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Partial differential equations

Symmetry results for solutions of equations involving zero-order operators



Symétries des solutions d'équations impliquant des opérateurs d'ordre zéro

Disson dos Prazeres^a, Ying Wang^b

^a Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, UMR2071, CNRS-UCHile, Universidad de Chile, Chile

^b Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China

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ABSTRACT

In this note, we study symmetry results of solutions to equation (E) $-\mathcal{I}_\epsilon[u] = f(u)$ in B_1 with the condition $u = 0$ in \bar{B}_1^c , where $\mathcal{I}_\epsilon[u](x) = \int_{\mathbb{R}^N} \frac{u(y)-u(x)}{\epsilon^{N+2\sigma} + |y-x|^{N+2\sigma}} dy$, with $\epsilon > 0$ and $\sigma \in (0, 1)$, is a zero-order nonlocal operator, which approaches the fractional Laplacian when $\epsilon \rightarrow 0$. The function f is locally Lipschitz continuous. We analyzed that the symmetry properties of solutions depend on the Lipschitz constant of f . When the Lipschitz constant is controlled by $C_{N,\sigma}\epsilon^{-2\sigma}$, any solution $u \in C(\bar{B}_1)$ of (E) satisfying $u > c$ in B_1 and $u = c$ on ∂B_1 is radially symmetric.

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RÉSUMÉ

Soit $\mathcal{I}_\epsilon[u](x) = \int_{\mathbb{R}^N} \frac{u(y)-u(x)}{\epsilon^{N+2\sigma} + |y-x|^{N+2\sigma}} dy$, avec $\epsilon > 0$ et $\sigma \in (0, 1)$, un opérateur non local d'ordre zéro qui approche le laplacien fractionnaire lorsque ϵ tend vers 0. Nous étudions dans cette Note les symétries des solutions de l'équation (E) : $-\mathcal{I}_\epsilon[u] = f(u)$ dans la boule unité ouverte B_1 avec la condition $u = 0$ sur le complémentaire de la boule unité fermée. Nous observons que les propriétés de symétrie dépendent de la constante de Lipschitz de f . Lorsque cette constante de Lipschitz est majorée par $C_{N,\sigma}\epsilon^{-2\sigma}$, toute solution $u \in C(\bar{B}_1)$ de (E) satisfaisant $u > c$ dans B_1 et $u = c$ sur le bord ∂B_1 est radialement symétrique.

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1. Introduction

Our purpose of this note is to study symmetry results of solutions to equations

$$\begin{cases} -\mathcal{I}_\epsilon[u] = f(u) & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \bar{B}_1, \end{cases} \quad (1.1)$$

E-mail addresses: dissonmathufc@gmail.com (D. dos Prazeres), yingwang00@126.com (Y. Wang).

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where B_1 is the open unit ball centered at the origin in \mathbb{R}^N , $\epsilon > 0$, \mathcal{I}_ϵ is a nonlocal operator approaching the fractional Laplacian as ϵ tends to 0, which has a form

$$\mathcal{I}_\epsilon[u](x) = \int_{\mathbb{R}^N} [u(y) - u(x)] K_\epsilon(y - x) dy,$$

where, for $\sigma \in (0, 1)$ fixed, the kernel K_ϵ is given by

$$K_\epsilon(z) = \frac{1}{\epsilon^{N+2\sigma} + |z|^{N+2\sigma}}.$$

Notice that for $\epsilon > 0$, K_ϵ is integrable in \mathbb{R}^N with L^1 norm equal to $C_{N,\sigma}\epsilon^{-2\sigma}$. We point out that operator \mathcal{I}_ϵ is a particular case of a broad class of nonlocal operators, is known in the literature [4] as a zero-order nonlocal operator.

The study of radial symmetry of positive solutions to the second order elliptic equations in bounded domains was initiated by Serrin [7] and Gidas, Ni and Nirenberg [5] that associated the maximum principle with the method of moving planes introduced by Alexandrov in [1]. For the equations involving nonlocal operators, specially, the fractional Laplacian, Felmer–Wang in [3] studied the radial symmetry of positive classical solutions of (1.1) by the method of moving planes as in [2] based on the Maximum Principle for small domains, which is derived by the Aleksandrov–Bakelman–Pucci (ABP) estimate in [6]. We observe that the operator $-I_\epsilon$ is the fractional Laplacian operator when $\epsilon = 0$. In the case of $\epsilon > 0$, for the operator $-I_\epsilon$, the ABP estimate is not available; to this end, we introduce a key lemma, i.e. Lemma 2.1, to prove the Maximum Principle for a small domain where the Lipschitz constant of nonlinearity plays an important role. Then we do the moving planes as in [3] to obtain the symmetry results to solutions to (1.1). Here we say that a bounded function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, continuous in \bar{B}_1 is a solution to Eq. (1.1) if u satisfies (1.1) in a pointwise sense. Inspired by the results in [4], we notice that any positive solution u to problem (1.1) jumps at ∂B_1 . In fact, when u is a positive solution to problem (1.1), where the function f is nonnegative and satisfies some extra conditions, we have that

