C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1–6

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Number theory/Algebraic geometry

The scaling site

Le site des fréquences

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ARTICLE INFO

Article history:
Received 24 September 2015
Accepted 29 September 2015
Available online 3 November 2015

Presented by Alain Connes

Keywords:
Arithmetic site
Scaling site
Adele class space
Topos
Characteristic 1

Mots-clés :
Site arithmétique
Site des fréquences
Classes d'adèles
Topos
Caractéristique 1

ABSTRACT

We investigate the semi-ringed topos obtained from the arithmetic site $\mathscr{A}$ of [3,4], by extension of scalars from the smallest Boolean semifield $\mathbb{B}$ to the tropical semifield $\mathbb{R}_+^{\max}$. The obtained site $[0,\infty) \rtimes \mathbb{N}^\times$ is the semi-direct product of the Euclidean half-line and the monoid $\mathbb{N}^\times$ of positive integers acting by multiplication. Its points are the same as the points $\mathscr{A}(\mathbb{R}_+^{\max})$ of $\mathscr{A}$ over $\mathbb{R}_+^{\max}$ and form the quotient of the adele class space of $\mathbb{Q}$ by the action of the maximal compact subgroup $\hat{\mathbb{Z}}^*$ of the idèle class group. The structure sheaf of the scaling topos endows it with a natural structure of tropical curve over the topos $\widehat{\mathbb{N}^\times}$. The restriction of this structure to the periodic orbits of the scaling flow gives, for each prime p, an analogue C_p of an elliptic curve whose Jacobian is $\mathbb{Z}/(p-1)\mathbb{Z}$. The Riemann–Roch formula holds on C_p and involves real-valued dimensions and real degrees for divisors.

RÉSUMÉ

Le site des fréquences $[0,\infty) \rtimes \mathbb{N}^\times$ est obtenu à partir du site arithmétique $\mathscr{A}$ de [3,4] par extension des scalaires du semicorps booléen $\mathbb{B}$ au semicorps tropical $\mathbb{R}_+^{\max}$. C'est le produit semi-direct de la demi-droite euclidienne $[0,\infty)$ par l'action du semi-groupe $\mathbb{N}^\times$ des entiers positifs par multiplication. Ses points sont les mêmes que ceux du site arithmétique définis sur $\mathbb{R}_+^{\max}$ et forment le quotient de l'espace des classes d'adèles de $\mathbb{Q}$ par l'action du sous-groupe compact maximal du groupe des classes d'idèles. Le faisceau structural du site des fréquences en fait une courbe tropicale dans le topos $\widehat{\mathbb{N}^\times}$. La restriction de cette structure aux orbites périodiques donne, pour chaque nombre premier p, un analogue C_p d'une courbe elliptique dont la jacobienne est $\mathbb{Z}/(p-1)\mathbb{Z}$. La formule de Riemann–Roch

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[1] The second author thanks the Collège de France for hospitality and financial support.

http://dx.doi.org/10.1016/j.crma.2015.09.027
1631-073X/

pour C_p fait apparaître des dimensions à valeurs réelles et les degrés des diviseurs sont des nombres réels.

