measurable and whenever $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$ is such that $i_q \le 0$ for all $1 \le q \le d$ and at least one inequality is strict, then $E(f \mid \mathcal{F}_{\underline{i}}) = 0$.

Notice that $U_{\underline{i}} f$ is then $\mathcal{F}_{\underline{i}}$-measurable, $\underline{i} = (i_1 \dots, i_d) \in \mathbb{Z}^d$, and if $\underline{j} = (j_1 \dots, j_d) \in \mathbb{Z}^d$ is such that $j_k \le i_k$ for all $1 \le k \le n$ and at least one inequality is strict, $E(U_{\underline{i}} f \mid \mathcal{F}_{\underline{j}}) = 0$.

Notice that by commutativity, if $U_{\underline{i}} f$ are martingale differences then

$$E(f \mid \mathcal{F}_{-1}^{(q)}) = 0$$

for all $1 \le q \le d$. $(f \circ T_{e_q}^j)_j$ is thus a sequence of martingale differences for the filtration of $\mathcal{F}_j^{(q)}$. In particular, for $d = 2$, $(f \circ T_{e_2}^j)$ is a sequence of martingale differences for the filtration of $\mathcal{F}_{\infty,j} = \mathcal{F}_j^{(2)}$.

Recall that a measure-preserving transformation T of $(\Omega, \mu, \mathcal{A})$ is said to be *ergodic* if for any $A \in \mathcal{A}$ such that $T^{-1}A = A$, $\mu(A) = 0$ or $\mu(A) = 1$. Similarly, a $\mathbb{Z}^d$ action $(T_{\underline{i}})_{\underline{i}}$ is ergodic if for any $A \in \mathcal{A}$ such that $T_{-\underline{i}} A = A$, $\mu(A) = 0$ or $\mu(A) = 1$.

A classical result by Billingsley and Ibragimov says that if $(f \circ T^i)_i$ is an ergodic sequence of martingale differences, the central limit theorem holds. The result does not hold for random fields, however.

Example. As noticed in a paper by Wang, Woodroofe [21], for a 2-dimensional random field, $Z_{i,j} = X_i Y_j$, where X_i and Y_j, $i, j \in \mathbb{Z}$, are mutually independent $\mathcal{N}(0,1)$ random variables, we get a convergence towards a non-normal law. The random field of $Z_{i,j}$ can be represented by an ergodic action of $\mathbb{Z}^2$.

Let $(\Omega, \mu, \mathcal{A})$ be a product of probability spaces $(\Omega', \mu', \mathcal{A}')$ and $(\Omega'', \mu'', \mathcal{A}'')$ equipped with ergodic measure preserving transformations T' and T''. On Ω we then define a measure preserving $\mathbb{Z}^2$ action $T_{i,j}(x, y) = (T'^i x, T''^j y)$. The σ-algebras $\mathcal{A}', \mathcal{A}''$ are generated by $\mathcal{N}(0,1)$, iid sequences of random variables $(e' \circ T'^i)_i$ and $(e'' \circ T''^i)_i$, respectively. The dynamical systems $(\Omega', \mu', \mathcal{A}', T')$ and $(\Omega'', \mu'', \mathcal{A}'', T'')$ are then Bernoulli hence ergodic (cf. [4]). On the other hand, for any $A' \in \mathcal{A}'$, $A' \times \Omega''$ is $T_{0,1}$-invariant hence $T_{0,1}$ is not an ergodic transformation. Similarly we get that $T_{1,0}$ is not an ergodic transformation either. By ergodicity of T', T'', $A' \times \Omega''$, $A' \in \mathcal{A}'$, are the only $T_{0,1}$-invariant measurable subsets of Ω and $A'' \times \Omega'$, $A'' \in \mathcal{A}''$, are the only $T_{1,0}$-invariant measurable subsets of Ω (modulo measure μ). Therefore, the only measurable subsets of Ω, which are invariant both for $T_{0,1}$ and for $T_{1,0}$, are of measure 0 or of measure 1, i.e. the $\mathbb{Z}^2$ action $T_{i,j}$ is ergodic.

