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Partial differential equations

Applications of Bourgain–Brézis inequalities to fluid mechanics and magnetism ☆

Applications des inégalités de Bourgain–Brézis à la mécanique des fluides et au magnétisme

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ABSTRACT

As a consequence of inequalities due to Bourgain–Brézis, we obtain local-in-time well-posedness for the two-dimensional Navier–Stokes equation with velocity bounded in spacetime and initial vorticity in bounded variation. We also obtain spacetime estimates for the magnetic field vector through improved Strichartz inequalities.

RÉSUMÉ

À partir d'inégalités de Bourgain–Brézis, nous démontrons le caractère bien posé localement dans le temps des équations de Navier–Stokes avec vitesse bornée en espace-temps et un tourbillon initial à variation bornée. Nous obtenons également des estimations en espace-temps pour le champ magnétique grâce à des inégalités de Strichartz améliorées.

1. Incompressible Navier–Stokes flow

Let $\mathbf{v}(x,t) \in \mathbb{R}^2$ be the velocity and $p(x,t)$ be the pressure of a fluid of viscosity $\nu > 0$ at position $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}$, governed by the incompressible two-dimensional Navier–Stokes equation:

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} - \nabla p, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1}$$

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When the viscosity coefficient ν degenerates to zero, (1) becomes the Euler equation. In two spatial dimensions, the vorticity of the flow is a scalar, defined by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$$

where we wrote $\mathbf{v} = (v_1, v_2)$. In the sequel, when we consider the Navier–Stokes equation, without loss of generality we set the viscosity coefficient $\nu = 1$.

The vorticity associated with the incompressible Navier–Stokes flow in two dimensions propagates according to the equation

$$\omega_t - \Delta\omega = -\nabla \cdot (\mathbf{v}\omega). \tag{2}$$

This follows from (1) by taking the curl of both sides. We express the velocity $\mathbf{v}$ in the Navier–Stokes equation in terms of the vorticity through the Biot–Savart relation:

$$\mathbf{v} = (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega). \tag{3}$$

This follows formally by differentiating $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$, and using that $\nabla \cdot \mathbf{v} = 0$.

Our theorem states:

Theorem 1. *Consider the two-dimensional vorticity equation (2) and an initial vorticity $\omega_0 \in W^{1,1}(\mathbb{R}^2)$ at time $t = 0$. If*

$$\|\omega_0\|_{W^{1,1}(\mathbb{R}^2)} \le A_0,$$

then there exists a unique solution to the vorticity equation (2) for all time $t \le t_0 = C/A_0^2$, such that

$$\sup_{t \le t_0} \|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \le cA_0.$$

Moreover, the solution ω depends continuously on the initial data ω_0, in the sense that if $\omega_0^{(i)}$ is a sequence of initial data converging in $W^{1,1}(\mathbb{R}^2)$ to ω_0, then the corresponding solutions $\omega^{(i)}$ to the vorticity equation (2) satisfy

$$\sup_{t \le t_0} \|\omega^{(i)}(\cdot, t) - \omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \to 0$$

as $i \to \infty$.

Finally, the velocity vector $\mathbf{v}$ defined by the Biot–Savart relation (3) solves the 2-dimensional incompressible Navier–Stokes equation (1), and satisfies

$$\sup_{t \le t_0} \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t \le t_0} \|\nabla\mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^2)} \le cA_0.$$

Via the Gagliardo–Nirenberg inequality, we can conclude from our theorem that

$$\sup_{0 \le t \le t_0} \|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} \le C, \quad 1 \le p \le 2.$$

In particular, this is enough to apply Theorem II of Kato [8] to express the velocity vector in the Navier–Stokes equation (1) in terms of the vorticity via the Biot–Savart relation displayed above.

In [7,8], it was proved that under the hypothesis that the initial vorticity is a measure, there is a global solution that is well-posed to the vorticity and Navier–Stokes equation; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brézis [4]. The velocity constructed then satisfies the estimate [8, (0.5)]:

$$\|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-\frac{1}{2}}, \; t \to 0. \tag{4}$$

In contrast, in Theorem 1 we have $\mathbf{v} \in L_t^\infty L_x^\infty$, $x \in \mathbb{R}^2$, though we are assuming that the initial vorticity has bounded variation, that is, its gradient is a measure.

