C. R. Acad. Sci. Paris, Ser. I 354 (2016) 283–289

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Numerical analysis

A slack approach to reduced-basis approximation and error estimation for variational inequalities

Approximation bases réduites et estimateur d'erreur pour les inéquations variationnelles via une approche par variable d'écart

Zhenying Zhang, Eduard Bader, Karen Veroy

Aachen Institute for Advanced Study in Computational Engineering Science (AICES), RWTH Aachen University, Germany

ARTICLE INFO

Article history:
Received 22 July 2015
Accepted 21 October 2015
Available online 4 February 2016

Presented by Philippe G. Ciarlet

ABSTRACT

We propose a novel approach for computing certified reduced-basis approximations to solutions to variational inequalities of the first kind. The proposed approach has three components: (*i*) a slack-based approximation for the solution; (*ii*) a primal approximation for the Lagrange multiplier; and (*iii*) *a posteriori* bounds for the error in the combined primal-slack variable approximation. The strict feasibility of the primal-slack approximations leads to two significant improvements upon existing methods. First, it provides *a posteriori* error bounds that are significantly sharper than existing bounds. Second, it enables a full offline–online computational decomposition, in which the online cost to compute the error bound is completely independent of the dimension of the original (high-dimensional) problem. Our numerical results allow us to compare the performance of the proposed and existing approaches.

RÉSUMÉ

Nous proposons une nouvelle approche pour le calcul d'approximations bases réduites pour des inégalités variationnelles du premier type. Les trois principales composantes de cette approche sont : (i) une approximation utilisant des variables d'écart pour la solution ; (ii) une approximation primale pour le multiplicateur de Lagrange ; (iii) une borne supérieure *a posteriori* de l'erreur sur la solution approchée. La stricte faisabilité de l'approximation primale par variable d'écart nous permet deux améliorations majeures par rapport aux méthodes existantes. La première est de pouvoir borner, *a posteriori*, de façon précise, l'erreur commise. La deuxième est l'utilisation d'une décomposition hors ligne/en ligne grâce à laquelle le coût de calcul de cette borne reste complètement indépendant de la (grande) dimension originale du problème. Les résultats numériques présentent une comparaison des performances entre cette nouvelle approche et les méthodes existantes.

E-mail addresses: zhang@aices.rwth-aachen.de (Z. Zhang), bader@aices.rwth-aachen.de (E. Bader), veroy@aices.rwth-aachen.de (K. Veroy).

http://dx.doi.org/10.1016/j.crma.2015.10.024

1. Preliminaries

We begin by presenting below several (different but equivalent) formulations of our problem: a general (standard) variational inequality of the first kind (1), a mixed formulation (2), and a mixed complementarity problem (3). Such problems occur in a wide variety of applications: contact mechanics, flow through porous media, lubrication, finance, optimal control, and so on. In this note, our goal is to develop *a posteriori* error bounds for reduced basis (RB) approximations [7] to the solutions to such problems. We are thus concerned with the parameterised forms of these problems; however, the parameter has been omitted in (1)–(3) for both succinctness and clarity.

Variational inequality (VI)

Find $u \in \mathcal{K}$ s.t.

$$\langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \forall\, v \in \mathcal{K}, \tag{1a}$$

$$\mathcal{K} := \{v \in \mathcal{V} \mid \langle Bv, q\rangle \leq \langle g, q\rangle \;\forall\, q \in \mathcal{M}\}. \tag{1b}$$

Mixed Problem (MP)

Find $u \in \mathcal{V},\ \lambda \in \mathcal{M}$ s.t.

$$\langle Au, v\rangle + \langle Bv, \lambda\rangle = \langle f, v\rangle \quad \forall\, v \in \mathcal{V}, \tag{2a}$$

$$\langle Bu, q-\lambda\rangle \leq \langle g, q-\lambda\rangle \quad \forall\, q \in \mathcal{M}. \tag{2b}$$

Mixed Complementarity Problem (MC)

Find $u \in \mathcal{V},\ \lambda \in \mathcal{Q}$ s.t.

$$\langle Au, v\rangle + \langle Bv, \lambda\rangle = \langle f, v\rangle \quad \forall\, v \in \mathcal{V}, \tag{3a}$$

$$\langle g, q\rangle - \langle Bu, q\rangle \geq 0 \quad \forall\, q \in \mathcal{M}, \tag{3b}$$

$$\lambda \geq 0, \tag{3c}$$

$$\langle g - Bu, \lambda\rangle = 0. \tag{3d}$$

Slack Problem (SP)

Find $s \in \mathcal{M}'$ s.t.

$$\langle \tilde{A}s, \sigma - s\rangle \geq \langle \tilde{f}, \sigma - s\rangle \quad \forall\, \sigma \in \mathcal{M}', \tag{4}$$

where $\tilde{A} := B^{-T} A B^{-1}$,

$\tilde{f} := B^{-T} A B^{-1} g - B^{-T} f$.