On Ω we define random variables X, Y by $X(x, y) = e'(x)$ and $Y(x, y) = e''(y)$. The random field of $(XY) \circ T_{i,j}$ then has the same distribution as the random field of $Z_{i,j} = X_i Y_j$ described above. The natural filtration of $\mathcal{F}_{i,j} = \sigma\{(XY) \circ T_{i',j'} : i' \le i, j' \le j\}$ is commuting and $((XY) \circ T_{i,j})_{i,j}$ is a field of martingale differences.

A very important particular case of a $\mathbb{Z}^d$ action is the case when the σ-algebra $\mathcal{A}$ is generated by iid random variables $U_{\underline{i}} e$, $\underline{i} \in \mathbb{Z}^d$. The σ-algebras $\mathcal{F}_{\underline{j}} = \sigma\{U_{\underline{i}} : i_k \le j_k, k = 1, \dots, d\}$ are then a completely commuting filtration and if $U_{\underline{i}} f$, $\underline{i} \in \mathbb{Z}^d$ is a martingale difference random field, the central limit theorem takes place (cf. [21]). This fact enabled to prove a variety of limit theorems by martingale approximations (cf., e.g., [18,21]).

For Bernoulli random fields, other methods of proving limit theorems have been used, cf., e.g., [2,5,20].

The aim of this paper is to show that for a martingale difference random field, the CLT can hold under assumptions weaker than Bernoullicity.

2. Main result

Let $T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, be a $\mathbb{Z}^d$ action of measure preserving transformations on $(\Omega, \mathcal{A}, \mu)$, $(\mathcal{F}_{\underline{i}})_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$, be a commuting filtration. By $e_i \in \mathbb{Z}^d$ we denote the vector $(0, \dots, 1, \dots, 0)$ having 1 on the i-th place and 0 at all other places, $1 \le i \le d$.

Theorem. *Let $f \in L^2$, be such that $(f \circ T_{\underline{i}})_{\underline{i}}$ is a field of martingale differences for a completely commuting filtration $\mathcal{F}_{\underline{i}}$. If at least one of the transformations T_{e_i}, $1 \le i \le d$, is ergodic then the central limit theorem holds, i.e. for $n_1, \dots, n_d \to \infty$ the distributions of*

$$\frac{1}{\sqrt{n_1 \dots n_d}} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} f \circ T_{(i_1,\dots,i_d)}$$

weakly converge to $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \|f\|_2^2$.

Remark 1. The results from [18] remain valid for $\mathbb{Z}^d$ actions satisfying the assumptions of the Theorem, only convergence of finite distributions is to be proved. Bernoullicity thus can be replaced by ergodicity of one of the transformations T_{e_i}. Under the assumptions of the Theorem, we thus also get a weak invariance principle. [18] implies many earlier results, cf. references there and in [21].

The central limit theorem for a summation over more general sets has been treated for the Bernoulli case, cf. [12]. In the general (ergodic) case the result does not hold even in dimension one because the CLT for martingale difference sequences need not remain true for subsequences (it does if the dynamical system is Bernoulli), see [19].

Proof. We prove the theorem for $d=2$. Proof of the general case follows by induction.

We suppose that the transformation $T_{0,1}$ is ergodic and $\|f\|_2 = 1$. To prove the central limit theorem for the random field it is sufficient to prove that for $m_k, n_k \to \infty$ as $k \to \infty$,

$$\frac{1}{\sqrt{m_k n_k}} \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f \circ T_{i,j} \quad \text{converge in distribution to } \mathcal{N}(0,1). \tag{1}$$

Recall the central limit theorem by D.L. McLeish (cf. [13]) saying that if $X_{n,i}$, $i=1,\dots,k_n$, is an array of martingale differences such that

(i) $\max_{1\le i\le k_n} |X_{n,i}| \to 0$ in probability,
(ii) there is an $L<\infty$ such that $\max_{1\le i\le k_n} X_{n,i}^2 \le L$ for all n, and
(iii) $\sum_{i=1}^{k_n} X_{n,i}^2 \to 1$ in probability,

then $\sum_{i=1}^{k_n} X_{n,i}$ converge to $\mathcal{N}(0,1)$ in law.

