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Differential geometry

Lower bounds for the eigenvalues of the Spin^c Dirac operator on manifolds with boundary

Minorations des valeurs propres de l'opérateur de Dirac sur les variétés Spin^c à bord

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ABSTRACT

We extend the Friedrich inequality for the eigenvalues of the Dirac operator on Spin^c manifolds with boundary under different boundary conditions. The limiting case is then studied and examples are given.

RÉSUMÉ

Nous étendons l'inégalité de Friedrich pour les valeurs propres de l'opérateur de Dirac sur les variétés Spin^c à bord pour différentes conditions à bord. Le cas limite est étudié et des exemples sont donnés.

Version française abrégée

En 1980, T. Friedrich [3] a minoré la première valeur propre λ_1 de l'opérateur de Dirac défini sur une variété riemannienne compacte Spin à courbure scalaire positive R. En effet, il a montré que

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{R}. \tag{1}$$

Le cas limite est caractérisé par l'existence d'un spineur de Killing. Plus tard, cette minoration a été établie [7,8] pour la première valeur propre de l'opérateur de Dirac défini sur une variété compacte Spin à bord et sous différents types de conditions à bord. Dans cette note, on établit l'inégalité de Friedrich dans le cas des variétés compactes Spin^c à bord. En effet, on montre le théorème suivant.

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Théorème. *On considère une variété riemannienne compacte* Spin^c *à bord de dimension n et on note par* $\mathrm{i}\Omega$ *la courbure du fibré en droites associé à la structure* Spin^c*. On suppose, sous les conditions à bord gAPS données par* $P_{\geqslant b}$ *pour* $b \leqslant 0$*, qu'il existe deux fonctions a et u telles que* $b + a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H$*, sur* ∂M*, où* ν *designe le vecteur normal unitaire et H la courbure moyenne du bord* ∂M*. Sous les conditions mgAPS* $P^m_{\geqslant b}$ *pour* $b \leqslant 0$ *ou CHI, on suppose qu'il existe deux fonctions a et u tel que* $a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H$*, sur* ∂M*. Alors, toute valeur propre de l'opérateur de Dirac* D^M *de M satisfait*

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \sup_{a,u} \inf_M \left(R_{a,u} - c_n|\Omega|\right).$$

Ici, $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$*,* $R_{a,u} = R - 4a\Delta u + 4 < \nabla a, \nabla u > -4(1-\frac{1}{n})a^2|\nabla u|^2$*. Sous les conditions mgAPS et CHI, le cas limite est caractérisé par l'existence d'un spineur de Killing sur M et le bord* ∂M *est minimal. Pour la condition gAPS, le cas limite ne peut être atteint.*

Enfin, pour la condition MIT, nous démontrons la minoration optimale suivante :

$$|\lambda|^2 \geqslant \frac{n}{4(n-1)} \inf_M (R - c_n|\Omega|) + nH_0 \Im m(\lambda),$$

où H_0 est l'infimum de la courbure moyenne sur ∂M et $\Im m(\lambda)$ la partie imaginaire de λ. De plus, l'égalité a lieu si et seulement si M admet un spineur de Killing imaginaire et ∂M est totalement ombilique.

1. Introduction

The spectrum of the Dirac operator on compact Spin manifolds with or without boundary has been extensively studied over the past three decades. First, the intrinsic aspect has been systematically studied, then the extrinsic aspect has been intensively exploited by many authors in order to study the geometry and the topology of submanifolds in general, and hypersurfaces in particular (including boundaries of domains). In [3], Friedrich proved that the first eigenvalue of the Dirac operator on a closed manifold (M^n, g) of positive scalar curvature R satisfies

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{R}. \tag{2}$$

The equality case is characterized by the existence of a real Killing spinor. The existence of such a spinor leads to geometric restrictions on the manifold. For example, the manifold is Einstein and in dimension 4, it has constant sectional curvature. The classification of simply connected Riemannian Spin manifolds carrying real Killing spinors gives, in some dimensions, other examples than the sphere. These examples are relevant to physicists in general relativity where the Dirac operator plays a central role. In [7,8], and under different boundary conditions, the lower bound (2) was established for the first eigenvalue of the Dirac operator defined on compact Riemannian Spin manifolds with boundary.

In this note, we extend the lower bound (2) for the first eigenvalue of the Spin^c Dirac operator defined on manifolds with boundary under different boundary conditions. In fact, we prove the following theorem.