In the above, $\mathcal{V}$ and $\mathcal{Q}$ are two separable Hilbert spaces with inner products, $(\cdot,\cdot)_{\mathcal{V}}$, $(\cdot,\cdot)_{\mathcal{Q}}$, and associated norms, $\|\cdot\|_{\mathcal{V}} = \sqrt{(\cdot,\cdot)_{\mathcal{V}}}$, $\|\cdot\|_{\mathcal{Q}} = \sqrt{(\cdot,\cdot)_{\mathcal{Q}}}$, respectively. Here, $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$, is a bounded Lipschitz domain. The corresponding dual spaces are denoted by $\mathcal{V}'$ and $\mathcal{Q}'$, and we denote a general duality pairing as $\langle\cdot,\cdot\rangle$.

Let $\mathcal{D} \subset \mathbb{R}^p$ be a prescribed p-dimensional, compact parameter set, and let μ be a parameter in $\mathcal{D}$. The parametrised operator, $A(\mu): \mathcal{V} \to \mathcal{V}'$, is then the induced linear map of a continuous, coercive bilinear form, $a(\cdot,\cdot;\mu): \mathcal{V} \times \mathcal{V} \to \mathbb{R}$; the associated continuity and coercivity constants are

$$\gamma(\mu) \equiv \sup_{w\in\mathcal{V}} \sup_{v\in\mathcal{V}} \frac{\langle A(\mu)w, v\rangle}{\|w\|_{\mathcal{V}}\|v\|_{\mathcal{V}}} < \infty, \qquad \alpha(\mu) \equiv \inf_{v\in\mathcal{V}} \frac{\langle A(\mu)v, v\rangle}{\|v\|^2_{\mathcal{V}}} > 0. \tag{5}$$

The parameterised linear functional, $f(\mu)$, is assumed to be bounded, $f(\mu) \in \mathcal{V}'$ for all $\mu \in \mathcal{D}$.

We also assume that $\mathcal{K}(\mu)$, a non-empty closed convex subset of $\mathcal{V}$, is given by (1b). Here, $B: \mathcal{V} \to \mathcal{Q}'$ is the induced linear map of a continuous bilinear form, $b(\cdot,\cdot): \mathcal{V} \times \mathcal{Q} \to \mathbb{R}$; and $g(\mu)$ is a bounded linear functional, $g(\mu) \in \mathcal{Q}'$. The set $\mathcal{M}$ is a proper positive cone of the space $\mathcal{Q}$, and $\mathcal{M}'$, the corresponding dual cone in the dual space $\mathcal{Q}'$, is defined as

$$\mathcal{M}' := \{\, \zeta \in \mathcal{Q}' \mid \langle \zeta, q\rangle \geq 0, \quad \forall\, q \in \mathcal{M}\,\}.$$

In this note, we consider only the simplest case, in which B is parameter independent and bijective; the latter ensures that B^{-1} is well defined. The case where B is only surjective will be addressed in a future publication. For more details on the above definitions and problem formulations, see (for example) [3,5].

It is well known that the efficiency of the RB method relies on an offline–online computational strategy that requires that all bilinear and linear forms depend affinely on the parameter μ. We thus assume that

$$A(\mu) = \sum_{k=1}^{Q_a} \Theta_a^k(\mu) A^k, \qquad f(\mu) = \sum_{k=1}^{Q_f} \Theta_f^k(\mu) f^k, \qquad g(\mu) = \sum_{k=1}^{Q_g} \Theta_g^k(\mu) g^k, \tag{6}$$

where $Q_a,\ Q_f,\ Q_g \in \mathbb{N}$ are assumed to be small, and the parameter-dependent coefficient functions, $\Theta_a^k(\mu)$, $\Theta_f^k(\mu)$, and $\Theta_g^k(\mu)$, are continuous over the parameter set $\mathcal{D}$. We also assume that the mappings $A^k: \mathcal{V} \to \mathcal{V}'$, $f^k: \mathcal{V} \to \mathbb{R}$, and $g^k: \mathcal{Q} \to \mathbb{R}$ are parameter independent, linear, and continuous.