1. Introduction

This note describes the Scaling Site as the algebraic geometric space obtained from the arithmetic site $\mathscr{A}$ of [3,4] by extension of scalars from the Boolean semifield $\mathbb{B}$ to the tropical semifield $\mathbb{R}_+^{\max}$. The underlying site $[0,\infty) \rtimes \mathbb{N}^\times$ inherits, from its sheaf of regular functions, a natural structure of tropical curve allowing one to define the sheaf of rational functions and to investigate an adequate version of the Riemann–Roch theorem in characteristic 1. We test this structure by restricting it to the periodic orbits of the scaling flow, *i.e.* to the points over the image of $\operatorname{Spec}\mathbb{Z}$ (*cf.* [4], §5.1). We find that for each prime p the corresponding circle of length $\log p$ is endowed with a quasi-tropical structure which turns this orbit into the analogue $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ of a classical elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$. In particular the notions of rational functions, divisors, etc. are all meaningful. A new feature is that the degree of a divisor can now be any real number. We determine the Jacobian of the curve C_p, *i.e.* the quotient $J(C_p)$ of the group of divisors of degree 0 by principal divisors and show in Theorem 6.4 that it is a cyclic group of order $p-1$. For each divisor D on C_p we define the corresponding Riemann–Roch problem with solution space $H^0(D) := H^0(C_p, \mathcal{O}(D))$. We introduce the continuous dimension $\operatorname{Dim}_{\mathbb{R}}(H^0(D))$ of this $\mathbb{R}_{\max}$-module using a limit of normalized topological dimensions and find that $\operatorname{Dim}_{\mathbb{R}}(H^0(D))$ is a real number. Finally, in Theorem 6.6 we prove that the Riemann–Roch formula holds for C_p. The appearance of arbitrary positive real numbers as continuous dimensions in this formula is due to the density in $\mathbb{R}$ of the subgroup $H_p \subset \mathbb{Q}$ of fractions with denominators a power of p and the fact that continuous dimensions are obtained as limits of normalized dimensions $p^{-n}\dim_{\mathrm{top}}(H^0(D)^{p^n})$. We view this outcome as the analogue in characteristic 1 of what happens for matroid C^*-algebras and the type-II normalized traces as in [5].

1.1. Notations

For any abelian ordered group H we let $H_{\max} = H \cup \{-\infty\}$ be the semifield obtained from H by applying the max-plus construction, *i.e.* the addition is given by the max, and the multiplication by the addition in H. In particular $\mathbb{R}_{\max}$ is isomorphic to $\mathbb{R}_+^{\max}$ by the exponential map (*cf.* [7]).

2. The scaling site

The scaling site $[0,\infty) \rtimes \mathbb{N}^\times$ is, as a site, given by a small category C endowed with a Grothendieck topology J [1]. The objects of C are the (possibly empty) bounded open intervals $\Omega \subset [0,\infty)$. The morphisms between two objects are defined by $\operatorname{Hom}_C(\Omega,\Omega') = \{n \in \mathbb{N}^\times \mid n\Omega \subset \Omega'\}$, if $\Omega \neq \emptyset$ and by $\operatorname{Hom}_C(\emptyset,\Omega') := \{*\}$, *i.e.* the one point set, for any object of C. Thus the empty set is the initial object of C. The category C admits pullbacks. Indeed, let $\Omega_j \neq \emptyset$ $(j = 1,2)$ and consider two morphisms $\phi_j : \Omega_j \to \Omega$ given by integers $n_j \in \operatorname{Hom}_C(\Omega_j,\Omega)$. Let $n = \operatorname{lcm}(n_j)$ be their lowest common multiple, write $n = a_j n_j$ and let $\Omega' := \{\lambda \in [0,\infty) \mid a_j\lambda \in \Omega_j,\ \ j = 1,2\}$. If $\Omega' = \emptyset$ the initial object is the pullback. Otherwise this gives an object Ω' of C and morphisms $a_j \in \operatorname{Hom}_C(\Omega',\Omega_j)$ such that $\phi_1 \circ a_1 = \phi_2 \circ a_2$. One sees that (Ω', a_j) is the pullback of the pair $\phi_j : \Omega_j \to \Omega$. Since the category C has pullbacks we can describe a Grothendieck topology J on C by providing a basis (*cf.* [8], Definition III.2).

Proposition 2.1. (i) *For each object Ω of C, let $K(\Omega)$ be the collection of all ordinary covers $\{\Omega_i \subset \Omega, i \in I \mid \cup\Omega_i = \Omega\}$ of Ω. Then K defines a Grothendieck topology J on C.*

(ii) *The category $\mathfrak{Sh}(C,J)$ of sheaves on (C,J) is canonically isomorphic to the category of $\mathbb{N}^\times$-equivariant sheaves on $[0,\infty)$.*

Definition 2.2. The *scaling site* $[0,\infty) \rtimes \mathbb{N}^\times$ is the small category C endowed with the Grothendieck topology J. The scaling topos is the category $\mathfrak{Sh}(C,J)$.