$$u(x) \geq c_0 \|u\|_{L^1(B_1)} \epsilon^{2\sigma}, \quad \forall x \in \bar{B}_1 \quad (1.2)$$

for some $c_0 > 0$ independent of u and ϵ .

Now we state our main theorem in this note.

Theorem 1.1. *Suppose that $u \in C(\bar{B}_1)$ is a solution of (1.1) with $\epsilon > 0$ such that*

$$\lim_{x \in B_1, x \rightarrow \partial B_1} u(x) = c, \quad (1.3)$$

where $c \geq 0$ is a constant and $u > c$ in B_1 . If f is Lipschitz continuous, with a Lipschitz constant C satisfying

$$C \leq C_{N,\sigma} \epsilon^{-2\sigma}, \quad (1.4)$$

then u is radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0, 1)$.

We notice that the condition (1.4) is obvious for any Lipschitz continuous function f when $\epsilon = 0$; in other words, here we have a new way to prove the Maximum Principle for small domain for the equations involving fractional Laplacian in [3]. It is important to note that the geometric assumptions here are similar to the classical Gidas–Ni–Nirenberg result, since we admit that the minimum in \bar{B}_1 is attained on ∂B_1 and $\{x \in \bar{B}_1; u(x) = \min_{\bar{B}_1} u\} = \partial B_1$.

Before continuing, we would like to discuss about our assumptions (1.3) and (1.4) in Theorem 1.1 and some open questions arising from them. We first consider assumption (1.4) appearing as a sufficient condition. Is it optimal? That is, is it possible to prove that if (1.4) does not hold, then there exists a solution satisfying the other hypotheses of Theorem 1.1 and which is not symmetric? We next look at our assumption regarding the boundary condition. We observe that in [3], the symmetry of positive solutions to $(-\Delta)^\alpha u = f(u)$ in B_1 is obtained assuming the boundary condition $u = 0$ in $\mathbb{R}^N \setminus B_1$, which can be interpreted as follows: the solution has zero boundary data in $\mathbb{R}^N \setminus B_1$ and the solution has limit zero as x tends to the boundary from inside. Thus, in the zeroth-order problem (1.1), the boundary condition can be seen as $u = 0$ in $\mathbb{R}^N \setminus B_1$ and the inner limit is a constant c , where we assume $c \geq 0$ because the solutions of equation (1.1) may jump at ∂B_1 . Then, a natural question is to ask if it is necessary to consider $u = 0$ in $\mathbb{R}^N \setminus B_1$ and u is constant at ∂B_1 to obtain the symmetry of solutions or, under suitable assumptions, just one of them is sufficient. We do not know the answer to these questions.

2. Symmetry result

In this section, we prove the main result of this note by the method of moving planes. Before, we will introduce the Maximum Principle for a small domain, which is a key tool in the proceeding. To this end, we first give the following lemma.

Lemma 2.1. Let Ω be an open bounded subset of \mathbb{R}^N . Suppose that $h : \Omega \rightarrow \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N)$ is a solution to

$$\begin{cases} \mathcal{I}_\epsilon[w](x) \leq h(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.1}$$

Then

$$-\inf_{\Omega} w \leq \|h\|_{L^\infty(\Omega)} C(\epsilon, |\Omega|), \tag{2.2}$$

where $C(\epsilon, |\Omega|) = \left(\int_{c_1|\Omega|^{\frac{1}{N}}}^{\infty} \frac{s^{N-1}}{\epsilon^{N+2\sigma} + s^{N+2\sigma}} ds \right)^{-1}$ for some $c_1 > 0$.