We assume that $\mathcal{V}$, $\mathcal{Q}$ are finite-element (FE) spaces with dimension $\mathcal{N}_{\mathcal{V}}$, $\mathcal{N}_{\mathcal{Q}}$, and basis functions Φ_i, Ψ_j, respectively: $\mathcal{V} = \mathrm{span}\{\Phi_i,\ 1 \leq i \leq \mathcal{N}_{\mathcal{V}}\}$, $\mathcal{Q} = \mathrm{span}\{\Psi_j,\ 1 \leq j \leq \mathcal{N}_{\mathcal{Q}}\}$. We further assume that the basis functions, Ψ_j, are non-negative; the convex cone, $\mathcal{M}$, is then given by

$$\mathcal{M} = \mathrm{span}_+\{\Psi_j\} = \{q \in \mathcal{Q} \mid q = \sum_{j=1}^{\mathcal{N}_{\mathcal{Q}}} \underline{q}_j \Psi_j \text{ and } \underline{q} \in \mathbb{R}_+^{\mathcal{N}_{\mathcal{Q}}}\}, \qquad \text{where } \mathbb{R}_+ := \{c \in \mathbb{R} \mid c \geq 0\}.$$

We now briefly summarise some results on the equivalence of the first three problem formulations (1), (2), and (3), as well as on the existence, uniqueness, and boundedness of their solution. (We defer the discussion of (4) to Section 2.2.) We begin with

Lemma 1.1. *Let $A : \mathcal{V} \to \mathcal{V}'$ satisfy (5) and let $\mathcal{K}$ be a non-empty, closed, convex set of $\mathcal{V}$. Suppose further that there exists a constant $\beta_0 > 0$ such that*

$$\beta \equiv \inf_{q \in \mathcal{Q}} \sup_{v \in \mathcal{V}} \frac{\langle Bv, q\rangle}{\|q\|_{\mathcal{Q}} \|v\|_{\mathcal{V}}} \geq \beta_0 > 0. \tag{7}$$

Then (2) has a unique solution. Furthermore, if (u, λ) solves (2), then u solves (1).

Proof. This statement follows directly from Thm. 2.1 of [6] and Thm. 2.1 of [2]. See also [4]. □

Lemma 1.2. *Under the assumptions of Lemma 1.1, there exists a unique solution $(u, \lambda) \in \mathcal{V} \times \mathcal{Q}$ to (3). Furthermore, the pair (u, λ) solves (2) if and only if it solves (3).*

Proof. We need only to show that (*i*) (2) implies (3b) and (3d), and that (*ii*) (3) implies (2b). To show (*i*), we take $q = q' + \lambda$ in (2b) to obtain (3b). Furthermore, (3d) follows from taking $q = 2\lambda$ and $q = 0$ in (2b). To prove (*ii*), we subtract (3d) from (3b) to obtain (2b). The desired result then follows from the definition of $\mathcal{M}$ and Lemma 1.1. □

We now develop certified RB approximations to solutions to parameterised variational inequalities.

2. Approximation and error estimation: a slack approach

We present in this section an approach that provides *a posteriori* error bounds that are sharper and (computationally) more efficient than existing bounds. Our approach has three components: (*i*) a primal approximation, $\lambda_n^{\rm pr}(\mu)$, for the Lagrange multiplier, $\lambda(\mu)$; (*ii*) a slack-based approximation, $u_n^{\rm sl}(\mu)$, for the solution, $u(\mu)$; and (*iii*) a posteriori bounds for the error in the approximation, $(u_n^{\rm sl}(\mu), \lambda_n^{\rm pr}(\mu))$.

2.1. The primal problem

We begin with the first component. The primal approach, summarised in (8)–(11), was first introduced in [4] in the context of the reduced basis method for elliptic VIs. Following [4], we let the approximation spaces, $\mathcal{V}_n \subset \mathcal{V}$ and $\mathcal{Q}_n \subset \mathcal{Q}$, be given by (8) and (9), respectively, where the basis functions, φ_i, ψ_j are assumed to be linearly independent. In standard RB methods, the basis functions are typically derived — via a Gram–Schmidt orthogonalisation procedure — from snapshots of the solutions at different parameter values. However, there are three additional issues that must be considered in our problem setting.

RB Approximation for the primal problem

$$\mathcal{V}_n = \mathrm{span}\{\varphi_i \in \mathcal{V},\ 1 \leq i \leq n_{\mathcal{V}}\}$$
$$= \mathrm{span}\{u(\mu_j), t_k,\ 1 \leq j \leq n,\ 1 \leq k \leq n_{\rm stab}\} \tag{8}$$
$$\mathcal{Q}_n = \mathrm{span}\{\psi_j \in \mathcal{M},\ 1 \leq j \leq n_{\mathcal{Q}}\}$$
$$= \mathrm{span}\{\lambda(\mu_k),\ 1 \leq k \leq n\} \tag{9}$$
$$\mathcal{M}_n = \{q_n \in \mathcal{Q}_n \,|\, q_n = \sum_{j=1}^{n_{\mathcal{Q}}} \underline{q}_{n,j}\psi_j \text{ and } \underline{q}_n \in \mathbb{R}_+^{n_{\mathcal{Q}}}\} \tag{10}$$