3. The points of the scaling topos

We recall from [4] that the space $\mathscr{A}(\mathbb{R}_+^{\max})$ of points of the arithmetic site $\mathscr{A}$ over $\mathbb{R}_+^{\max}$ is the disjoint union of the following two spaces:

(i) The points which are defined over $\mathbb{B}$: they correspond to the points of $\widehat{\mathbb{N}^\times}$ and are in canonical bijection with the space $\mathbb{Q}_+^\times \backslash \mathbb{A}^f / \hat{\mathbb{Z}}^*$ of adele classes whose Archimedean component vanishes.

(ii) The points of $\mathscr{A}(\mathbb{R}_+^{\max}) \setminus \mathscr{A}(\mathbb{B})$ are in canonical bijection with the space $\mathbb{Q}_+^\times \backslash \left((\mathbb{A}^f/\hat{\mathbb{Z}}^*) \times \mathbb{R}_+^*\right)$ of adele classes whose Archimedean component does not vanish. Equivalently, these points correspond to the space $\mathfrak{R}$ of rank-one subgroups of $\mathbb{R}$ through the map

$$(a, \lambda) \mapsto \lambda H_a\,,\ \ \forall a \in \mathbb{A}_f/\hat{\mathbb{Z}}^*,\ \lambda \in \mathbb{R}_+^*,\ \ H_a := \{q \in \mathbb{Q} \mid qa \in \hat{\mathbb{Z}}\}.$$

The next statement shows that the points of the scaling topos $\mathfrak{Sh}(C, J)$ are in canonical bijection with $\mathscr{A}(\mathbb{R}_+^{\max})$. We recall that the points of a topos of the form $\mathfrak{Sh}(C, J)$ are equivalently described as flat, continuous functors $F : C \to \mathfrak{Sets}$ (*cf.* [8, VII.6, Corollary 4]). In our context, we define the support of such a functor as the complement of the union of the open intervals I such that $F(I) = \emptyset$.

Theorem 3.1. (i) *The category of points of the scaling topos with support* $\{0\}$ *is the same as the category of points of* $\widehat{\mathbb{N}^\times}$.

(ii) *The category of points of the scaling topos with support different from* $\{0\}$ *is canonically equivalent to the category of rank-one subgroups of* $\mathbb{R}$.

The proof of the above theorem follows from the next four lemmas.

Lemma 3.2. (i) *Let* $H \subset \mathbb{R}$ *be a rank-one subgroup. Then* $F_H(V) := V \cap H_+$ *defines a flat, continuous functor* $F_H : C \to \mathfrak{Sets}$.

(ii) *The map* $H \mapsto \mathfrak{p}_H$*, which associates with a rank-one subgroup of* $\mathbb{R}$ *the point of* $\mathfrak{Sh}(C, J)$ *represented by the flat continuous functor* F_H *is an injection of* $\mathfrak{R}$ *in the space of points of the scaling topos up to isomorphism.*

The next lemma shows that the category of points of the scaling topos with support $\{0\}$ is the same as the category of points of $\widehat{\mathbb{N}^\times}$.

Lemma 3.3. *Let* $F : C \to \mathfrak{Sets}$ *be a flat continuous functor. Assume that* $F(V) = \emptyset$ *when* $0 \notin V$*. Then there exists a unique flat functor* $X : \mathbb{N}^\times \to \mathfrak{Sets}$ *such that* $F(V) = X$ *for any object* V *of* C *containing* 0.

The next two lemmas show that the category of points of the scaling topos with support $\neq \{0\}$ is equivalent to the category of rank-one subgroups of $\mathbb{R}$.

Lemma 3.4. *Let* $F : C \to \mathfrak{Sets}$ *be a flat continuous functor. Let* $\lambda \in (0, \infty)$ *and* $F_\lambda := \varprojlim_{\lambda \in J} F(J)$ *be the co-stalk of* F *at* λ*. Then there exists at most one element in the set* F_λ *and for any bounded open interval* $V \subset (0, \infty)$*,* $F(V)$ *is the disjoint union* $\cup_{\lambda \in V} F_\lambda$.

