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Analytic geometry

A singular Demailly–Păun theorem

Un théorème de Demailly–Păun singulier

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A B S T R A C T

We give a numerical characterization of the Kähler cone of a possibly singular compact analytic variety that is embedded in a smooth ambient space.

R É S U M É

On donne une caractérisation numérique du cône kählérien d'une variété analytique compacte qui est plongée dans un espace ambiant lisse.

1. Introduction

The classical Nakai–Moishezon ampleness criterion (see, e.g., [8] and references therein) characterizes ample line bundles on a projective variety as those that have positive intersection against all subvarieties. This was later extended to $\mathbb{R}$-divisors by Campana and Peternell [1]. In a groundbreaking paper, Demailly and Păun [7] proved a vast generalization of this result, which holds for all real $(1,1)$ classes on a compact Kähler manifold. More precisely, they proved that the Kähler cone of a compact Kähler manifold is one of the connected components of the positive cone, consisting of classes that have positive intersection against all analytic subvarieties. Very recently, a new proof of this theorem was obtained by combining the main result of our previous work [3] with a result of Chiose [2].

In this note, we prove an extension of the Demailly–Păun theorem [7] to singular varieties that are embedded in a smooth ambient space. A $(1,1)$ class on the variety is just taken to be the restriction of a $(1,1)$ class from the ambient space, and such a class is Kähler if it is so in a neighborhood of the variety inside the ambient space. This is in fact equivalent to the more intrinsic definition of a Kähler class on a compact analytic space as given for example in [12], as shown by Păun [9], and this allows us to avoid discussing these more technical notions. With these observations in mind, our main theorem is the following:

Theorem 1.1. *Let (M,ω) be a smooth (but possibly noncompact and incomplete) Kähler manifold, and $E \subset M$ be a compact analytic subvariety. Let α be a closed smooth real $(1,1)$ form on M such that*

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$$\int_V \alpha^k \wedge \omega^{\dim V - k} > 0, \tag{1.1}$$

for all positive-dimensional irreducible analytic subvarieties $V \subset E$, and for all $1 \leqslant k \leqslant \dim V$. Then there exist an open neighborhood U of E in M and a smooth function $\varphi : U \to \mathbb{R}$ such that $\alpha + \sqrt{-1}\partial\bar\partial\varphi$ is a Kähler metric on U. If M is an open subset of the regular locus of some projective variety, then the inequalities

$$\int_V \alpha^{\dim V} > 0, \tag{1.2}$$

for all V as above suffice to reach the same conclusion.

This theorem answers a question that was posed to us by R.J. Conlon and H.-J. Hein, in relation to their paper [5] (see also [4, 1.3.5]). Applications of this result to the study of the Kähler cone of asymptotically conical Calabi–Yau manifolds will appear in a forthcoming revision of [5].

The main tools we use are the Demailly–Păun theorem itself, for smooth compact Kähler manifolds, and our recent theorem [3] that shows that the non-Kähler locus of a nef and big class on a compact complex manifold equals the null locus of the class. The idea is to work by induction on the dimension on E (as in [7]), and to prove the result by working on a resolution of singularities (as in [3]). This way we avoid any technical discussion of currents on singular analytic spaces.

In future work, we hope to address the extension of the Demailly–Păun theorem [7] as well as the main result of our previous work [3] to general compact Kähler (reduced and irreducible) analytic spaces.

2. Proof of Theorem 1.1

This section contains the proof of Theorem 1.1.

Clearly we may assume that no component of E is zero-dimensional, since for those the result is trivial.

Let us first assume that E is irreducible and 1-dimensional. Let $\nu : \tilde M \to M$ be an embedded resolution of singularities of $E \subset M$, so that $\tilde M$ is smooth, connected and Kähler, and the proper transform $\tilde E$ of E is a smooth compact Riemann surface. We will also write $\nu : \tilde E \to E$ for the induced map, so that $\nu^*\alpha$ is a smooth closed real $(1,1)$ form with $\int_{\tilde E} \nu^*\alpha > 0$. Therefore the class $[\nu^*\alpha]$ on $\tilde E$ is Kähler, and we can find a smooth function ψ on $\tilde E$ such that $\nu^*\alpha + \sqrt{-1}\partial\bar\partial\psi > 0$ on $\tilde E$. It is elementary to find an open neighborhood $\tilde U$ of $\tilde E$ in $\tilde M$ and a smooth extension of ψ to $\tilde U$ (still denoted by ψ) such that $\nu^*\alpha + \sqrt{-1}\partial\bar\partial\psi > 0$ on $\tilde U$ (see, e.g., [9, Lemme 1, p. 416]). Note that $U = \nu(\tilde U)\backslash E_{\text{sing}}$ is an open neighborhood of E_{reg} inside M, but in general it is not the case that $\nu(\tilde U)$ is an open neighborhood of E inside M, because it may "pinch off" near E_{sing}. Furthermore, even if $\nu(\tilde U)$ happens to be an open neighborhood of E, the pushforward function $\nu_*\psi$ is not well defined wherever different branches of $\tilde E$ come together under the map ν. Therefore, we need to work a bit harder to achieve our goal.

