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Differential geometry

Harmonic vector fields on Finsler manifolds

Champs de vecteurs harmoniques sur les variétés finslériennes

Alireza Shahi, Behroz Bidabad

Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran

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ABSTRACT

Let (M, F) be a compact boundaryless Finsler manifold. In this work, a sufficient condition for a vector field on (M, F) to be harmonic is obtained. Next the harmonic vector fields on Finsler manifolds are characterized and an upper bound for the first horizontal de Rham cohomology group of the sphere bundle SM is obtained.

RÉSUMÉ

Soit (M, F) une variété finslérienne compacte sans bord. Dans cet article, nous donnons une condition suffisante pour qu'un champ de vecteurs sur (M, F) soit harmonique. Par ailleurs, nous obtenons une caractérisation des champs de vecteurs harmoniques sur les variétés finslériennes, ainsi qu'une borne supérieure pour le premier groupe horizontal de cohomologie de de Rham du fibré en sphères SM.

1. Introduction

On a Riemannian manifold a harmonic vector field is a vector field for which the Hodge Laplacian operator of its corresponding 1-form vanishes. Harmonic vector fields are extensively studied by different authors in Riemannian geometry, regarding its applications, see for instance, [5,6,8] and [9]. Akbar–Zadeh has considered a natural horizontal Laplacian operator on SM and generalized the Bochner and Yano's techniques for Finsler manifolds cf., [1] and [2] page 241. Bao and Lackey in [4] construct a Laplace operator on differential forms and study harmonic forms on the underlying Finsler manifold.

Recently, the present authors have studied harmonic vector fields on Landsberg manifolds and found a necessary and sufficient condition for a vector field to be harmonic. Furthermore the harmonic vector fields on Landsberg manifolds are characterized, cf., [7].

In the present work we characterize the harmonic vector fields on Finsler manifolds through the two scalar functions I and J with significant geometric interpretations defined by (9) and (10), respectively. More intuitively, the following theorems are proved.

E-mail addresses: alirezashahi@aut.ac.ir (A. Shahi), bidabad@aut.ac.ir (B. Bidabad).

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Theorem 1. *Let (M, F) be a compact Finsler manifold without boundary and X a vector field on M. If $I \geq 0$ and $\frac{\partial X_i}{\partial y^j} = 0$, then X is a harmonic vector field.*

Theorem 2. *Let (M, F) be a compact Finsler manifold without boundary.*

- *If $J = 0$ for a harmonic vector field X, then X is parallel in Berwald connection.*
- *If $J > 0$, then there is no non-zero harmonic vector field.*

Theorem 3. *Let (M, F) be an n-dimensional compact Finsler manifold without boundary and $H^1_{\mathrm{dR}}(SM)$ the first horizontal de Rham cohomology group of SM.*

- *If $J = 0$ for all harmonic vector fields, then $\dim H^1_{\mathrm{dR}}(SM) \leq n$.*
- *If $J > 0$, then $H^1_{\mathrm{dR}}(SM) = 0$.*

We show also, if the complete lift of a harmonic vector field X coincides with its canonical lift on SM, then X is Killing.

2. Preliminaries

Let (M, F) be a Finsler manifold, $\pi : TM_0 \to M$ the bundle of non-zero tangent vectors and π^*TM the pullback bundle. We often use notations and terminologies of [2] and sometimes those of [3]. The covariant derivatives of Cartan and Berwald connections are denoted here by ∇ and D, respectively. Let $TTM_0 = HTM \oplus VTM$, be the Whitney sum, where HTM and VTM are *horizontal* and *vertical bundle* respectively and for any $\hat{X} \in TTM_0$, $\hat{X} = H\hat{X} + V\hat{X}$. If X and Y are sections of π^*TM, then the Cartan and Berwald connections are related by

$$D_{H\hat{X}}Y = \nabla_{H\hat{X}}Y + y^i(\nabla_i T)(X, Y), \tag{1}$$

$$D_{V\hat{X}}Y = V\hat{X}.Y, \tag{2}$$

where T is the *Cartan tensor* with the components $T_{kij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and $y = y^i \frac{\partial}{\partial x^i} \in T_xM$. The Finsler manifold (M, F) is called *Landsberg* manifold if the hv-curvature of the Cartan connection vanishes everywhere or equivalently $\nabla_0 T = 0$. By means of (1), we have $D_k g_{ij} = -2\nabla_0 T_{kij}$, where the index 0 denotes the contracted multiplication by y, hence $D_0 g_{ij} = y^k D_k g_{ij} = -2y^k \nabla_0 T_{kij} = 0$. Equation (2) is written locally $D_{\dot{\partial}_j} Y^i = \frac{\partial Y^i}{\partial y^j}$, which leads to the following Ricci identities for the Berwald connection:

$$D_l D_k X^i - D_k D_l X^i = X^r R^i_{rlk} - \frac{\partial X^i}{\partial y^r} R^r_{0lk}\,, \tag{3}$$

$$D_i D_k X_j - D_k D_i X_j = -X_l R^l_{jik} - \frac{\partial X_j}{\partial y^r} R^r_{0ik}\,, \tag{4}$$

where R^i_{rlk} is the hh-curvature tensor of the Berwald connection, cf., [2], page 19. Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on M. We associate with X the 1-form $\tilde{X}$ on SM defined by $\tilde{X} = X_i(z)\,\mathrm{d}x^i + \dot{X}_i\,\mathrm{d}y^i$, where $\dot{X}_i = \frac{1}{F}(D_0 X_i - y_i D_0(y^j X_j)F^{-2})$, cf., [2] page 231. The horizontal part of the associated 1-form on SM is denoted again in this paper by $X = X_i(z)\,\mathrm{d}x^i$, where $z \in SM$. The differential and co-differential operators of the horizontal 1-form X are given by

$$\mathrm{d}X = \frac{1}{2}(D_i X_j - D_j X_i)\,\mathrm{d}x^i \wedge \mathrm{d}x^j - \frac{\partial X_i}{\partial y^j}\,\mathrm{d}x^i \wedge \mathrm{d}y^j, \tag{5}$$

$$\delta X = -(\nabla^j X_j - X_j \nabla_0 T^j) = -g^{ij} D_i X_j\,, \tag{6}$$

respectively, where the co-differential operator δ is the formal adjoint of d, in the global scalar product over SM, cf. [2], pages 223 and 239. Let (M, F) be a compact Finsler manifold without boundary, the divergence formula for a horizontal 1-form $X = X_i(z)\,\mathrm{d}x^i$ is given by

$$\int_{SM} (\delta X)\eta = -\int_{SM} (g^{ij} D_i X_j)\eta = 0, \tag{7}$$

where η is a volume form on SM, cf., [2], page 66.

3. Harmonic vector fields on Finsler manifolds

Let (M, F) be a Finsler manifold. A vector field X on M is said to be *harmonic* if its corresponding horizontal 1-form on SM satisfies $\Delta X = \mathrm{d}\delta(X) + \delta\,\mathrm{d}(X) = 0$ or $\mathrm{d}X = 0$ and $\delta X = 0$, equivalently

$$D_i X_j = D_j X_i, \quad g^{ij} D_i X_j = 0, \quad \frac{\partial X_i}{\partial y^j} = 0\,. \tag{8}$$

Let us consider the two significant scalar function I and J defined respectively by

$$I = X_i\{g^{jk} D_k D_j X^i - (R^i_k + T^{it}_r R^r_{0kt})X^k - 2\big(D_j(X^k \nabla_0 T^{ji}_k) + 2X_j \nabla_0 T^{jk}_t \nabla_0 T^{it}_k - X_l D^i \nabla_0 T^l\big)\}, \tag{9}$$

$$J = (R_{jk} + T^t_{rj} R^r_{0kt})X^j X^k + 2X_t\{\nabla_0 T^l D^t X_l + X_l D_j \nabla_0 T^{tjl}\}. \tag{10}$$

On Landsberg manifolds, for the canonical lift of a harmonic vector field X on SM, these scalar functions reduce to $I = g^{jk} X_i D_k D_j X^i - R_{jk} X^j X^k$ and $J = R_{jk} X^j X^k$. Therefore on a Landsberg manifold of positive Ricci directional curvature (i.e. $R_{jk} X^j X^k > 0$), we have $J > 0$.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. A straightforward computation using the Ricci identities leads to the following formula. For more details see [7], Eq. (14):

$$D_j(X^k D_k X^j) - D_k(X^k D_j X^j) = (R_{jk} + T^t_{rj} R^r_{0kt})X^j X^k + D_k X^j D_j X^k - D_k X^k D_j X^j. \tag{11}$$