Lemma 3.5. *Let* $F : C \to \mathfrak{Sets}$ *be a flat continuous functor. Assume that* $F(V) \neq \emptyset$ *for some open interval* V *not containing* 0*. Then the set* $H_F^+ := \{\lambda \in (0, \infty) \mid F_\lambda \neq \emptyset\}$ *is the positive part of a rank-one subgroup* H_F *of* $\mathbb{R}$.

4. The structure sheaf of the scaling site

The Legendre transform allows one to describe the reduced semiring $\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_+^{\max}$ involved in the extension of scalars of the arithmetic site $\mathscr{A}$ from $\mathbb{B}$ to $\mathbb{R}_+^{\max}$ in terms of $\mathbb{R}_{\max}$-valued functions on $[0, \infty)$ which are convex, piecewise affine functions with integral slopes. We first discuss an analogous result that holds when $\mathbb{Z}_{\max}$ is replaced by the semiring $H_{\max}$ associated by the max-plus construction with a rank-one subgroup $H \subset \mathbb{R}$.

4.1. The Legendre transform

Let us fix a rank-one subgroup $H \subset \mathbb{R}$ and consider the tensor product $H_{\max} \otimes_{\mathbb{B}} \mathbb{R}_{\max}$ and the associated multiplicatively cancellative semiring $R = H_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max}$ whose elements are viewed as Newton polygons with vertices pairs $(x, y) \in H \times \mathbb{R}$ [4]. Let $Q = H_+ \times \mathbb{R}_+$. Any element of R is given by the convex hull N in $\mathbb{R}^2$ of the union of finitely many quadrants $(x_j, y_j) - Q$. This convex hull N is the intersection of half planes $P \subset \mathbb{R}^2$ of the form $P_{\lambda,u} := \{(x, y) \mid \lambda x + y \leq u\}$, $P^v := \{(x, y) \mid x \leq v\}$, where $\lambda \in \mathbb{R}_+$ and $u, v \in \mathbb{R}$. This description shows that N is uniquely determined by the function $\ell_N(\lambda) := \min\{u \in \mathbb{R} \mid N \subset P_{\lambda,u}\}$ and that this function is given in terms of the finitely many vertices (x_j, y_j) of the Newton polygon N by the formula

$$\ell_N(\lambda) = \max_j \lambda x_j + y_j. \tag{1}$$

Proposition 4.1. *Let* $H \subset \mathbb{R}$ *be a subgroup of rank one. The map* $N \mapsto \ell_N$ *is an isomorphism of the multiplicatively cancellative semiring* $R = H_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max}$ *with the semiring* $\mathcal{R}(H)$ *of convex, piecewise affine continuous functions on* $[0, \infty)$ *with slopes in* $H \subset \mathbb{R}$ *and only finitely many singularities. The operations are the pointwise operations of* $\mathbb{R}_{\max}$*-valued functions.*

4.2. The stalks of $\mathcal{O}$

Proposition 4.1 gives the relation between the reduced semiring $\mathbb{Z}_{\max}\hat{\otimes}_{\mathbb{B}}\mathbb{R}_+^{\max}$ involved in the extension of scalars of the arithmetic site from $\mathbb{B}$ to $\mathbb{R}_+^{\max}$, and the semiring $\mathcal{R}(\mathbb{Z})$. The structure sheaf $\mathcal{O}$ of $[0,\infty)\rtimes\mathbb{N}^\times$ is defined by localizing the semiring $\mathcal{R}(\mathbb{Z})$. The sections $\xi\in\mathcal{O}(\Omega)$ on an open set $\Omega\subset[0,\infty)$ are convex, piecewise affine continuous functions on Ω with slopes in $\mathbb{Z}\subset\mathbb{R}$. The action of $\mathbb{N}^\times$ on $\mathcal{O}$ is given by the morphisms

$$\gamma_n:\mathcal{O}(\Omega)\to\mathcal{O}(\frac{1}{n}\Omega),\ \ \gamma_n(\xi)(\lambda):=\xi(n\lambda)\,,\ \ \forall\lambda\in[0,\infty),\ n\in\mathbb{N}^\times. \tag{2}$$

For $\xi(\lambda)=\max\{\lambda h_j+s_j\}$ as in (1) one has $\xi(n\lambda)=\max\{\lambda n h_j+s_j\}$ so that $\gamma_n(\xi)\in\mathcal{O}(\frac{1}{n}\Omega)$. Note that these maps are not invertible.