Find $(u_n^{\rm pr}, \lambda_n^{\rm pr}) \in \mathcal{V}_n \times \mathcal{Q}_n$ s.t.

$$\langle Au_n^{\rm pr}, v_n\rangle + \langle Bv_n, \lambda_n^{\rm pr}\rangle = \langle f, v_n\rangle \quad \forall\, v_n \in \mathcal{V}_n, \tag{11a}$$
$$\langle g, q_n\rangle - \langle Bu_n^{\rm pr}, q_n\rangle \geq 0 \quad \forall\, q_n \in \mathcal{Q}_n, \tag{11b}$$
$$\lambda_n^{\rm pr} \geq 0, \tag{11c}$$
$$\langle g - Bu_n^{\rm pr}, \lambda_n^{\rm pr}\rangle = 0. \tag{11d}$$

RB Approximation for the slack problem

$$\Sigma_n := \mathrm{span}\{\zeta_i \in \mathcal{M}',\ 1 \leq i \leq n_{\mathcal{S}}\}$$
$$= \mathrm{span}\{s(\mu_i),\ 1 \leq i \leq n\} \tag{12}$$
$$\mathcal{S}_n := \{\sigma_n \in \Sigma_n \,|\, \sigma_n = \sum_{i=1}^{n_{\mathcal{Q}}} \underline{\sigma}_{n,i}\zeta_i \text{ and } \underline{\sigma}_n \in \mathbb{R}_+^{n_{\mathcal{Q}}}\} \tag{13}$$

Find $s_n \in \mathcal{S}_n$ s.t.

$$\langle \tilde{A} s_n, \sigma_n - s_n\rangle \geq \langle \tilde{f}, \sigma_n - s_n\rangle \quad \forall\, \sigma_n \in \mathcal{S}_n. \tag{14a}$$

Then

$$u_n^{\rm sl} := B^{-1}(g - s_n). \tag{14b}$$

First, it is known that $\mathcal{V}_n$ and $\mathcal{Q}_n$ cannot be chosen arbitrarily. Following [8] and [4], we thus state:

Corollary 2.1. *Suppose there exists a constant $\beta_0 > 0$ such that*

$$\beta_n \equiv \inf_{q_n \in \mathcal{Q}_n} \sup_{v_n \in \mathcal{V}_n} \frac{\langle B v_n, q_n \rangle}{\|q_n\|_{\mathcal{Q}} \|v_n\|_{\mathcal{V}}} \geq \beta_0 > 0, \tag{15}$$

for $n \in \mathbb{N}$. Then there exists a unique solution (u_n, λ_n) to (11).

Proof. The result directly follows from Lemma 1.1. □

We thus derive the basis functions, φ_i, from (i) snapshots, $u(\mu_j)$, of the solution at different values of the parameter; and (ii) stabilising functions, t_k, that ensure (15) is satisfied. We refer the reader to [8,4] for more details on the choice of the stabilising functions. Second, note from (9) that we require $\psi_j \in \mathcal{M}$: we thus cannot use Gram–Schmidt orthogonalisation, and instead directly use the snapshots, $\lambda(\mu_k)$, of the Lagrange multiplier [4]. Finally, we let

$$\mathcal{K}_n(\mu) := \{ v_n \in \mathcal{V}_n \mid \langle B v_n, q_n \rangle \leq \langle g(\mu), q_n \rangle \; \forall \, q_n \in \mathcal{M}_n \}, \tag{16}$$

and observe that (in general) $\mathcal{K}_n(\mu) \not\subset \mathcal{K}(\mu)$ since the inequality in (16) holds only in $\mathcal{M}_n$, not $\mathcal{M}$. With these considerations, the *primal* RB approximation, $(u_n^{\mathrm{pr}}(\mu), \lambda_n^{\mathrm{pr}}(\mu))$, is then computed via (11).

2.2. The slack problem

The second component of our proposed approach is based on an auxiliary *slack* problem. We introduce a slack variable, $s(\mu) \in \mathcal{M}' \subset Q'$, that satisfies

$$\langle s(\mu), q \rangle = \langle g(\mu) - B u(\mu), q \rangle \quad \forall \, q \in \mathcal{Q} \qquad \Longleftrightarrow \qquad u(\mu) = B^{-1}(g(\mu) - s(\mu)), \tag{17}$$

where the latter follows since B is by assumption bijective. By substituting $u(\mu)$ in (17) into (1), we then readily obtain the slack problem in (4). It thus follows that (1) and (4) are equivalent by virtue of (17).