Let $\{p_1, \dots, p_N\} \subset \tilde E$ be the exceptional locus of ν intersected with $\tilde E$, so that $\{\nu(p_1), \dots, \nu(p_N)\}$ equals the singular set of E. For each point p_j, we add to ψ a function of the form $\varepsilon\theta(z)\log|z - p_j|$, where $\varepsilon > 0$ is small enough, where $z = (z_1, \dots, z_{\dim M})$ are local coordinates for $\tilde M$ near p_j, and θ is a smooth cutoff function supported in a small neighborhood of p_j in $\tilde M$, so that we obtain a new function $\tilde\psi$, which is smooth away from the p_j's and goes to $-\infty$ there, and such that $\nu^*\alpha + \sqrt{-1}\partial\bar\partial\tilde\psi$ is a Kähler current on $\tilde U$.

Then the smooth function $\hat\psi = \nu_*\tilde\psi$ on U satisfies $\alpha + \sqrt{-1}\partial\bar\partial\hat\psi > 0$, but we are not done yet because U does not contain the singular points of E. Let $\{\nu(p_1), \dots, \nu(p_k)\}$ be all the singular points of E (so $k \leqslant N$), and fix charts U_j for M centered at $\nu(p_j)$ for $1 \leqslant j \leqslant k$, with coordinates so that each U_j is the Euclidean ball of radius 2. Call U'_j the Euclidean ball of radius 1 in these coordinates, and let A be the minimum of $\hat\psi$ on the compact set

$$\bigcup_{j=1}^k \overline{(\partial U'_j) \cap U},$$

which is a finite number because $\hat\psi$ is smooth there. Choose a large constant $B > 0$ such that on each U_j we have $\alpha + B\sqrt{-1}\partial\bar\partial|z|^2 > 0$. On $U \cap U_j$, then we have that $\hat\psi$ and $B|z|^2 + A - B - 1$ are both strictly α-plurisubharmonic, with $\hat\psi$ approaching $-\infty$ at the center of the ball U_j, and with $\hat\psi > B|z|^2 + A - B - 1$ on a neighborhood of $(\partial U'_j) \cap U$. If $\widetilde{\max}$ denotes a regularized maximum function (see, e.g., [6, I.5.18]), then

$$\psi_g = \widetilde{\max}(\hat\psi, B|z|^2 + A - B - 1)$$

is smooth and strictly α-plurisubharmonic on $U_j \cap U$, it equals $\hat\psi$ in a neighborhood of $(\partial U'_j) \cap U$, and it equals $B|z|^2 + A - B - 1$ as we approach the origin. Therefore the function ψ_g trivially glues to $\hat\psi$ outside U'_j, and we can extend it to be equal to $B|z|^2 + A - B - 1$ in a small neighborhood of the origin in U_j. Repeating this construction for all j, and gluing each of them to $\hat{\psi}$, we finally obtain an open neighborhood $\overline{U}$ of E in M and a smooth function φ on $\overline{U}$ such that $\alpha+\sqrt{-1}\partial\overline{\partial}\varphi$ is a Kähler metric on $\overline{U}$, as required.

Next, we assume that E has pure dimension 1, but need not be irreducible anymore. Then, writing $E=\cup_j E_j$ with E_j irreducible, we can apply the result to each E_j and obtain U_j, φ_j as above, and "glue" them all together using [9, Lemme, p. 419], and obtain the desired Kähler potential φ on some neighborhood $\overline{U}$ of E.

We now deal with the general case, by induction on $\dim E$ (which is by definition the max of the dimensions of the irreducible components of E). The base of the induction is what we have just proved. For the induction step, let $\dim E=n$ and assume the result holds in all dimensions $<n$. As we just did, it is enough to prove the theorem in the case when E is irreducible, since if there are several components then we work on each one separately, and in the end glue the resulting metrics as before. So we will assume that E is irreducible. Take $\nu:\tilde{M}\to M$ to be an embedded resolution of singularities of $E\subset M$, obtained as a composition of blowups with smooth centers, so that $\tilde{M}$ is smooth and Kähler, and the proper transform $\tilde{E}$ of E is smooth.