Next, we will suppose $k_n = n$; we will denote $U_{i,j} f = f \circ T_{i,j}$. For a given positive integer v and positive integers u, n define

$$F_{i,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{i,j} f, \qquad X_{n,i} = X_{v,n,i} = \frac{1}{\sqrt{n}} F_{i,v}, \quad i=1,\dots,n$$

(the $X_{n,i}$ depends on v). Clearly, $X_{n,i}$ are martingale differences for the filtration $(\mathcal{F}_{i,\infty})_i$. We will verify the assumptions of McLeish's theorem.

The conditions (i) and (ii) are well known to follow from stationarity. For the reader's convenience, we recall their proofs.

(i) For $\epsilon > 0$ and any integer $v \ge 1$,

$$\mu(\max_{1\le i\le n} |X_{n,i}| > \epsilon) \le \sum_{i=1}^{n} \mu(|X_{n,i}| > \epsilon) = n\mu\left(\left|\frac{1}{\sqrt{nv}} \sum_{j=1}^{v} U_{0,j} f\right| > \epsilon\right)$$
$$\le \frac{1}{\epsilon^2} E\left(\left(\frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{0,j} f\right)^2 1_{|\sum_{j=1}^{v} U_{0,j} f| \ge \epsilon\sqrt{nv}}\right) \to 0$$

as $n \to \infty$; this proves (i). From the CLT for $\frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{0,j} f$ it follows that $(\frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{0,j} f)^2$ are uniformly integrable hence the convergence in (i) is uniform for v.

To see (ii) we note

$$\left(\max_{1\le i\le n} |X_{n,i}|\right)^2 \le \sum_{i=1}^{n} X_{n,i}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{i,j} f\right)^2$$

which implies $E(\max_{1\le i\le n} |X_{n,i}|)^2 \le 1$.

It remains to prove (iii).

Let us fix a positive integer m and for constants $a_1, \dots, a_m$ consider the sums

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{v} U_{i,j} f, \quad v \to \infty.$$

Then $(\sum_{i=1}^{m} a_i U_{i,j} f)_j$, $j=1,2,\dots$, are martingale differences for the filtration $(\mathcal{F}_{\infty,j})_j$ and by the central limit theorem of Billingsley and Ibragimov [3,10] (we can also use the McLeish's theorem)

$$\frac{1}{\sqrt{v}}\sum_{j=1}^{v}\Big(\sum_{i=1}^{m}a_iU_{i,j}f\Big)$$

converge in law to $\mathcal{N}(0,\sum_{i=1}^m a_i^2)$. Notice that here we use the assumption of ergodicity of $T_{0,1}$.

From this it follows that the random vectors $(F_{1,v},\dots,F_{m,v})$ converge in law to a vector $(W_1,\dots,W_m)$ of m mutually independent and $\mathcal{N}(0,1)$ distributed random variables. For a given $\epsilon>0$, if $m=m(\epsilon)$ is sufficiently big then we have $\big\|1-(1/m)\sum_{u=1}^m W_u^2\big\|_1<\epsilon/2$. Using a truncation argument we can from the convergence in law of $(F_{u,v},\dots,F_{m,v})$ towards $(W_1,\dots,W_m)$ deduce that for $m=m(\epsilon)$ sufficiently big and v bigger than some $v(m,\epsilon)$,