Theorem 4.2. (*i*) *Let $H\subset\mathbb{R}$ be a rank-one subgroup of $\mathbb{R}$ and $\mathfrak{p}_H$ be the associated point of the scaling topos. The stalk of the structure sheaf $\mathcal{O}$ at $\mathfrak{p}_H$ is the semiring $\mathcal{O}_H$ of germs of $\mathbb{R}_{\max}$-valued, piecewise affine, convex continuous functions with slope in H.*

(*ii*) *Let H be an abstract rank-one ordered group and $\mathfrak{p}_H^0$ the point of the scaling topos with support $\{0\}$, associated with H. The stalk of the structure sheaf $\mathcal{O}$ at $\mathfrak{p}_H^0$ is the semiring $\mathcal{Z}_H=(\mathbb{R}\times H)_{\max}$ associated by the max-plus construction with the totally ordered group $\mathbb{R}\times H$ endowed with the lexicographic order.*

The germs at $\lambda=1$ of $\mathbb{R}_{\max}$-valued, piecewise affine, convex continuous functions $f(\lambda)$ with slopes in H are characterized by a triple (x,h_+,h_-), such that $f(1\pm\epsilon)=x\pm h_\pm\epsilon$ for $\epsilon\geq 0$ small enough. Here, one has $x\in\mathbb{R}$, $h_\pm\in H$, $h_+\geq h_-$. The only additional element of this semiring $\mathcal{R}_H$ corresponds to the germ of the constant function $-\infty$. This function is the zero element of the semiring. The algebraic rules for non-zero elements in $\mathcal{R}_H$ are as follows. The addition $\vee$ in $\mathcal{R}_H$ is given by the max of the two germs:

$$(x,h_+,h_-)\vee(x',h'_+,h'_-):=\begin{cases}(x,h_+,h_-) & \text{if } x>x'\\ (x',h'_+,h'_-) & \text{if } x'>x\\ (x,h_+\vee h'_+,h_-\wedge h'_-) & \text{if } x=x'\end{cases}$$

The product in $\mathcal{R}_H$ is given by the sum of the two germs $(x,h_+,h_-)\bullet(x',h'_+,h'_-):=(x+x',h_++h'_+,h_-+h'_-)$.

We shall denote by $\hat{\mathscr{A}}$ the semi-ringed topos $([0,\infty)\rtimes\mathbb{N}^\times,\mathcal{O})$. We view it as a relative topos over $\mathbb{R}_+^{\max}$ in the sense that the structure semirings are over $\mathbb{R}_+^{\max}$. Likewise, for the arithmetic site, the structure sheaf has no non-constant global sections.

5. The points of $\hat{\mathscr{A}}$ over $\mathbb{R}_+^{\max}$

The next theorem states that extension of scalars from $\mathscr{A}$ to $\hat{\mathscr{A}}$ does not affect the points over $\mathbb{R}_+^{\max}$.

Theorem 5.1. *The canonical projection from points of $\hat{\mathscr{A}}$ defined over $\mathbb{R}_+^{\max}$ to points of the scaling topos is bijective.*

6. The real valued Riemann–Roch theorem on periodic orbits

To realize the notion of rational functions in our context we proceed as in the definition of Cartier divisors and consider the sheaf obtained from the structure sheaf $\mathcal{O}$ by passing to the semifield of fractions.

Proposition 6.1. *For any object Ω of C the semiring $\mathcal{O}(\Omega)$ is multiplicatively cancellative and the canonical morphism to its semifield of fractions $\mathcal{K}(\Omega)$ is the inclusion of convex, piecewise affine, continuous functions among continuous, piecewise affine functions, endowed with the two operations of max and plus.*

The natural action of $\mathbb{N}^\times$ on $\mathcal{K}$ defines a sheaf of semifields in the scaling topos. One determines its stalks in the same way as for the structure sheaf $\mathcal{O}$. The local convexity no longer holds, *i.e.* the difference $h_+-h_-\in H\subset\mathbb{R}$ is no longer required to be positive.