Then $\nu^*\alpha$ is a smooth closed real $(1,1)$ form on $\tilde{E}$, and we claim that its class $[\nu^*\alpha]$ on $\tilde{E}$ is nef. If we assume that M is an open subset of the regular locus of some projective variety, then this holds because we have $\int_V(\nu^*\alpha)^{\dim V}\geqslant 0$ for all positive-dimensional irreducible subvarieties V in $\tilde{E}$ (using (1.1)), and so [7, Theorem 4.5(ii)] gives that the class $[\nu^*\alpha]$ on $\tilde{E}$ is nef. However, in our general setup (where there may be no projective compactification), to use [7, Theorem 4.3(ii)], we would have to check instead that

$$\int_V \nu^*\alpha^k\wedge\tilde{\omega}^{\dim V-k}\geqslant 0,$$

for all positive-dimensional irreducible subvarieties $V\subset\tilde{E}$, for some Kähler form $\tilde{\omega}$ on $\tilde{E}$ and for all $1\leqslant k\leqslant\dim V$, and it does not seem easy to check this directly. Instead, we argue as follows. We have

$$\int_{\tilde{E}} \nu^*(\alpha^k\wedge\omega^{\dim E-k})>0,$$

for $1\leqslant k\leqslant\dim E$, because $\nu:\tilde{E}\to E$ is a modification, and using (1.1). Since the class $[\nu^*\omega]$ is nef on $\tilde{E}$, we can find Kähler classes on $\tilde{E}$ arbitrarily close to it, and therefore there exists a Kähler metric $\tilde{\omega}$ on $\tilde{E}$ such that

$$\int_{\tilde{E}} \nu^*\alpha^k\wedge\tilde{\omega}^{\dim E-k}>0, \tag{2.1}$$

for $1\leqslant k\leqslant\dim E$. Now for $t\geqslant 0$ sufficiently large, the class $[\nu^*\alpha+t\tilde{\omega}]$ is Kähler on $\tilde{E}$. Let t_0 be the minimum value of t such that the class $[\nu^*\alpha+t\tilde{\omega}]$ is nef on $\tilde{E}$, and suppose for a contradiction that $t_0>0$. By definition, the class $[\nu^*\alpha+t_0\tilde{\omega}]$ is not Kähler on $\tilde{E}$. Thanks to [7, Theorem 0.1], there exists a positive-dimensional irreducible analytic subvariety $V\subset\tilde{E}$, such that

$$\int_V (\nu^*\alpha+t_0\tilde{\omega})^{\dim V}=0, \tag{2.2}$$

since if we had strict positivity for all such V then the class $[\nu^*\alpha+t_0\tilde{\omega}]$ would be Kähler. Also V must be properly contained in $\tilde{E}$, because we have

$$\int_{\tilde{E}} (\nu^*\alpha+t_0\tilde{\omega})^{\dim E}>0,$$

by (2.1). Then $\nu(V)$ is an irreducible analytic subvariety of E (possibly a point), of dimension strictly less than $\dim E$, and with the same positivity property (1.1), so by induction we can find an open neighborhood W of $\nu(V)$ in M and a smooth function η on W such that $\alpha+\sqrt{-1}\partial\overline{\partial}\eta>0$. Therefore, in the open neighborhood $\nu^{-1}(W)$ of V the smooth function $\nu^*\eta$ satisfies $\nu^*\alpha+\sqrt{-1}\partial\overline{\partial}(\nu^*\eta)\geqslant 0$. Since $\tilde{\omega}$ is Kähler on $\tilde{E}$ and $t_0>0$, this implies that

$$\int_V (\nu^*\alpha+t_0\tilde{\omega})^{\dim V}>0,$$

contradicting (2.2). Therefore we must have $t_0\leqslant 0$, and so the class $[\nu^*\alpha]$ is indeed nef on $\tilde{E}$.