The first and second terms in the left-hand side are written respectively:

$$\begin{aligned} D_j(X^k D_k X^j) &= D_j(X^k D_k(g^{jl} X_l)) = D_j[X^k g^{jl} D_k X_l + X^k X_l D_k g^{jl}] \\ &= g^{jl} D_j(X^k D_k X_l) + D_j g^{jl} X^k D_k X_l + D_j(X^k X_l D_k g^{jl}) \\ &= g^{jl} D_j(X^k D_k X_l) - 2(\nabla_0 T^l)(X^k D_k X_l) - 2D_j(X^k X_l \nabla_0 T^{jl}_k), \end{aligned} \tag{12}$$

and

$$\begin{aligned} D_k(X^k D_j X^j) &= D_k(g^{kt} X_t D_j X^j) = g^{kt} D_k(X_t D_j X^j) + X_t D_j X^j (D_k g^{kt}) \\ &= g^{kt} D_k(X_t D_j X^j) - 2X_t D_j X^j \nabla_0 T^t. \end{aligned} \tag{13}$$

Plugging (12) and (13) into (11) yields

$$\begin{aligned} D_j(X^k D_k X^j) - D_k(X^k D_j X^j) &= g^{jl} D_j(X^k D_k X_l) - g^{kt} D_k(X_t D_j X^j) \\ &\quad - 2(\nabla_0 T^l)(X^k D_k X_l) - 2D_j(X^k X_l \nabla_0 T^{jl}_k) + 2X_t D_j X^j \nabla_0 T^t. \end{aligned} \tag{14}$$

Both first two terms on the right-hand side of (14) are divergence. In fact, considering $\omega_l = X^k D_k X_l$ as a 1-form, $D^l \omega_l$ is a divergence function. Integrating (11) over SM, using (14) and the divergence formula (7) we obtain:

$$\begin{aligned} \int_{SM} &[(R_{jk} + T^t_{rj} R^r_{0kt})X^j X^k + D_k X^j D_j X^k - D_k X^k D_j X^j \\ &+ 2\big((\nabla_0 T^l)(X^k D_k X_l) + D_j(X^k X_l \nabla_0 T^{jl}_k) - X_t D_j X^j \nabla_0 T^t\big)]\eta = 0. \end{aligned} \tag{15}$$

Consider the scalar function $\phi = X_i X^i$ on SM. Contracting its second Berwald covariant derivative yields

$$g^{jk} D_k D_j \phi = g^{jk}[D_k X_i D_j X^i + X_i D_k D_j X^i + D_k X^i D_j X_i + X^i D_k D_j X_i]. \tag{16}$$

Note that

$$\begin{aligned} g^{jk} X_i D_k D_j X^i &= g^{jk} X_i D_k[g^{it} D_j X_t + X_t D_j g^{it}] \\ &= g^{jk} X_i[g^{it} D_k D_j X_t + (D_k g^{it})(D_j X_t) + D_k(X_t D_j g^{it})] \\ &= g^{jk} X^t D_k D_j X_t + g^{jk}[X_i(D_k g^{it})(D_j X_t) + D_k(X_i X_t D_j g^{it}) - (D_k X_i) X_t D_j g^{it}] \\ &= g^{jk} X^t D_k D_j X_t + X_i(D^j g^{it})(D_j X_t) + g^{jk} D_k(X_i X_t D_j g^{it}) - X_t(D_k X_i) D^k g^{it} \\ &= g^{jk} X^t D_k D_j X_t + g^{jk} D_k(X_i X_t D_j g^{it}). \end{aligned} \tag{17}$$

On the other hand,

$$g^{jk}D_kX_iD_jX^i = D_kX_iD^kX^i = g^{jk}D_jX_iD_kX^i. \tag{18}$$

Therefore plugging (17) and (18) in (16) we get

$$g^{jk}D_kD_j\phi = 2X_ig^{jk}D_kD_jX^i + 2D^jX^iD_jX_i - g^{jk}D_k(X_iX_tD_jg^{it}). \tag{19}$$