Definition 6.2. Let $\mathfrak{p}_H$ be the point of the scaling topos associated with the rank-one subgroup $H\subset\mathbb{R}$ and let f be an element of the stalk of $\mathcal{K}$ at $\mathfrak{p}_H$. The order of f at H is defined as $\mathrm{Order}(f)=h_+-h_-\in H\subset\mathbb{R}$ where $h_\pm=\lim_{\epsilon\to 0\pm}(f((1+\epsilon)H)-f(H))/\epsilon$.

Let p be a prime, consider the subspace C_p of points $\mathfrak{p}_H$ of $[0,\infty)\rtimes\mathbb{N}^\times$ corresponding to subgroups $H\subset\mathbb{R}$ which are abstractly isomorphic to the subgroup $H_p\subset\mathbb{Q}$ of fractions with denominator a power of p.

Lemma 6.3. (*i*) *The map $\mathbb{R}_+^* \to C_p$, $\lambda \mapsto \lambda H_p$ induces a topological isomorphism $\eta_p : \mathbb{R}_+^*/p^{\mathbb{Z}} \to C_p$. The pullback by η_p of the structure sheaf $\mathcal{O}$ is the sheaf $\mathcal{O}_p$ on $\mathbb{R}_+^*/p^{\mathbb{Z}}$ of piecewise affine, continuous convex functions, with slopes in H_p.*

(*ii*) *The sheaf of quotients $\mathcal{K}_p$ of the sheaf of semirings $\mathcal{O}_p$ is the sheaf (on $\mathbb{R}_+^*/p^{\mathbb{Z}}$) of piecewise affine, continuous functions with slopes in H_p, endowed with the operations of max and plus.*

(*iii*) *The sheaf $\mathcal{K}_p$ admits global sections and for any $f \in H^0(\mathbb{R}_+^*/p^{\mathbb{Z}}, \mathcal{K}_p)$ one has:*

$$\sum_{\lambda \in \mathbb{R}_+^*/p^{\mathbb{Z}}} Order_\lambda(f) = 0, \ \text{ where } \ Order_\lambda(f) := Order_{\lambda H_p}(f \circ \eta_p^{-1}) \in \lambda H_p\,, \ \forall \lambda \in \mathbb{R}_+^*/p^{\mathbb{Z}}.$$

A divisor D is a section $H \mapsto D(H) \in H \subset \mathbb{R}$, vanishing except on a finite subset, of the projection on the base from the total space of the bundle formed by pairs (H, h) where $H \subset \mathbb{R}$ is a subgroup abstractly isomorphic to the subgroup $H_p \subset \mathbb{Q}$ and where $h \in H$. The degree of a divisor D is the finite sum $\deg(D) := \sum_{H \in C_p} D(H) \in \mathbb{R}$. Next, we define an invariant of divisors with values in the group $H_p/(p-1)H_p \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. Note that given $H \in C_p$, the elements $\lambda \in \mathbb{R}_+^*$ such that $H = \lambda H_p$ determine maps $\lambda^{-1} : H \to H_p$ differing from each other by multiplication by a power of p, thus the corresponding map $\chi : H \to H_p/(p-1)H_p \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ is canonical. For any divisor D on C_p we define $\chi(D) := \sum_{H \in C_p} \chi(D(H)) \in \mathbb{Z}/(p-1)\mathbb{Z}$.