We are now equipped to develop our slack-based RB approximation, $u_n^{\mathrm{sl}}(\mu)$, to the solution, $u(\mu)$. Let the slack approximation space, $\mathcal{S}_n \subset \mathcal{M}'$, be given by (13), where the basis functions, ζ_i, are assumed to be linearly independent. Similar to the approximation of the Lagrange multiplier in the primal approach, the basis functions, ζ_i, are derived from non-orthogonalised snapshots, $s(\mu_i)$, of the slack variable. We then compute an RB approximation, $s_n(\mu)$, to the slack variable, $s(\mu)$, via (14a). The *slack-based* RB approximation, $u_n^{\mathrm{sl}}(\mu)$, to the solution, $u(\mu)$, is then given by (14b).

Given the primal and slack RB approximations, our *combined* primal-slack approximation to the solution, $(u(\mu), \lambda(\mu))$, is then $(u_n^{\mathrm{sl}}(\mu), \lambda_n^{\mathrm{pr}}(\mu))$, where $u_n^{\mathrm{sl}}(\mu)$ is obtained from (14), and $\lambda_n^{\mathrm{pr}}(\mu))$ from (11).

We now remark on some fundamental differences between the primal approximation, (11), and the slack-based approximation, (14). We note that although (1) and (4) are equivalent, the approximations (11) and (14) are not. If we assume that (1) corresponds to an optimisation problem (i.e., A is coercive), then the RB approximation, (11), is obtained through an *optimise-then-reduce* approach: we derive the optimality system of (1) to obtain (3), and subsequently introduce an RB approximation, (11). On the other hand, (14) is derived using a *transform-then-reduce* approach: we re-write the problem in terms of the slack variable to obtain (4), and then *directly* introduce an RB approximation (14).

The approximations resulting from the two approaches thus differ in two ways. First, the primal approach provides an RB approximation to the Lagrange multiplier, $\lambda(\mu)$, whereas the slack-based approach does not. Second, the slack-based approximation $u_n^{\mathrm{sl}}(\mu)$ is feasible in the sense of (3b): $u_n^{\mathrm{sl}}(\mu) \in \mathcal{K}(\mu)$, since

$$s_n(\mu) = g(\mu) - B u_n^{\mathrm{sl}}(\mu) \geq 0 \qquad \forall \, \mu \in \mathcal{D}; \tag{18}$$

this follows from (14b) and the fact that $s_n(\mu) \in \mathcal{S}_n \subset \mathcal{M}'$. On the other hand, the approximation, $u_n^{\mathrm{pr}}(\mu)$, is not necessarily feasible: $u_n^{\mathrm{pr}}(\mu) \in \mathcal{K}_n(\mu)$ does not imply that $u_n^{\mathrm{pr}}(\mu) \in \mathcal{K}(\mu)$. As we shall see shortly, the latter has important consequences in the development of sharp *a posteriori* error bounds.

To derive error bounds for the approximation, $(u_n^{\mathrm{sl}}(\mu), \lambda_n^{\mathrm{pr}}(\mu))$, we first define the residual, $r_n \in \mathcal{V}'$ as

$$r_n(v; \mu) := \langle f(\mu), v \rangle - \langle A(\mu) u_n^{\mathrm{sl}}(\mu), v \rangle - \langle B v, \lambda_n^{\mathrm{pr}}(\mu) \rangle \qquad \forall \, v \in \mathcal{V}. \tag{19}$$

We then state

Proposition 2.1. *For any $\mu \in \mathcal{D}$, let $\hat{\alpha}(\mu)$ be a lower bound to $\alpha(\mu)$, $\hat{\gamma}(\mu)$ be an upper bound to $\gamma(\mu)$,*

$$d_1(\mu) := \frac{\|r_n(\,\cdot\,; \mu)\|_{\mathcal{V}'}}{2\hat{\alpha}(\mu)}, \qquad \textit{and} \qquad d_2(\mu) := \frac{\langle s_n(\mu), \lambda_n^{\mathrm{pr}}(\mu) \rangle}{\hat{\alpha}(\mu)}. \tag{20}$$

The errors can then be bounded by

$$\|u(\mu) - u_n^{\rm sl}(\mu)\|_{\mathcal{V}} \le \Delta_n^u(\mu) := d_1(\mu) + \sqrt{d_1^2(\mu) + d_2(\mu)}, \tag{21a}$$

$$\|\lambda(\mu) - \lambda_n^{\rm pr}(\mu)\|_{\mathcal{Q}} \le \Delta_n^\lambda(\mu) := \frac{1}{\beta}\left(\|r_n(\mu)\|_{\mathcal{V}'} + \hat{\gamma}(\mu)\Delta_n^u(\mu)\right). \tag{21b}$$