The estimate (4) is indeed sharp as can be seen by the famous example of the *Lamb–Oseen vortex* [9], which consists of an initial vorticity $\omega_0 = \alpha_0\delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ with strength α_0. The constant α_0 is called the total circulation of the vortex. A unique solution to the vorticity equation (2) can be obtained by setting

$$\omega(x, t) = \frac{\alpha_0}{4\pi t} \mathrm{e}^{-\frac{|x|^2}{4t}}, \quad \mathbf{v}(x, t) = \frac{\alpha_0}{2\pi} \frac{(-x_2, x_1)}{|x|^2}\left(1 - \mathrm{e}^{-\frac{|x|^2}{4t}}\right).$$

It can be seen from the identities above that

$$\|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \sim ct^{-\frac{1}{2}}, \; t \to 0.$$

Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorem 1. Thus to get uniform-in-time, L^∞ space bounds all the way to $t = 0$, we need a stronger hypothesis and one such is vorticity in BV (bounded variation).

It is also helpful to further compare our result with that of Kato [8], who establishes in (0.4) of his paper that given that the initial vorticity is a measure, one has for the vorticity at further time

$$\|\nabla\omega(\cdot,t)\|_{L^q(\mathbb{R}^2)} \le ct^{\frac{1}{q}-\frac{3}{2}}, \quad 1 < q \le \infty.$$

In contrast, we obtain uniform-in-time bounds for $q = 1$, as opposed to singular bounds for $q > 1$ when $t \to 0$.

It is an open question whether there is a global version of Theorem 1 of our paper.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brézis [2,3]. A part of this proposition also holds in three dimensions. Recall that if $\mathbf{v}(x,t) \in \mathbb{R}^3$ is the velocity of a fluid at a point $x \in \mathbb{R}^3$ at time t, then the vorticity of $\mathbf{v}$ is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}.$$

Under the assumption that the flow is incompressible, the Biot–Savart relation reads

$$\mathbf{v} = (-\Delta)^{-1}(\nabla \times \boldsymbol{\omega}). \tag{5}$$

Proposition 2.

(a) *Consider the velocity* $\mathbf{v}$ *in three spatial dimensions. Assume that* $\mathbf{v}$ *satisfies the Biot–Savart relation* (5). *Then at any fixed time* t,

$$\|\mathbf{v}(\cdot,t)\|_{L^3(\mathbb{R}^3)} + \|\nabla\mathbf{v}(\cdot,t)\|_{L^{3/2}(\mathbb{R}^3)} \le C\|\nabla \times \boldsymbol{\omega}(\cdot,t)\|_{L^1(\mathbb{R}^3)}$$

where C *is a constant independent of* t, $\mathbf{v}$, *and* $\boldsymbol{\omega}$.

(b) *Consider the velocity* $\mathbf{v}$ *in two spatial dimensions. Assume that* $\mathbf{v}$ *satisfies the Biot–Savart relation* (3). *Then at any fixed time* t,

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\omega(\cdot,t)\|_{L^1(\mathbb{R}^2)}.$$

where C *is a constant independent of* t, $\mathbf{v}$ *and* ω.

We remark that in two dimensions, by the Poincaré inequality, it follows from $\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)} < \infty$, that $\mathbf{v}$ lies in $VMO(\mathbb{R}^2)$, i.e. has vanishing mean oscillation.

Proof of Proposition 2. Note that

$$\nabla \cdot (\nabla \times \boldsymbol{\omega}) = 0.$$

Thus we can immediately apply the result of Bourgain–Brézis [3] (see also [2,5,10]) to the Biot–Savart formula (5) and get the desired conclusions in part (a).

To consider the 2-dimensional flow, note that $(-\partial_{x_2}\omega, \partial_{x_1}\omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. In view of the two-dimensional Biot–Savart relation (3), we can then use the two-dimensional Bourgain–Brézis result [3], and we obtain (b). □

We note further that the proposition applies to both the Euler (inviscid) or the Navier–Stokes (viscous) flow.

