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On the distribution of the product of correlated normal random variables

Sur la distribution exacte du produit de variables aléatoires normales corrélées

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A R T I C L E I N F O

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A B S T R A C T

We solve a problem that has remained unsolved since 1936 – the exact distribution of the product of two correlated normal random variables. As a by-product, we derive the exact distribution of the mean of the product of correlated normal random variables.

R É S U M É

Dans cette Note, on résout un problème, posé depuis 1936, sur la distribution exacte du produit de variables aléatoires normales corrélées. Comme résultat supplémentaire, on déduit la distribution exacte de la moyenne du produit de variables aléatoires normales corrélées.

1. Introduction

Let (X, Y) denote a bivariate normal random vector with means (μ_1, μ_2), variances (σ_1^2, σ_2^2), and correlation coefficient ρ. The exact distribution of $Z = XY$ has been studied since 1936 [4]. But no one has been able to derive a closed form expression for the exact probability density function (PDF) of Z for cases other than $\rho = 0, 1$.

Craig [4] derived the moment generating function and the first four moments of Z for the general case $-1 < \rho < 1$. Craig [4] gave a closed-form expression for the exact PDF of Z when $\rho = 0$. This involved the modified Bessel function of the second kind of order zero. He expressed the exact PDF in the general case as the difference of two integrals and as an infinite series of modified Bessel functions. Haldane [6] derived the cumulant generating function, the first four moments and the first four cumulants of Z in the general case. Aroian [1] showed that the distribution of Z in the general case can be approximated by a normal distribution when μ_1/σ_1 and μ_2/σ_2 are large. Aroian [1] also showed that the distribution of Z when $\rho = 0$ can be approximated by a Gram–Charlier type-A series. Hayya and Ferrara [7] proposed approximating

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the distribution of Z in the general case by gamma and normal distributions. Aroian et al. [2] expressed the PDF of Z in the general case as an integral. They gave closed form expressions for the PDF for the cases: i) $\mu_1 = \mu_2 = 0$, $\rho = 1$; ii) $\mu_1/\sigma_1 = \mu_2/\sigma_2$, $\rho = 1$; iii) $\rho = 0$. Meeker et al. [10, pp. 129–144] provided numerical tables for the distribution of Z. Bandi and Connaughton [3] stated that the PDF of Z cannot "generally be expressed in terms of elementary functions, except in the case of zero correlation" [3, p. 5].

The applications of the distribution of $Z = XY$ (when X and Y are correlated normal random variables) have been too numerous and date back to 1936. Some recent applications have included: product confidence limits for indirect effects [9]; statistics of Lagrangian power in two-dimensional turbulence [3]; statistical mediation analysis [8, Chapter 4].

The aim of this short note is to derive the first closed-form expression for the exact PDF of $Z = XY$ applicable for any $-1 < \rho < 1$. As a by-product, we also derive closed-form expressions for the exact PDF of the mean $\overline{Z} = (1/n)(Z_1 + Z_2 \cdots + Z_n)$ when $Z_1, Z_2, \ldots, Z_n$ are independent and identical copies of Z. The mean of the product of correlated normal random variables arises in many areas. For instance, Ware and Lad [11] show that the sum of the product of correlated normal random variables arises in "Differential Continuous Phase Frequency Shift Keying" (a problem in electrical engineering). They propose an approximation to determine the distribution of the sum.

The main results of this short note are given in Section 2. Their proofs are sketched in Section 3. Throughout, we assume without loss of generality that $\sigma_1 = \sigma_2 = 1$. We also assume that $\mu_1 = \mu_2 = 0$. A future work is to extend the main results to non-zero means.

2. Main results

Our main results are Theorems 2.1 and 2.2. Theorem 2.1 derives the exact PDF of the product of two correlated normal random variables. Theorem 2.2 derives the exact PDF of the mean of the product of correlated normal random variables.

Theorem 2.1. *Let (X, Y) denote a bivariate normal random vector with zero means, unit variances and correlation coefficient ρ. Then, the PDF of $Z = XY$ is*

$$f_Z(z) = \frac{1}{\pi\sqrt{1-\rho^2}} \exp\left[\frac{\rho z}{1-\rho^2}\right] K_0\left(\frac{|z|}{1-\rho^2}\right) \tag{1}$$

for $-\infty < z < \infty$, where $K_0(\cdot)$ denotes the modified Bessel function of the second kind of order zero.