Proof. We omit here the parameter μ for the sake of brevity. From (3a), (5), and (19), we have $\hat{\alpha}\|u - u_n^{\rm sl}\|_{\mathcal{V}}^2 \le r_n(u - u_n^{\rm sl}) - \langle B(u - u_n^{\rm sl}), \lambda - \lambda_n\rangle$. We then note from (3d) and (18) that

$$\langle B(u - u_n^{\rm sl}), \lambda\rangle = \langle g - Bu_n^{\rm sl}, \lambda\rangle \ge 0, \quad \text{and} \quad \langle B(u - u_n^{\rm sl}), \lambda_n^{\rm pr}\rangle \le \langle g, \lambda_n^{\rm pr}\rangle - \langle g - s_n, \lambda_n^{\rm pr}\rangle = \langle s_n, \lambda_n^{\rm pr}\rangle. \tag{22}$$

It thus follows that $\hat{\alpha}\|u - u_n^{\rm sl}\|_{\mathcal{V}}^2 \le \|r_n\|_{\mathcal{V}'}\|u - u_n^{\rm sl}\|_{\mathcal{V}} + \langle s_n, \lambda_n\rangle$. Using (20) and solving the quadratic inequality, we obtain (21a). The remaining result (21b) follows from (19), (3d), (5), and (7) (or directly from the result and proof of Prop. 1.3 (Sec. II.1) in [1]). See also [4]. □

We now briefly remark on the differences between our proposed bounds above, and existing bounds in [4]. We note that in our proposed approach, the feasibility (see (18)) of the slack-based RB approximation, $u_n^{\rm sl}(\mu)$, was used in (22) to obtain the term $\langle s_n(\mu), \lambda_n^{\rm pr}(\mu)\rangle$. Comparison with (3d) and (14b) indicates that this term will be close to zero, i.e., the approximations, $s_n(\mu)$ and $\lambda_n^{\rm pr}(\mu)$, will be near-orthogonal.

On the other hand, in [4], only the *primal* approximations, $(u_n^{\rm pr}(\mu), \lambda_n^{\rm pr}(\mu))$, were used. As we already pointed out, $u_n^{\rm pr}(\mu)$ is in general not in $\mathcal{K}(\mu)$. This lack of feasibility thus requires [4] to introduce further approximations that (*i*) require knowledge of where $u_n^{\rm pr}(\mu)$ violates the constraint, (3b), and, perhaps more importantly, (*ii*) subsequently miss out on the near-orthogonality of the term $\langle g - Bu_n^{\rm pr}(\mu), \lambda_n^{\rm pr}(\mu)\rangle$. The former leads to *less efficient* bounds due to (online) computations that scale with $\mathcal{N}$ (the dimension of the FE space), while the latter leads to *pessimistic* bounds due to the loss of sharpness.

We shall return to these comments in the next section on numerical results.

3. Numerical results

We now consider a model problem presented in [4], and compare the performance of the slack approach to the primal-only approach. We thus consider a one-dimensional elastic rope over a rigid obstacle with domain $\Omega = (0, 1)$; we impose homogeneous Dirichlet boundary conditions. The (single) parameter is the rope "stiffness," $\mu \in \mathcal{D}$. Following [4], we seek $u(\mu)$ satisfying (1) where, for any $w, v \in \mathcal{V}, q \in \mathcal{Q}$,

$$\langle A(\mu)v, w\rangle = \mu \int_\Omega v_x w_x \,\mathrm{d}x, \qquad \langle Bv, q\rangle = -\langle q, v\rangle, \qquad \langle f, v\rangle = -\int_\Omega v \,\mathrm{d}x, \qquad \langle g, q\rangle = \sum_{i=1}^{\mathcal{N}} \underline{q}_i h(x_i),$$

where $h(x) := 5x - 10$, $q = \sum_{i=1}^{\mathcal{N}} \underline{q}_i \psi_i$, and the x_i are FE nodes. We consider two parameter ranges, $\mathcal{D}_1 := [0.001, 0.01]$ and $\mathcal{D}_2 := [0.01, 0.03]$. For $\mathcal{D}_1$, the system exhibits contact (i.e., the constraint is active over some non-empty region of Ω) over the whole parameter range. On the other hand, $\mathcal{D}_2$ exhibits contact only for roughly a third of the parameter range; in this range there is thus little to no contact.

As in [4], we use for $\mathcal{V}$ a triangulation of 200 elements and employ a standard conforming first-order nodal basis, and for $\mathcal{Q}$ the biorthogonal functions of the basis functions of $\mathcal{V}$. To solve (1), we use the quadratic optimisation capabilities of MATLAB through the built-in function `quadprog`. We generate the RB-spaces through a greedy procedure [9] using a training sample of 1000 uniformly distributed parameters. For this simple example, we stabilise $\mathcal{V}_n$ using the procedure in [4] for which $n_{\rm stab} = 1$. We further use a test sample, $\mathcal{F}$, of 250 randomly chosen parameters uniformly distributed over $\mathcal{D}$.