Proof of Theorem 1. Now set K_t for the heat kernel in two dimensions, i.e.

$$K_t(x) = \frac{1}{4\pi t}\mathrm{e}^{-\frac{|x|^2}{4t}}.$$

Rewriting (2) as an integral equation for ω using Duhamel's theorem, where ω_0 is the initial vorticity, we have

$$\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]\,\mathrm{d}s \tag{6}$$

where $\mathbf{v}$ is given by (3).

We apply a Banach fixed point argument to the operator T given by

$$T\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]\,\mathrm{d}s, \tag{7}$$

where again $\mathbf{v}$ is given by (3). Let us set

$$E = \left\{ g \mid \sup_{0<t<t_0} \|g(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \le A \right\}.$$

We will first show that T maps E into itself, for t_0 chosen as in the theorem.

Differentiating (7) in the space variable once, we get

$$(T\omega(x,t))_x = K_t \star f_0(x) + \int_0^t \partial_x K_{t-s} \star \left(\mathbf{v}_x \omega\right) ds + \int_0^t \partial_x K_{t-s} \star \left(\mathbf{v}\omega_x\right) \mathrm{d}s.$$

Here we denote by f_0 the spatial derivative of the initial vorticity ω_0. Using Young's convolution inequality, we have

$$\|(T\omega(\cdot,t))_x\|_{L^1(\mathbb{R}^2)} \le \|f_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2} (\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)})\,\mathrm{d}s.$$

Now we apply Proposition 2(b) to each of the terms on the right. For the first term, we have, by Cauchy–Schwartz,

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \le C\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega\|_{L^2(\mathbb{R}^2)}.$$

The Gagliardo–Nirenberg inequality applies as $\omega \in E$ and so $\omega(\cdot,t) \in L^1(\mathbb{R}^2)$ and so,

$$\|\omega\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\omega\|_{L^1(\mathbb{R}^2)},$$

and to $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}$ we apply Proposition 2(b). Similarly, for the second term,

$$\|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)} \le \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)}.$$

Again we apply Proposition 2(b) to $\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}$. Hence in all we have,

$$\|(T\omega)_x\|_{L^1(\mathbb{R}^2)} \le \|f_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}\|\nabla\omega\|^2_{L^1(\mathbb{R}^2)}\,\mathrm{d}s.$$

Thus setting $\|f_0\|_{L^1(\mathbb{R}^2)} = \|\omega_0\|_{\dot{W}^{1,1}(\mathbb{R}^2)} \le A_0$, we get for $t \le t_0$ and since $\omega \in E$,

$$\|\nabla(T\omega)(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le A_0 + Ct_0^{1/2}A^2.$$

Next from Young's convolution inequality it follows from (7) that

$$\|T\omega(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le A_0 + \int_0^t (t-s)^{-1/2}\|\mathbf{v}\omega(\cdot,s)\|_1\,\mathrm{d}s.$$

But by Proposition 2(b) again,

$$\|\mathbf{v}\omega\|_1 \le \|\mathbf{v}\|_\infty\|\omega\|_1 \le cA^2.$$

Thus

$$\|T\omega(\cdot,t)\|_1 \le A_0 + c\,t^{1/2}A^2.$$

So, adding the estimates for $T\omega$ and $\nabla(T\omega)$, we have:

$$\sup_{t\le t_0}\|T\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \le 2A_0 + ct_0^{1/2}A^2.$$

By choosing A so that $A_0 = A/8$ and $t < t_0 = C/A_0^2$, we can assure that if $\omega \in E$, then

$$\sup_{t\le t_0}\|(T\omega)(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \le \frac{A}{2}.$$

Thus $T\omega \in E$, if $\omega \in E$. If we establish that T is a contraction, then we are done.