A moment's thought shows that in the equation (19) the left-hand side and the last term of the right-hand side are divergence. Hence, by integration over SM and using the divergence formula (7), we obtain

$$\int_{SM} [X_ig^{jk}D_kD_jX^i + D^jX^kD_jX_k]\eta = 0. \tag{20}$$

Therefore by means of equations (15) and (20), we obtain

$$\begin{aligned}\int_{SM} [X_ig^{jk}D_kD_jX^i &- (R_{jk} + T^t_{rj}R^r_{0kt})X^jX^k + (D^jX^kD_jX_k - D_kX^jD_jX^k)\\ &+ D_kX^kD_jX^j - 2\big(X^kD_kX_l\nabla_0T^l + D_j(X^kX_l\nabla_0T^{jl}_k) - X_tD_jX^j\nabla_0T^t\big)]\eta = 0.\end{aligned} \tag{21}$$

On the other hand $(R_{jk} + T^t_{rj}R^r_{0kt})X^jX^k = (R^i_k + T^{it}_rR^r_{0kt})X_iX^k$, and $D_j(X^kX_l\nabla_0T^{jl}_k) = X_iD_j(X^k\nabla_0T^{ji}_k) + X_i\nabla_0T^{jli}(D_jX_l)$. Direct computations show that

$$D^jX^kD_jX_k - D_kX^jD_jX^k = \frac{1}{2}(D^jX^k - D^kX^j)(D_jX_k - D_kX_j) + 2X_i\nabla_0T^{it}_kD_tX^k,$$

and

$$\nabla_0T^{it}_kD_tX^k = D_tX_j\nabla_0T^{jit} - 2X_j\nabla_0T^{jk}_t\nabla_0T^{it}_k.$$

Replacing these terms in (21) leads to

$$\begin{aligned}\int_{SM} [X_i\{g^{jk}D_kD_jX^i &- (R^i_k + T^{it}_rR^r_{0kt})X^k - 2D_j(X^k\nabla_0T^{ji}_k) - 4X_j\nabla_0T^{jk}_t\nabla_0T^{it}_k\\ &- 2(\nabla_0T^l)(D^iX_l) + 2D_jX^j\nabla_0T^i\} + \frac{1}{2}(D^jX^k - D^kX^j)(D_jX_k - D_kX_j) + D_kX^kD_jX^j]\eta = 0.\end{aligned} \tag{22}$$

Consider the following norm on SM

$$\begin{aligned}\parallel (D_jX_k - D_kX_j) \parallel^2 &:= g^{lj}g^{tk}(D_lX_t - D_tX_l)(D_jX_k - D_kX_j)\\ &= (g^{lj}[D_lX^k - X_t(D_lg^{tk})] - g^{tk}[D_tX^j - X_l(D_tg^{lj})])[D_jX_k - D_kX_j]\\ &= [D^jX^k - X_tD^jg^{tk} - D^kX^j + X_lD^kg^{lj}][D_jX_k - D_kX_j]\\ &= [D^jX^k - D^kX^j + 2X_t\nabla_0T^{jtk} - 2X_l\nabla_0T^{klj}][D_jX_k - D_kX_j]\\ &= [D^jX^k - D^kX^j][D_jX_k - D_kX_j],\end{aligned} \tag{23}$$

and

$$\begin{aligned}D_kX^kD_jX^j &= (g^{ik}D_kX_i + X_iD_kg^{ik})(g^{jl}D_jX_l + X_lD_jg^{jl})\\ &= (g^{jl}D_jX_l)^2 - 2X_l\nabla_0T^lg^{ik}D_kX_i - 2X_i\nabla_0T^ig^{jl}D_jX_l + 4X_iX_l\nabla_0T^i\nabla_0T^l\\ &= (g^{jl}D_jX_l)^2 - 4X_l\nabla_0T^lg^{ik}D_kX_i + 4X_iX_l\nabla_0T^i\nabla_0T^l\\ &= (g^{jl}D_jX_l)^2 - 4X_l\nabla_0T^l(D_kX^k - X_iD_kg^{ik}) + 4X_iX_l\nabla_0T^i\nabla_0T^l\\ &= (g^{jl}D_jX_l)^2 - 4X_l\nabla_0T^l(D_kX^k) - 8X_iX_l\nabla_0T^i\nabla_0T^l + 4X_iX_l\nabla_0T^i\nabla_0T^l\\ &= (g^{jl}D_jX_l)^2 - 4X_l\nabla_0T^l(D_jX^j) - 4X_iX_l\nabla_0T^i\nabla_0T^l.\end{aligned} \tag{24}$$