$$\Big\|1-\frac{1}{m}\sum_{u=1}^{m}F_{u,v}^2\Big\|_1<\epsilon.$$

Any integer $N\ge 0$ can be expressed as $N=pm+q$ where $0\le q\le m-1$. Therefore

$$\frac{1}{N}\sum_{i=1}^{N}F_{i,v}^2-1=\frac{m}{N}\sum_{k=0}^{p-1}\Big(\frac{1}{m}\sum_{i=km+1}^{(k+1)m}F_{i,v}^2-1\Big)+\frac{1}{N}\sum_{i=mp+1}^{N}F_{i,v}^2-\frac{q}{N}. \tag{2}$$

There exists an N_ϵ such that for $N\ge N_\epsilon$ we have $\|\frac{1}{N}\sum_{i=mp+1}^{N}F_{i,v}^2-\frac{q}{N}\|_1<\epsilon$. Hence if $v\ge v(m,\epsilon)$ and $N\ge N_\epsilon$ then

$$\Big\|1-\frac{1}{N}\sum_{i=1}^{N}F_{i,v}^2\Big\|_1=\Big\|1-\frac{1}{Nv}\sum_{i=1}^{N}\Big(\sum_{j=1}^{v}U_{i,j}f\Big)^2\Big\|_1<2\epsilon. \tag{3}$$

This proves that for $\epsilon>0$ there are positive integers $v(m(\epsilon/2),\epsilon/2)$ and $N_{\epsilon/2}$ such that for $M\ge v(m(\epsilon/2),\epsilon/2)$ and $n\ge N_{\epsilon/2}$, for $X_{n,i}=(1/\sqrt{n})F_{i,M}$

$$\Big\|\sum_{i=1}^{n}X_{n,i}^2-1\Big\|_1=\Big\|\sum_{i=1}^{n}\Big(\frac{1}{\sqrt{nM}}\sum_{j=1}^{M}U_{i,j}f\Big)^2-1\Big\|_1<\epsilon.$$

In the general case, we can suppose that T_{e_d} is ergodic (we can permute the coordinates). Instead of $T_{i,j}$ we will consider transformations $T_{\underline{i},j}$ where $\underline{i}\in\mathbb{Z}^{d-1}$ and in (3), instead of segments $\{km+1,\dots,km+m\}$ we take boxes of $(k_1m+i_1,\dots,k_{d-1}m+i_{d-1})$, $i_1,\dots,i_{d-1}\in\{1,\dots,m\}$.

Let us give a sketch of an induction leading to a proof for dimension $d>2$.

Without loss of generality we can suppose that T_{e_1} is ergodic. We will suppose that the Theorem is true for the random field generated by $T_{e_1},\dots,T_{e_r}$, $2\le r<d$. We denote the transformation $T_{e_1}^{i_1}\dots T_{e_r}^{i_r}T_{e_{r+1}}^{i}$ by $T_{i,\underline{j}}$ and $f\circ T_{i,\underline{j}}$ by $U_{i,\underline{j}}f$, where $\underline{j}=(i_1,\dots,i_r)$.

For $v=\{1,\dots,N_1\}\times\dots\times\{1,\dots,N_r\}$ with cardinality $|v|=N_1\cdot\ldots\cdot N_r$ we denote $F_{i,v}=(1/|v|^{1/2})\sum_{\underline{j}\in v}U_{i,\underline{j}}f$, $i\in\mathbb{Z}$, and $X_{n,i}=X_{v,n,i}=(1/\sqrt{n})F_{i,v}$, $i=1,\dots,n$. $(X_{n,i})_i$ is a martingale difference sequence and (i), (ii) can be verified as before.