Theorem 6.4. *The map χ vanishes on principal divisors and it induces an isomorphism of groups $\chi : J(C_p) \to \mathbb{Z}/(p-1)\mathbb{Z}$ of the quotient $J(C_p) = \mathrm{Div}(C_p)^0/\mathcal{P}$ of the group of divisors of degree 0 by the subgroup $\mathcal{P}$ of principal divisors.*

Since the group law on divisors is given by pointwise addition of sections, both the maps $\deg : \mathrm{Div}(C_p) \to \mathbb{R}$ and $\chi : \mathrm{Div}(C_p) \to \mathbb{Z}/(p-1)\mathbb{Z}$ are group homomorphisms and the pair $(\deg, \chi)$ is an isomorphism of groups

$$(\deg, \chi) : \mathrm{Div}(C_p)/\mathcal{P} \to \mathbb{R} \times (\mathbb{Z}/(p-1)\mathbb{Z}). \tag{3}$$

Given a divisor $D \in \mathrm{Div}(C_p)$ one defines the following module over $\mathbb{R}_{\max}$:

$$H^0(D) = \Gamma(C_p, \mathcal{O}_p(D)) = \{f \in \mathcal{K}_p \mid D + (f) \geq 0\}.$$

Definition 6.5. Let $f \in \Gamma(C_p, \mathcal{K}_p)$. One sets $\|f\|_p := \max\{|h(\lambda)|_p/\lambda \mid \lambda \in C_p\}$, and $h(\lambda) \in H_p$ is the slope[2] of f at λ, where $|\ |_p$ is the p-adic norm.

Let $D \in \mathrm{Div}(C_p)$ be a divisor. We introduce the following increasing filtration of $H^0(D)$ by $\mathbb{R}_{\max}$-submodules: $H^0(D)^\rho := \{f \in H^0(D) \mid \|f\|_p \leq \rho\}$. We denote by $\dim_{\mathrm{top}}(\mathcal{E})$ the topological covering dimension of an $\mathbb{R}_{\max}$-module $\mathcal{E}$ (*cf.* [10]) and define

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n\to\infty} p^{-n} \dim_{\mathrm{top}}(H^0(D)^{p^n}). \tag{4}$$

One shows that the above limit exists: indeed, one has the following

Theorem 6.6. (*i*) *Let $D \in \mathrm{Div}(C_p)$ be a divisor with $\deg(D) \geq 0$. Then the limit in (4) converges and one has $\mathrm{Dim}_{\mathbb{R}}(H^0(D)) = \deg(D)$.*

(*ii*) *The following Riemann–Roch formula holds*

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) - \mathrm{Dim}_{\mathbb{R}}(H^0(-D)) = \deg(D), \qquad \forall D \in \mathrm{Div}(C_p).$$

One can compare the above Riemann–Roch theorem with the tropical Riemann–Roch theorem of [2,6,9] and its variants. More precisely, let us consider an elliptic tropical curve C, given by a circle of length L. In this case, the structure of the group $\mathrm{Div}(C)/\mathcal{P}$ of divisor classes is inserted into an exact sequence of the form $0 \to \mathbb{R}/L\mathbb{Z} \to \mathrm{Div}(C)/\mathcal{P} \xrightarrow{\deg} \mathbb{Z} \to 0$ (*cf.* [9]). This sequence is very different from the split exact sequence associated with C_p and deduced from (3), *i.e.* $0 \to \mathbb{Z}/(p-1)\mathbb{Z} \to \mathrm{Div}(C_p)/\mathcal{P} \xrightarrow{\deg} \mathbb{R} \to 0$. The reason for this difference is due to the nature of the structure sheaf of C_p, when this sheaf is written in terms of the variable $u = \log \lambda$. This choice is dictated by the requirement that the periodicity condition $f(px) = f(x)$ becomes translation invariance by $\log p$. The condition for f of being piecewise affine in the parameter λ is expressed in the variable u by the piecewise vanishing of $\Delta' f$, where Δ' is the elliptic translation invariant operator $\Delta'(f) := \left(\frac{\partial}{\partial u}\right)^2 f - \frac{\partial}{\partial u} f$. In terms of the variable $\lambda = e^u$, this operator takes the form $\lambda^2 \left(\frac{\partial}{\partial \lambda}\right)^2$ and this fact explains why the structure sheaf of C_p is considered as tropical (in terms of the variable λ).

[2] At a point of discontinuity of the slopes one takes the max of the two values $|h_\pm(\lambda)|_p/\lambda$.

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