Replacing (23) and (24) in (22) yields

$$\begin{aligned}\int_{SM} [X_i\{g^{jk}D_kD_jX^i &- (R^i_k + T^{it}_rR^r_{0kt})X^k - 2\big(D_j(X^k\nabla_0T^{ji}_k) + 2X_j\nabla_0T^{jk}_t\nabla_0T^{it}_k + (\nabla_0T^l)(D^iX_l) + D_jX^j\nabla_0T^i\\ &+ 2X_l\nabla_0T^i\nabla_0T^l\big)\} + \frac{1}{2} \parallel (D_jX_k - D_kX_j) \parallel^2 + (g^{jl}D_jX_l)^2]\eta = 0.\end{aligned} \tag{25}$$

Finally note that

$$D_jX^j\nabla_0T^i = g^{jk}D_jX_k\nabla_0T^i - 2X_k\nabla_0T^k\nabla_0T^i. \tag{26}$$

By similar computation, we get

$$X_i\nabla_0T^lD^iX_l + X_i\nabla_0T^iD^kX_k = g^{ik}D_k(\nabla_0T^lX_iX_l) - X_iX_lD^i(\nabla_0T^l). \tag{27}$$

The first term on the right-hand side of (27) is divergence. Plugging (26) and (27) into (25), we have

$$\int_{SM}[X_i\{g^{jk}D_kD_jX^i - (R^i_k + T^{it}_rR^r_{0kt})X^k - 2\big(D_j(X^k\nabla_0T^{ji}_k) + 2X_j\nabla_0T^{jk}_t\nabla_0T^{it}_k - X_lD^i\nabla_0T^l\big)\}$$
$$+\frac{1}{2}\parallel(D_jX_k - D_kX_j)\parallel^2 + (g^{jl}D_jX_l)^2]\,\eta = 0. \tag{28}$$

By the assumption $I \geq 0$ and the fact that the last two terms are non-negative, we conclude the last two terms are zero. Therefore, we obtain $D_jX_k - D_kX_j = 0$ and $g^{jl}D_jX_l = 0$. Thus X is a harmonic vector field on M. This completes the proof. □

Proposition 1. *Let (M, F) be a Finsler manifold and X a harmonic vector field on M. If the complete lift of X coincides with its canonical lift on SM, then X is Killing.*

Proof. Let $X = X^j(x)\frac{\partial}{\partial x^j}$ be a harmonic vector field on (M, F) and $X^j(x, y) = X^j \circ \pi(x, y)$ the canonical lift of X on SM, where $\pi : SM \to M$ is the projection map. The corresponding horizontal 1-form related to X on SM is given by $X = X_i(x, y)\,dx^i := g_{ij}X^j(x, y)\,dx^i$. By definition of harmonic vector fields we have $\frac{\partial X_i}{\partial y^k} = 0$. The vertical derivation leads to $T^k_{ij}X^j = 0$, and by contraction we have $\nabla_0(T^iX_i) = 0$. Let $\tilde{X} = X_i(z)\,dx^i + \dot{X}_i\,dy^i$, where $\dot{X}_i = \frac{1}{F}(D_0X_i - y_iD_0(y^jX_j)F^{-2})$, be the 1-form corresponding to the complete lift of X on SM. If it coincides with the corresponding 1-form associated with the canonical lift $X = X_i(x, y)\,dx^i$, then, by a proposition in [2] (page 242), we conclude that X is a conformal vector field. Therefore, we have $L_{\hat{X}}g_{ij} = 2\phi g_{ij}$, where ϕ is the potential function on M and $L_{\hat{X}}g_{ij} = D_iX_j + D_jX_i + D_0(\frac{\partial X_j}{\partial y^i})$ is the Lie derivative, cf., [2], page 227. Contracting with g^{ij} and using $\frac{\partial X_j}{\partial y^i} = 0$, yields $\phi = 0$, and hence X is Killing. This completes the proof. □

Remark 1. In Theorem 1, on a Landsberg manifold a vector field X on M with canonical lift, is harmonic if and only if $g^{jk}D_kD_jX^i = (R^i_k)X^k$ and $\frac{\partial X_i}{\partial y^k} = 0$, cf. [7].