Theorem 2.2. *Let $Z_1, Z_2, \ldots, Z_n$ be independent and identical random variables distributed according to (1). Let $\overline{Z}$ denote their sample mean. Then, the PDF of $\overline{Z}$ for $n \geq 2$ can be expressed as*

$$f_{\overline{Z}}(z) = \frac{n^{n/2} 2^{-n/2}}{\Gamma(n/2)} |z|^{n/2-1} \exp\left(\frac{\beta-\gamma}{2} z\right) W_{0, \frac{1-n}{2}}\left((\beta+\gamma)|z|\right) \tag{2}$$

or

$$f_{\overline{Z}}(z) = \frac{n^{(n+1)/2} 2^{(1-n)/2} |z|^{(n-1)/2}}{\sqrt{\pi\left(1-\rho^2\right)}\,\Gamma(n/2)} \exp\left(\frac{\beta-\gamma}{2} z\right) K_{\frac{1-n}{2}}\left(\frac{\beta+\gamma}{2}|z|\right) \tag{3}$$

for $-\infty < z < \infty$, where $\beta = n/(1-\rho)$, $\gamma = n/(1+\rho)$, $W_{\nu,\mu}(\cdot)$ denotes the Whittaker's W function and $K_\mu(\cdot)$ denotes the modified Bessel function of the second kind of order μ.

3. Proofs

The proofs of the two theorems require the following lemma.

Lemma 3.1. *Let*

$$I_{a,b}(t) = \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}xt)\,\mathrm{d}x}{\sqrt{(x-\mathrm{i}a)(x+\mathrm{i}b)}},$$

where $\mathrm{i} = \sqrt{-1}$, $a > 0$, $b > 0$ and $-\infty < t < \infty$ are fixed real numbers. Then,

$$I_{a,b}(t) = 2\exp\left(\frac{b-a}{2} t\right) K_0\left(\frac{a+b}{2}|t|\right).$$

Proof. Setting $x = y + \frac{\mathrm{i}}{2}(a-b)$ into the integrand of $I_{a,b}(t)$, we obtain

$$\begin{aligned} I_{a,b}(t) &= \exp\left(\frac{b-a}{2}t\right)\int_{-\infty}^{\infty}\exp(\mathrm{i}ty)\left[y^2+\left(\frac{a+b}{2}\right)^2\right]^{-1/2}\mathrm{d}y \\ &= 2\exp\left(\frac{b-a}{2}t\right)\int_{0}^{\infty}\cos(ty)\left[y^2+\left(\frac{a+b}{2}\right)^2\right]^{-1/2}\mathrm{d}y \\ &= 2\exp\left(\frac{b-a}{2}t\right)K_0\left(\frac{a+b}{2}\,|\,t\,|\right), \end{aligned}$$

where the final step follows by equation (3.754.3) in [5]. The proof is complete. □

Proof of Theorem 2.1. By equation (10) in [4], the characteristic function of Z is

$$E\left[\exp(\mathrm{i}tZ)\right] = \left[1-\mathrm{i}(1+\rho)t\right]^{-1/2}\left[1+\mathrm{i}(1-\rho)\right]^{-1/2}. \tag{4}$$

Hence, by the inversion theorem, the PDF of Z can be expressed as

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\left[1-\mathrm{i}(1+\rho)t\right]^{-1/2}\left[1+\mathrm{i}(1-\rho)\right]^{-1/2}\mathrm{d}t. \tag{5}$$

The result follows by applying Lemma 3.1 to calculate the integral in (5). □

Proof of Theorem 2.2. By (4), the characteristic function of $\overline{Z}$ can be expressed as

$$\begin{aligned} E\left[\exp\left(\mathrm{i}t\overline{Z}\right)\right] &= E\left[\exp\left(\mathrm{i}\frac{t}{n}\sum_{i=1}^{n}Z_i\right)\right] \\ &= \left\{E\left[\exp\left(\mathrm{i}\frac{t}{n}Z_1\right)\right]\right\}^n \\ &= \left[1-\mathrm{i}(1+\rho)t/n\right]^{-n/2}\left[1+\mathrm{i}(1-\rho)t/n\right]^{-n/2}. \end{aligned}$$

Hence, by the inversion theorem, the PDF of $\overline{Z}$ can be expressed as

$$\begin{aligned} f_{\overline{Z}}(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\mathrm{i}zt)\left[1-\mathrm{i}(1+\rho)t/n\right]^{-n/2}\left[1+\mathrm{i}(1-\rho)t/n\right]^{-n/2}\mathrm{d}t \\ &= \frac{n^n}{2\pi\left(1-\rho^2\right)^{n/2}}\int_{-\infty}^{\infty}\exp(-\mathrm{i}zt)\left[\gamma-\mathrm{i}t\right]^{-n/2}\left[\beta+\mathrm{i}t\right]^{-n/2}\mathrm{d}t \\ &= \frac{n^{n/2}2^{-n/2}}{\Gamma(n/2)}\,|\,z\,|^{n/2-1}\exp\left(\frac{\beta-\gamma}{2}z\right)W_{0,\frac{1-n}{2}}\left((\beta+\gamma)\,|\,z\,|\right), \end{aligned}$$

the same as (2), where the final step follows by equation (3.384.9) in [5]. The equivalent expression in (3) follows by using the fact that

$$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}K_\mu\left(\frac{z}{2}\right).$$

The proof is complete. □

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