Next we observe that the estimates in Proposition 2(b) are linear estimates. That is

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_\infty + \|\nabla\mathbf{v}_1 - \nabla\mathbf{v}_2\|_2 \le C\|\omega_1 - \omega_2\|_{W^{1,1}(\mathbb{R}^2)}.$$

We easily can see from the computations above, that we have

$$\sup_{t\le t_0} \|T\omega_1 - T\omega_2\|_{W^{1,1}(\mathbb{R}^2)} \le CAt_0^{1/2} \sup_{t\le t_0} \|\omega_1-\omega_2\|_{W^{1,1}(\mathbb{R}^2)}.$$

By the choice of t_0, it is seen that T is a contraction. Thus using the Banach fixed-point theorem on E, we obtain our operator T has a fixed point and so the integral equation (6) has a solution in E. The remaining part of our theorem follows easily from Proposition 2(b). □

We note in passing an estimate in $\mathbb{R}^3$ from Proposition 2(a) above for the Navier–Stokes or the Euler flow:

$$\sup_{t>0} \|\mathbf{v}\|_{L^3(\mathbb{R}^3)} + \sup_{t>0} \|\nabla \mathbf{v}\|_{L^{3/2}(\mathbb{R}^3)} \le C \sup_{t>0} \|\nabla\times\boldsymbol{\omega}\|_{L^1(\mathbb{R}^3)}. \tag{8}$$

2. Magnetism

We next turn to our results on magnetism. We denote by $\mathbf{B}(x,t)$ and $\mathbf{E}(x,t)$ the magnetic and electric field vectors at $(x,t)\in\mathbb{R}^3\times\mathbb{R}$. Let $\mathbf{j}(x,t)$ denote the current density vector. The Maxwell equations imply

$$\nabla\cdot\mathbf{B}=0, \tag{9}$$

$$\partial_t\mathbf{B}+\nabla\times\mathbf{E}=0, \tag{10}$$

$$\partial_t\mathbf{E}-\nabla\times\mathbf{B}=-\mathbf{j}. \tag{11}$$

Differentiating (10) in t and using (11), together with the vector identity $\nabla\times(\nabla\times\mathbf{B})=\nabla(\nabla\cdot\mathbf{B})-\Delta\mathbf{B}$ and (9), one obtains an inhomogeneous wave equation for $\mathbf{B}$:

$$\mathbf{B}_{ttT}-\Delta\mathbf{B}=\nabla\times\mathbf{j}. \tag{12}$$

The right side of (12) satisfies the vanishing divergence condition

$$\nabla\cdot(\nabla\times\mathbf{j})=0$$

for any fixed time t. Thus an improved Strichartz estimate, namely Theorem 1 in [6], applies. We point out that the Bourgain–Brézis inequalities play a key role in the proof of Theorem 1 in [6]. We conclude easily:

Theorem 3. *Let* $\mathbf{B}$ *satisfy* (12) *and let* $\mathbf{B}(x,0)=\mathbf{B}_0$, $\partial_t\mathbf{B}(x,0)=\mathbf{B_1}$ *denote the initial data at time* $t=0$. *Let* $s,k\in\mathbb{R}$. *Assume* $2\le q\le\infty$, $2<\tilde q\le\infty$ *and* $2\le r<\infty$. *Let* (q,r) *satisfy the wave compatibility condition*

$$\frac{1}{q}+\frac{1}{r}\le\frac{1}{2},$$

and assume the following scale invariance condition is verified:

$$\frac{1}{q}+\frac{3}{r}=\frac{3}{2}-s=\frac{1}{\tilde q'}+1-k.$$

Then, for $\frac{1}{\tilde q}+\frac{1}{\tilde q'}=1$, *we have*

$$\|\mathbf{B}\|_{L^q_tL^r_x}+\|\mathbf{B}\|_{C^0_t\dot H^s_x}+\|\partial_t\mathbf{B}\|_{C^0_t\dot H^{s-1}_x}\le C(\|\mathbf{B}_0\|_{\dot H^s}+\|\mathbf{B}_1\|_{\dot H^{s-1}}+\|(-\Delta)^{k/2}(\nabla_x\mathbf{j})\|_{L^{\tilde q'}_tL^1_x}).$$

The main point in the theorem above is that we have L^1 norm in space on the right side.

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