Proof of Theorem 2. Let X be a harmonic vector field on M, then

$$\begin{aligned}(D_kX^j)(D_jX^k) &= (g^{jl}D_kX_l + (D_kg^{jl})X_l)(g^{kt}D_jX_t + (D_jg^{kt})X_t)\\ &= \parallel D_kX_l\parallel^2 + 2g^{jl}X_t(D_kX_l)(D_jg^{kt}) + X_tX_l(D_kg^{jl})(D_jg^{kt})\\ &= \parallel D_kX_l\parallel^2 - 4X_t(D_kX_l)(\nabla_0T^{lkt}) + 4X_tX_l(\nabla_0T^{jl}_k)(\nabla_0T^{kt}_j)\\ &= \parallel D_kX_l\parallel^2 - 4X_t(D_kX_l)(\nabla_0T^{lkt}) + 4\parallel X_i\nabla_0T^{ji}_k\parallel^2.\end{aligned} \tag{29}$$

On the other hand, we compute the following terms in (15):

$$\begin{aligned}(D_kX^k)(D_jX^j) &= (g^{kt}D_kX_t + X_tD_kg^{kt})(g^{jl}D_jX_l + X_lD_jg^{jl})\\ &= (X_tD_kg^{kt})(X_lD_jg^{jl}) = 4X_tX_l(\nabla_0T^t)(\nabla_0T^l).\end{aligned} \tag{30}$$

We also have

$$X_tD_jX^j\nabla_0T^t = (X_tX_lD_jg^{jl})(\nabla_0T^t) = -2X_tX_l\nabla_0T^t\nabla_0T^l. \tag{31}$$

Moreover, we have

$$D_j(X^kX_l\nabla_0T^{jl}_k) = D_j(X_tX_l\nabla_0T^{tjl}) = 2X_t\nabla_0T^{tjl}(D_jX_l) + X_lX_tD_j(\nabla_0T^{tjl}). \tag{32}$$

And finally

$$(\nabla_0T^l)(X^kD_kX_l) = X_t(\nabla_0T^l)(D^tX_l). \tag{33}$$

Substituting (29), (30), (31), (32) and (33) into (15) yields

$$\int_{SM} [(R_{jk} + T^t_{rj} R^r_{0kt}) X^j X^k + 2X_t\big((\nabla_0 T^l)(D^t X_l) + X_l D_j(\nabla_0 T^{tjl})\big)$$
$$+ \| D_k X_l \|^2 + 4 \| X_i \nabla_0 T^{ki}_j \|^2]\eta = 0. \tag{34}$$

By definition of J or (10), the above equation is written

$$\int_{SM} [J + \| D_k X_l \|^2 + 4 \| X_i \nabla_0 T^{ki}_j \|^2]\, \eta = 0. \tag{35}$$

If the vector field X satisfies $J = 0$, by means of (35) we get, $D_k X_l = 0$. By the definition of the harmonic vector fields, we have $D_{\dot{\partial}_i} X_j = 0$, thus X is horizontally and vertically parallel in Berwald connection. This completes the proof of the first assertion.

If X satisfies $J > 0$, then it contradicts (35) and therefore there is not any non-zero harmonic vector field satisfying $J > 0$. This completes the proof of the second assertion. □

Remark 2. In Theorem 2, on a Landsberg manifold (M, F), the assumptions $J = 0$ and $J > 0$ for a harmonic vector field X with canonical lift on SM, reduce to $R_{jk} X^j X^k = 0$ and $R_{jk} X^j X^k > 0$, respectively, cf. [7].

Proof of Theorem 3. Let X be a harmonic vector field on (M, F) endowed with a Berwald connection that satisfies $J = 0$. By the definition of the harmonic vector fields and Theorem 2, we have $D_i X_j = D_{\dot{\partial}_i} X_j = 0$. Therefore, the 1-form $X = X_i(z)\, dx^i$ is parallel with respect to the Berwald connection on SM. It is well known that a parallel 1-form on a manifold is determined by its value at a point on the underlying manifold. Hence the dimension of the vector space of parallel horizontal 1-forms is at most equal to n, the dimension of its restriction to the horizontal 1-forms on the cotangent space $T^*_x(SM)$. Therefore, for the restriction of the first de Rham cohomology group $H^1_{\mathrm{dR}}(SM)$ of these 1-forms, we have $\dim H^1_{\mathrm{dR}}(SM) \le n$. This completes the proof of the first assertion.

Next, by means of Theorem 2, there is no non-zero harmonic vector field X that satisfies $J > 0$. Hence SM has no non-trivial harmonic 1-form. Thus $H^1_{\mathrm{dR}}(SM) = 0$ and this completes the proof of the second assertion. □

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