Proof. The result is obvious if $\inf_{\Omega} w \geq 0$. Now we assume that $\inf_{\Omega} w < 0$, then there exists $x_0 \in \Omega$ such that $w(x_0) = \inf_{\Omega} w < 0$. Combining with (2.1), we have that

$$\|h\|_{L^\infty(\Omega)} \geq h(x_0) \geq \mathcal{I}_\epsilon[w](x_0). \tag{2.3}$$

By the definition of I_ϵ , we have that

$$\begin{aligned} \mathcal{I}_\epsilon[w](x_0) &= \int_{\mathbb{R}^N} [w(y) - w(x_0)] K_\epsilon(y - x_0) dy \\ &= \int_{\Omega} [w(y) - w(x_0)] K_\epsilon(y - x_0) dy + \int_{\mathbb{R}^N \setminus \Omega} [w(y) - w(x_0)] K_\epsilon(y - x_0) dy \\ &\geq - \int_{\mathbb{R}^N \setminus \Omega} w(x_0) K_\epsilon(y - x_0) dy, \end{aligned}$$

let $r = c_1|\Omega|^{\frac{1}{N}}$ with $c_1 > 0$ such that $|\Omega| = |B_r(x_0)|$, combining with the fact that K_ϵ is radial and decreasing, we obtain that

$$\mathcal{I}_\epsilon[w](x_0) \geq - \int_{\mathbb{R}^N \setminus B_r(x_0)} w(x_0) K_\epsilon(y - x_0) dy = -w(x_0) \int_{c_1|\Omega|^{\frac{1}{N}}}^{\infty} \frac{s^{N-1}}{\epsilon^{N+2\sigma} + s^{N+2\sigma}} ds,$$

by (2.3), we have that

$$\|h\|_{L^\infty(\Omega)} \geq -w(x_0) \int_{c_1|\Omega|^{\frac{1}{N}}}^{\infty} \frac{s^{N-1}}{\epsilon^{N+2\sigma} + s^{N+2\sigma}} ds.$$

Therefore,

$$-\inf_{\Omega} w = -w(x_0) \leq \|h\|_{L^\infty(\Omega)} C(\epsilon, |\Omega|),$$

where $C(\epsilon, |\Omega|) = \left(\int_{c_1|\Omega|^{\frac{1}{N}}}^{\infty} \frac{s^{N-1}}{\epsilon^{N+2\sigma} + s^{N+2\sigma}} ds \right)^{-1}$. \square

Now we state the Maximum Principle for small domain as follows.

Lemma 2.2. Let Ω be an open bounded subset of \mathbb{R}^N . Suppose that $\varphi : \Omega \rightarrow \mathbb{R}$ is in $L^\infty(\Omega)$ satisfying

$$\|\varphi\|_{L^\infty(\Omega)} < C_{N,\sigma} \epsilon^{-2\sigma} \tag{2.4}$$

and $w \in L^\infty(\mathbb{R}^N)$ is a solution to

$$\begin{cases} \mathcal{I}_\epsilon[w](x) \leq \varphi(x) w(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.5}$$

Then there is $\delta > 0$ such that whenever $|\Omega| \leq \delta$, w has to be non-negative in Ω .

Proof. Let us define $\Omega^- = \{x \in \Omega \mid w(x) < 0\}$, we observe that

$$\begin{cases} \mathcal{I}_\epsilon[w](x) \leq \varphi(x)w(x), & x \in \Omega^-, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega^-. \end{cases}$$

Using Lemma 2.1 with $h(x) = \varphi(x)w(x)$, we have that

$$\|w\|_{L^\infty(\Omega^-)} = -\inf_{\Omega^-} w \leq \|\varphi\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega^-)} C(\epsilon, |\Omega^-|).$$

By (2.4), there exists $\delta > 0$ such that for $|\Omega| \leq \delta$, we have that $\|\varphi\|_{L^\infty(\Omega)} \cdot C(\epsilon, |\Omega^-|) < 1$, then $\|w\|_{L^\infty(\Omega^-)} = 0$, that is, Ω^- is empty, completing the proof. \square

Using the method of moving planes as in [3] based on Lemma 2.2 by the fact of (1.4), we can prove the radial symmetry result in Theorem 1.1.

Proof of Theorem 1.1. Let us define

$$\begin{aligned} \Sigma_\lambda &= \{x = (x_1, x') \in B_1 \mid x_1 > \lambda\}, \\ T_\lambda &= \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\}, \\ u_\lambda(x) &= u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \end{aligned}$$

where $\lambda \in (0, 1)$ and $x_\lambda = (2\lambda - x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$. For any subset A of \mathbb{R}^N , we write $A_\lambda = \{x_\lambda : x \in A\}$.