We show in Fig. 1 numerical results for (a, b) $\mathcal{D} = \mathcal{D}_1$ and (c, d) $\mathcal{D} = \mathcal{D}_2$. We plot the maximum relative error and relative error bound over $\mathcal{F}$ for (a, c) the primal variable, $u(\mu)$, and (b, d) the Lagrange multiplier, $\lambda(\mu)$, for both the slack approach and the primal-only approach. For the slack approach, the greedy procedure alternately employs $\Delta_n^u(\mu)$ and $\Delta_n^\lambda(\mu)$ of (21) for the error indicator. For the primal-only approach, we employ the same procedure, but use the analogous error bounds given in [4].

The results show that the error bounds from the slack approach are sharper than those of the primal-only approach. Furthermore, since the error bounds from the slack approach are sharper and their effectivities (i.e., the ratio of the bound to the error) relatively uniform over $\mathcal{D}$, the slack approach enables the greedy procedure to more optimally choose the parameters for the RB sample. This is partly evidenced by the more rapid convergence with respect to n of the RB approximation with the slack approach.

At this point we briefly remark on some seeming drawbacks of our proposed method. First, the slack approach requires the solution to an additional reduced problem, (14), to compute the combined RB approximation $(u_n^{\rm pr}(\mu), \lambda_n^{\rm sl}(\mu))$. However, in real applications, it is, in fact, the (computable) *error bound* and not the (unknown) error which determines the total cost. It can thus be expected that for large $\mathcal{N}$, the cost of this additional solve is offset by the sharpness of the error bounds in the slack approach and the $\mathcal{N}$-dependence of the online cost in the primal-only approach.

Second, we note that the non-negativity condition imposed by (13) in the slack approach is more restrictive than (11b) in two ways: it precludes orthogonalisation of the basis functions, ζ_i, and permits only conical linear combinations (i.e.,