This proves our claim that the class $[\nu^*\alpha]$ is nef on $\tilde{E}$, and since

$$\int_{\tilde{E}} (\nu^*\alpha)^{\dim E}=\int_E \alpha^{\dim E}>0,$$

by (1.1), we can apply [7, Theorem 2.12] and see that this class is also big, i.e. it contains a Kähler current $\nu^*\alpha + \sqrt{-1}\partial\overline{\partial}\psi$, which we may assume to have analytic singularities thanks to Demailly's regularization theorem (see [7, Theorem 3.2]). Also, if $V \not\subset \mathrm{Exc}(\nu) \cap \tilde{E}$, then $\nu(V)$ is an irreducible subvariety of E of the same dimension as V, and $\nu : V \to \nu(V)$ is bimeromorphic and so we have $\int_V (\nu^*\alpha)^{\dim V} = \int_{\nu(V)} \alpha^{\dim V} > 0$, thanks to assumption (1.1). This means that the null locus of the class $[\nu^*\alpha]$ on $\tilde{E}$ is contained in $\mathrm{Exc}(\nu)$, and so using [3, Theorem 1.1], we may choose ψ to be smooth on $\tilde{E}\backslash\mathrm{Exc}(\nu)$. We use [7, Lemma 2.1] to obtain a quasi-plurisubharmonic function with nontrivial analytic singularities along $\mathrm{Exc}(\nu)$, and add a small multiple of it to ψ, to obtain a function $\tilde{\psi}$ that is smooth on $\tilde{E}\backslash\mathrm{Exc}(\nu)$ and goes to $-\infty$ along $\mathrm{Exc}(\nu)$, and such that $\nu^*\alpha + \sqrt{-1}\partial\overline{\partial}\tilde{\psi}$ is a Kähler current on $\tilde{E}$ with analytic singularities along $\mathrm{Exc}(\nu)$.

As in the first part of the proof of [3, Theorem 3.2], up to modifying $\tilde{\psi}$ slightly (maintaining its same properties), we can find an extension $\tilde{\psi}'$ to an open neighborhood $\tilde{U}$ of $\tilde{E}\backslash\mathrm{Exc}(\nu)$ in $\tilde{M}$.

Here are some details for this construction (see [3] for full details). By a resolution of singularity arguments, we construct a modification $\mu : \hat{M} \to \tilde{M}$, which is a composition of blowups with smooth centers, such that $\mu(\mathrm{Exc}(\mu))$ is equal to $\mathrm{Exc}(\nu)$, such that the proper transform $\hat{E}$ of $\tilde{E}$ is smooth, and the pullback under μ of the ideal sheaf on $\tilde{E}$, which defines the singularities of the Kähler current $\nu^*\alpha + \sqrt{-1}\partial\overline{\partial}\tilde{\psi}$ is a principal ideal, supported along a simple normal crossings divisor, which is the restriction to $\hat{E}$ of a simple normal crossings divisor on $\hat{M}$ (which is equal to $\mathrm{Exc}(\mu)$), which has normal crossings with $\hat{E}$. We then cover $\hat{E}$ by finitely many coordinate charts $\{W_j\}$ for $\hat{M}$. To the pullback $\mu^*\tilde{\psi}$ we add a small multiple of $\sqrt{-1}\partial\overline{\partial}\log|s|_h^2$, where s defines $\mathrm{Exc}(\mu)$ (and h is chosen suitably), to obtain a strictly $\mu^*\nu^*\alpha$-plurisubharmonic function Ψ on $\hat{E}$, with analytic singularities as before (in particular, smooth away from $\mathrm{Exc}(\mu)$). For each j, we then extend $\Psi|_{W_j \cap \hat{E}}$ to a function ψ_j on W_j in an elementary fashion, still preserving strict $\mu^*\nu^*\alpha$-plurisubharmonicity. Then we use a gluing procedure inspired by a classical method of Richberg [10] (see, e.g., [11, Lemma 3.3]), but with the extra difficulty that now the functions ψ_j have poles. Nevertheless, arguing exactly as in [3, Proof of Theorem 3.2], we can obtain an open neighborhood U_1 of $\hat{E}$ in $\hat{M}$ and a strictly $\mu^*\nu^*\alpha$-plurisubharmonic function $\tilde{\Psi}$ on U_1, which restricts to Ψ on $\hat{E}$, and is smooth on $\hat{E}\backslash\mathrm{Exc}(\mu)$.

Here we highlight that since $\hat{E}$ is a complex submanifold of a complex manifold, constructing this extension $\tilde{\Psi}$ on an open neighborhood U_1 of $\hat{E}$ would be standard by Richberg [10] if Ψ were smooth (or even just continuous) on $\hat{E}$. On the other hand, if the singularities of Ψ were completely arbitrary, then such an extension would not be possible in general. The key property that saves us here is that the singular locus of Ψ is the intersection with $\hat{E}$ of a simple normal crossings divisor, $\mathrm{Exc}(\mu)$, in the ambient space $\hat{M}$.