We first prove that $w_\lambda \geq 0$ in Σ_λ if $\lambda \in (0, 1)$ is close to 1. Indeed, let us define $\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$ and

$$w_\lambda^+(x) = \begin{cases} w_\lambda(x), & x \in \Sigma_\lambda^-, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-, \end{cases} \quad w_\lambda^-(x) = \begin{cases} 0, & x \in \Sigma_\lambda^-, \\ w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-. \end{cases}$$

Since the kernel K_ϵ is symmetric and decreasing, then we can use the similar computation as in the proof of Theorem 1.1 in [3] to obtain that for all $0 < \lambda < 1$,

$$-\mathcal{I}_\epsilon[w_\lambda^-](x) \leq 0, \quad \forall x \in \Sigma_\lambda^-.$$

Combining with the fact of $w_\lambda^+ = w_\lambda - w_\lambda^-$ in \mathbb{R}^N and (1.1), we have that

$$\begin{aligned} -\mathcal{I}_\epsilon[w_\lambda^+](x) &\geq -\mathcal{I}_\epsilon[w_\lambda](x) = -\mathcal{I}_\epsilon[u_\lambda](x) + \mathcal{I}_\epsilon[u](x) = f(u_\lambda(x)) - f(u(x)) \\ &= \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} w_\lambda^+(x), \end{aligned}$$

for $x \in \Sigma_\lambda^-$. Let us define $\varphi(x) = -(f(u_\lambda(x)) - f(u(x)))/(u_\lambda(x) - u(x))$ for $x \in \Sigma_\lambda^-$. By the assumptions of f , we have that $\varphi \in L^\infty(\Sigma_\lambda^-)$ and satisfies (2.4). Choosing $\lambda \in (0, 1)$ close enough to 1 such that $|\Sigma_\lambda^-|$ is small, by $w_\lambda^+ = 0$ in $(\Sigma_\lambda^-)^c$ and Lemma 2.2, we obtain that $w_\lambda = w_\lambda^+ \geq 0$ in Σ_λ^- . Then Σ_λ^- is empty, that is, $w_\lambda \geq 0$ in Σ_λ .

We next claim that for $0 < \lambda < 1$, if $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in Σ_λ , then $w_\lambda > 0$ in Σ_λ . If this is not true, then there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$ and then

$$-\mathcal{I}_\epsilon[w_\lambda](x_0) = -\mathcal{I}_\epsilon[u_\lambda](x_0) + \mathcal{I}_\epsilon[u](x_0) = f(u_\lambda(x_0)) - f(u(x_0)) = 0. \tag{2.6}$$

On the other hand, defining $A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$, since $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and $w_\lambda(x_0) = 0$, then we have that

$$\begin{aligned} -\mathcal{I}_\epsilon[w_\lambda](x_0) &= -\int_{A_\lambda} \frac{w_\lambda(z)}{\epsilon^{N+2\sigma} + |x_0 - z|^{N+2\alpha}} dz - \int_{\mathbb{R}^N \setminus A_\lambda} \frac{w_\lambda(z)}{\epsilon^{N+2\sigma} + |x_0 - z|^{N+2\alpha}} dz \\ &= -\int_{A_\lambda} \frac{w_\lambda(z)}{\epsilon^{N+2\sigma} + |x_0 - z|^{N+2\alpha}} dz - \int_{A_\lambda} \frac{w_\lambda(z_\lambda)}{\epsilon^{N+2\sigma} + |x_0 - z_\lambda|^{N+2\alpha}} dz \\ &= -\int_{A_\lambda} w_\lambda(z) \left(\frac{1}{\epsilon^{N+2\sigma} + |x_0 - z|^{N+2\alpha}} - \frac{1}{\epsilon^{N+2\sigma} + |x_0 - z_\lambda|^{N+2\alpha}} \right) dz. \end{aligned}$$

Since $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in A_\lambda$, $w_\lambda(z) \geq 0$ and $w_\lambda(z) \not\equiv 0$ in A_λ , we get that

$$-\mathcal{I}_\epsilon[w_\lambda](x_0) < 0,$$

which contradicts (2.6). Thus, $w_\lambda > 0$ in Σ_λ if $\lambda \in (0, 1)$ is close to 1.

Then we apply the similar way as in the proof of Theorem 1.1 in [3], we obtain that $\lambda_0 := \inf\{\lambda \in (0, 1) \mid w_\lambda > 0 \text{ in } \Sigma_\lambda\} = 0$. Then we have that $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$ and then $u(-x_1, x') = u(x_1, x')$ for $x_1 \geq 0$. Repeating this procedure in all directions, we obtain the radial symmetry and the monotonicity of u . \square

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