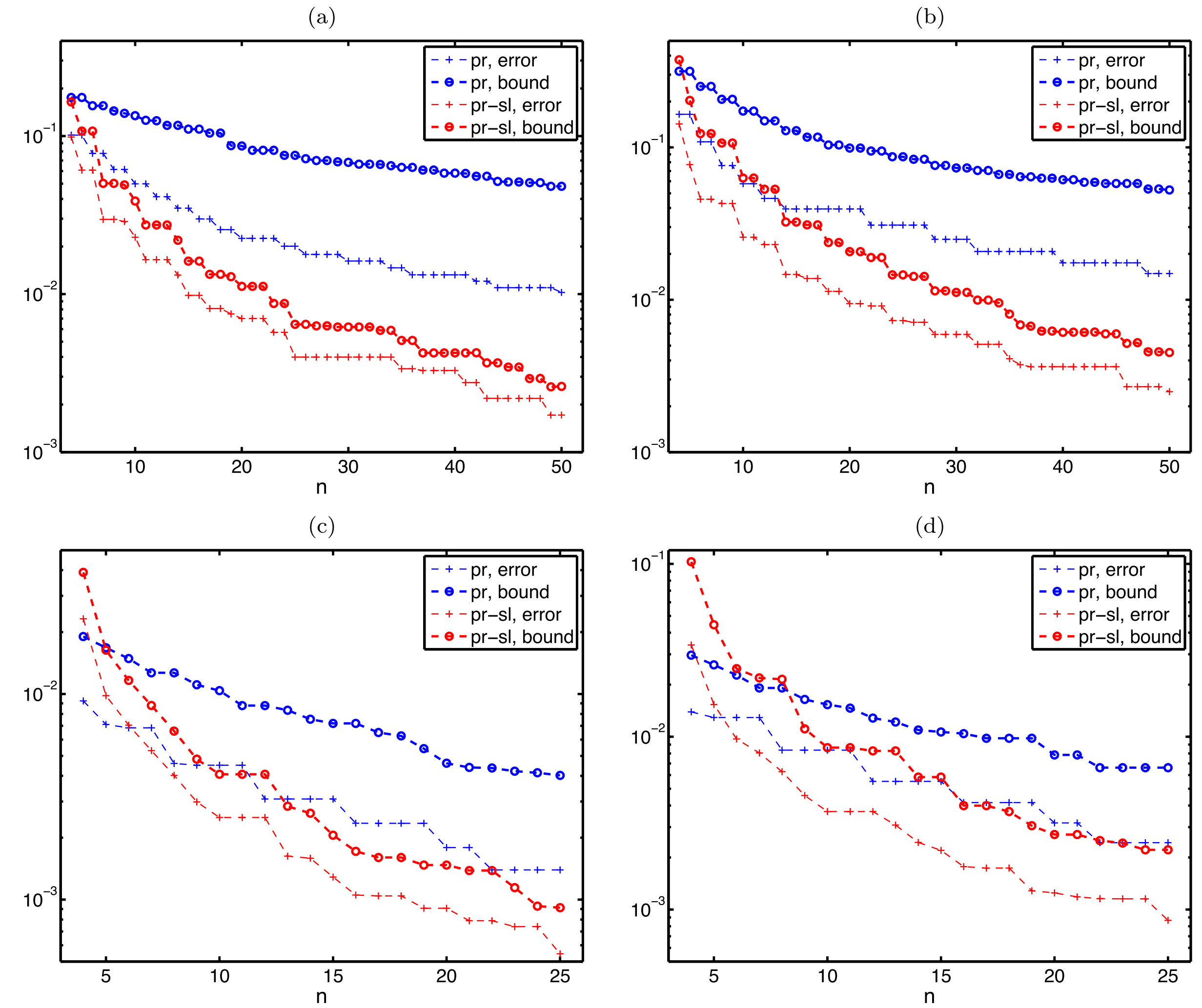

Fig. 1. Comparison between the primal-only approach of [4] and the slack approach. Numerical results are shown for the model problem with (a, b) $\mathcal{D}=\mathcal{D}_1$ (large contact region) and (c, d) $\mathcal{D}=\mathcal{D}_2$ (small contact region or no contact). We plot in (a, c) the maximum relative error, $\max_{\mu\in\mathcal{F}}(\|u(\mu)-u_n^m(\mu)\|_{\mathcal{V}}/\|u(\mu)\|_{\mathcal{V}})$, and maximum relative *a posteriori* error bound, $\max_{\mu\in\mathcal{F}}(\Delta_n^u(\mu)/\|u(\mu)\|_{\mathcal{V}})$; and in (b, d) the maximum relative error, $\max_{\mu\in\mathcal{F}}(\|\lambda(\mu)-\lambda_n^{\mathrm{pr}}(\mu)\|_{\mathcal{V}'}/\|\lambda(\mu)\|_{\mathcal{V}'})$ and maximum absolute error bound $\max_{\mu\in\mathcal{F}}\Delta_n^{\lambda}(\mu)$.

with only positive coefficients, $\underline{s}_{n,i}$). In the case of small or no contact (e.g., $\mathcal{D}=\mathcal{D}_2$), this restriction results in larger approximation errors for the slack approach for n very small (i.e., <5 in this example). However, the greedy approach automatically addresses this issue by choosing the appropriate parameters to add to the RB sample; its success is evidenced by the more rapid convergence of the slack approach even in the case of small or no contact (see Fig. 1c–d).

4. Summary and perspectives

We proposed a slack-based approach for computing RB approximations and associated *a posteriori* error bounds to solutions to VIs of the first kind. The proposed approach uses an additional approximation problem for the slack variable in order to obtain strictly feasible approximations. This in turn enables the development of *a posteriori* error bounds that are significantly sharper than existing bounds. The approach allows a *full* offline–online computational decomposition of the error bound; the online cost to compute the error bounds is thus completely independent of the FE dimension $\mathcal{N}$. Future work will focus on (i) the application of the method to more complex problems, and (ii) more general operators B.

Acknowledgements

We would like to thank M. Grepl and M. Kärcher of RWTH Aachen University, and A.T. Patera of MIT for the helpful discussions, comments, and their careful critique of this manuscript. We also thank J.-B. Wahl and C. Prud'homme of Université de Strasbourg for their kind help with the French translation. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) through Grant GSC 111.

References

[1] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, NY, USA, 1991.
[2] F. Brezzi, W. Hager, P. Raviart, Error estimates for the finite element solution of variational inequalities part ii. Mixed methods, Numer. Math. 31 (1978) 1–16.
[3] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer, 1984.
[4] B. Haasdonk, J. Salomon, B. Wohlmuth, A reduced basis method for parametrized variational inequalities, SIAM J. Numer. Anal. 50 (5) (2012) 2656–2676.
[5] N.A. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, Springer, 1988.
[6] J.L. Lions, G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math. 20 (3) (1967) 493–519.
[7] C. Prud'homme, D.V. Rovas, K. Veroy, L. Machiels, Y. Maday, A.T. Patera, G. Turinici, Reliable real-time solution of parametrized partial differential equations: reduced-basis output bound methods, J. Fluids Eng. 124 (1) (2002) 70–80.
[8] G. Rozza, K. Veroy, On the stability of the reduced basis method for Stokes equations in parametrized domains, Comput. Methods Appl. Mech. Eng. 196 (7) (2007) 1244–1260.
[9] K. Veroy, C. Prud'homme, D.V. Rovas, A.T. Patera, A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations, in: Proceedings of the 16th AIAA Computational Fluid Dynamics Conference, 2003, AIAA paper 2003-3847.