Then we take $\tilde{U} = \mu(U_1)$, and $\tilde{\psi}' = \mu_*\tilde{\Psi}$, which are as required. In particular, $\tilde{U}$ is an open neighborhood of $\tilde{E}\backslash\mathrm{Exc}(\nu)$ in $\tilde{M}$, and $\tilde{\psi}'$ is strictly $\nu^*\alpha$-plurisubharmonic, and it is smooth on $\tilde{U}\backslash\mathrm{Exc}(\nu)$.

On the open set $U = \nu(\tilde{U})\backslash E_{\mathrm{sing}}$ (which is a neighborhood of E_{reg} in M), we have the smooth function $\hat{\psi} = \nu_*\tilde{\psi}'$ with $\alpha + \sqrt{-1}\partial\overline{\partial}(\nu_*\tilde{\psi}')$ a smooth Kähler metric there, and with $\nu_*\tilde{\psi}'$ approaching $-\infty$ along E_{sing}. Now E_{sing} is a subvariety of M of dimension strictly less than n, with the same positivity property (1.1), so by induction we can find an open neighborhood W of E_{sing} in M and a smooth function $\hat{\varphi}$ on W with $\alpha + \sqrt{-1}\partial\overline{\partial}\hat{\varphi} > 0$ on W. We may also assume that $\hat{\varphi}$ is defined on a slightly larger open set, so that it is smooth up to ∂W.

If we let A be the minimum of $\hat{\psi}$ on the compact set $\overline{(\partial W) \cap U}$ and B be the maximum of $\hat{\varphi}$ on the same set, then $\hat{\psi} > \hat{\varphi} + A - B - 1$ holds on a neighborhood of $(\partial W) \cap U$. Then

$$\psi_g = \widetilde{\max}(\hat{\psi}, \hat{\varphi} + A - B - 1)$$

is smooth and strictly α-plurisubharmonic on $U \cap W$, equal to $\hat{\psi}$ near $(\partial W) \cap U$, and equal to $\hat{\varphi} + A - B - 1$ as we approach E_{sing}. Therefore ψ_g trivially glues to $\hat{\psi}$ outside W, and we can extend it to be equal to $\hat{\varphi} + A - B - 1$ in a neighborhood of E_{sing}. In this way, we obtain an open neighborhood $\overline{U}$ of E in M and a smooth function φ on $\overline{U}$ such that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi$ is a Kähler metric on $\overline{U}$, as required.

Lastly, the statement in the projective case follows from the Kähler one exactly as in [7], by choosing ω to be the curvature form of a very ample line bundle L on the projective variety which contains M as an open subset, and observing that

$$\int_V \alpha^k \wedge \omega^{\dim V - k} = \int_{V \cap H_1 \cap \cdots \cap H_{\dim V - k}} \alpha^k,$$

for generic members $H_1, \ldots, H_{\dim V - k}$ of the linear system $|L|$, so that $V \cap H_1 \cap \cdots \cap H_{\dim V - k}$ is an irreducible subvariety of dimension k.

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References

[1] F. Campana, T. Peternell, Algebraicity of the ample cone of projective varieties, J. Reine Angew. Math. 407 (1990) 160–166.
[2] I. Chiose, The Kähler rank of compact complex manifolds, J. Geom. Anal. (2015), http://dx.doi.org/10.1007/s12220-015-9564-z, in press.
[3] T. Collins, V. Tosatti, Kähler currents and null loci, Invent. Math. 202 (3) (2015) 1167–1198.
[4] R.J. Conlon, H.-J. Hein, Asymptotically conical Calabi–Yau manifolds, I, Duke Math. J. 162 (15) (2013) 2855–2902.
[5] R.J. Conlon, H.-J. Hein, Asymptotically conical Calabi–Yau manifolds, III, preprint, arXiv:1405.7140.
[6] J.-P. Demailly, Complex analytic and differential geometry, available on the author's webpage.
[7] J.-P. Demailly, M. Păun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. of Math. (2) 159 (3) (2004) 1247–1274.
[8] S.L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. (2) 84 (1966) 293–344.
[9] M. Păun, Sur l'effectivité numérique des images inverses de fibrés en droites, Math. Ann. 310 (3) (1998) 411–421.
[10] R. Richberg, Stetige streng pseudokonvexe Funktionen, Math. Ann. 175 (1968) 257–286.
[11] P.A.N. Smith, Smoothing plurisubharmonic functions on complex spaces, Math. Ann. 273 (3) (1986) 397–413.
[12] J. Varouchas, Kähler spaces and proper open morphisms, Math. Ann. 283 (1) (1989) 13–52.