By assumption, for any vector of $(a_1,\dots,a_m)\in\mathbb{R}^m$ and $m\in\mathbb{N}$ the CLT holds true also for the r-dimensional random field $(\sum_{i=1}^m a_iU_{i,\underline{j}}f)_{\underline{j}\in\mathbb{N}^r}$. We thus get a convergence in law of the random vectors $(F_{1,v},\dots,F_{m,v})$ towards a Gaussian vector of iid random variables $(W_1,\dots,W_m)$ and we deduce (iii) as before. The CLT for the random field generated by $T_{e_1},\dots,T_{e_r},T_{e_{r+1}}$ follows.

This finishes the proof of the Theorem. □

Remark 2. The conditions (i)–(iii) imply the CLT even if the martingale difference sequences $(X_{n,i})_i$ do not belong to the same probability spaces, in particular have not the same filtration. By taking $Y_{n,i}=X_{n,k_n-i+1}$, we thus can deduce the CLT for the case of a decreasing filtration. The Theorem therefore remains true if $(\mathcal{F}_{\underline{i}})_{\underline{i}}$ is a decreasing commuting filtration (it can be defined similarly as the increasing one).

Remark 3. For any positive integer d there exists a random field of martingale differences $(f\circ T_{\underline{i}})$ for a commuting filtration of $\mathcal{F}_{\underline{i}}$ where $T_{\underline{i}}$, $\underline{i}\in\mathbb{Z}^d$, is a non-Bernoulli $\mathbb{Z}^d$ action and all T_{e_i}, $1\le i\le d$, are ergodic.

To show this we take a Bernoulli $\mathbb{Z}^d$ action $T_{\underline{i}}$, $\underline{i}\in\mathbb{Z}^d$ on $(\Omega,\mathcal{A},\mu)$ generated by iid random variables $(e\circ T_{\underline{i}})$ as defined, e.g., in [21] or [18].

Then we take another $\mathbb{Z}^d$ action of irrational rotations on the unit circle (identified with the interval $[0,1)$) generated by $\tau_{e_i}=\tau_{\theta_i}$, $\tau_{\theta_i}x=x+\theta_i$ mod 1; θ_i, $1\le i\le d$, are linearly independent irrational numbers. The unit circle is equipped with the Borel σ-algebra $\mathcal{B}$ and the (probability) Lebesgue measure λ.

On the product $\Omega\times[0,1)$ with product σ-algebra and product measure, we define the product $\mathbb{Z}^d$ action $(T_{\underline{i}}\times\tau_{\underline{i}})(x,y)=(T_{\underline{i}}x,\tau_{\underline{i}}y)$. From the ergodicity of the product of two transformations where one is ergodic and the other Bernoulli (hence weakly mixing), we conclude that for every e_i, $1 \le i \le d$, $T_{e_i} \times \tau_{e_i}$ is ergodic (cf. [16]). The product $\mathbb{Z}^d$ action is not Bernoulli (it has irrational rotations for factors).

On $\Omega \times [0,1)$ we define a filtration $\mathcal{F}_{(i_1,\dots,i_d)} = \sigma\{U_{(i',\dots,i'_d)} e \circ \pi_1, i' - 1 \le i_1, \dots, i'_d \le i_d, \pi_2^{-1}\mathcal{B}\}$ where π_1, π_2 are the coordinate projection of $\Omega \times [0,1)$.

The filtration defined above is commuting and we can find a random field of martingale differences satisfying the assumptions of the Theorem.

Remark 4. In the one-dimensional central limit theorem, non-ergodicity implies a convergence towards a mixture of normal laws. This comes from the fact that using a decomposition of the measure μ into ergodic components, we get the "ergodic case" for each of the components (cf. [19]); the variance is given by the limit of $(1/n)\sum_{i=1}^n U^i f^2$, which by the Birkhoff Ergodic Theorem exists a.s. and in L^1 and is T-invariant. In the case of a $\mathbb{Z}^2$ action (taking $d = 2$ for simplicity), the limit for $T_{0,1}$ need not to be $T_{1,0}$-invariant. This is exactly the case described in the Example and eventually we got there a convergence towards a law, which is not normal.

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