Theorem 1.1. *Let* (M^n, g) *be a compact Riemannian* Spin^c *manifold with non-empty boundary* ∂M *and line bundle curvature* $\mathrm{i}\Omega$*. Let* λ *be an eigenvalue of the Dirac operator* D^M *on M. Under the gAPS boundary condition* $P_{\geqslant b}$ *for some* $b \leqslant 0$*, we assume that there exists some real functions a and u on M such that*

$$b + a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H,$$

on ∂M*, where* $(n-1)H$ *is the trace of the second fundamental form of the boundary. Under the mgAPS boundary condition* $P^m_{\geqslant b}$ *for some* $b \leqslant 0$ *or under the CHI boundary condition* P_{CHI}*, we assume that there exists some real functions a and u such that*

$$a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H,$$

on ∂M*. Then*

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a,u} \inf_M \left(R_{a,u} - c_n|\Omega|\right). \tag{3}$$

Under the gAPS boundary condition, the equality case cannot occur. Under the mgAPS or the CHI boundary conditions, equality occurs if and only if M carries a non-trivial real Killing spinor with Killing constant $-\frac{\lambda}{n}$ *and the boundary* ∂M *is minimal.*

Under the MIT bag condition P_{MIT}, we prove the following theorem.

Theorem 1.2. *Let (M^n, g) be a compact Riemannian* Spin^c *manifold with non-empty boundary. Let* $\mathrm{i}\Omega$ *be the curvature of the auxiliary line bundle associated with the* Spin^c *structure. Let λ be an eigenvalue of the Dirac operator D^M under the MIT bag condition P_{MIT}. Assume that the mean curvature H (with respect to the inner unit normal) of ∂M is strictly positive, then*

$$|\lambda|^2 \geqslant \frac{n}{4(n-1)} \inf_M (R - c_n|\Omega|) + nH_0 \Im m(\lambda), \tag{4}$$

where H_0 is the infimum of H on ∂M. When equality holds, the eigenspinor ψ is an imaginary Spin^c *Ω-Killing spinor and the boundary ∂M is totally umbilical with constant mean curvature equal to $H_0 = \dfrac{2\Im m(\lambda)}{n}$.*

At the end, we focus on examples satisfying the limiting case in (4) and (3), which are Spin^c but not Spin.

2. Manifolds with boundary

Let (M^n, g) be a compact Riemannian Spin^c manifold with non-empty compact boundary ∂M. We denote by ∇ the Levi Civita Spin^c connection of M, $< ., . >$ denotes the Hermitian scalar product on the Spin^c bundle ΣM and "γ" the Clifford multiplication on M. We denote by L the auxiliary line bundle associated with the Spin^c structure and $\mathrm{i}\Omega$ its curvature imaginary 2-form of some Hermitian connection (see [4]). We will consider boundary conditions in order to generalize the Friedrich eigenvalue estimate for the spectrum of the Spin^c Dirac operator on M.

APS and gAPS boundary conditions: The well-known APS boundary condition [1] was introduced by Atiyah, Patodi and Singer. Since the boundary is a closed manifold, its Dirac operator has a real discrete spectrum and we defined the projection $\pi_+ : \Gamma(\Sigma M_{|\partial M}) \longrightarrow \Gamma(\Sigma M_{|\partial M})$ onto the subspace of $\Gamma(\Sigma M_{|\partial M})$ spanned by the eigenspinors associated with nonnegative eigenvalues. It is a classical fact ([8]) that this gives a self-adjoint elliptic boundary condition for the Dirac operator and so it has a real discrete spectrum. The generalized Atiyah–Patodi–Singer condition, denoted by gAPS, is a generalization of the APS condition: for any real number b, we consider the projection $P_{\geqslant b}$ onto the subspace of $\Gamma(\Sigma M)$ spanned by the eigenspinors φ_k associated with eigenvalues $\lambda_k \geqslant b$. As mentioned in [2], this is also a self-adjoint elliptic boundary condition for any nonpositive b. We remark that the gAPS boundary condition for $b = 0$ is just the standard APS condition. Moreover, for more convenience, we will use the following useful notations; $P_{>b}$ is defined in the same way, but the projection is onto the subspace spanned by the eigenspinors φ_k associated with eigenvalues $\lambda_k > b$. We also define $P_{<b} = Id - P_{\geqslant b}$.

mAPS and mgAPS boundary conditions: the mAPS and mgAPS boundary conditions are modifications of the APS and gAPS, respectively, in the following way. For $\varphi \in \Gamma(\Sigma M)$, we have $P^m_{\geqslant b}\varphi = P_{\geqslant b}(Id + \gamma(\nu))\varphi$. For $b = 0$, this condition is just the modified APS condition (mAPS) introduced by Hijazi, Montiel, and Roldan [8].

Boundary condition associated with a chirality operator: in contrast with the above boundary conditions, we consider the following local boundary condition associated with a chirality operator which is subject to the existence of such an operator. So we consider a linear map $G : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ such that

$$G^2 = Id, \quad \langle G\varphi, G\psi\rangle = \langle \varphi, \psi\rangle, \quad \nabla_X(G\varphi) = G\nabla_X\varphi, \quad \gamma(X)G\varphi = -G\gamma(X)\varphi \tag{5}$$

for any vector X tangent to M. Such an operator is called a *chirality* operator because in the even dimensional case, an example is $G = \gamma(\omega_n)$, the Clifford multiplication by the complex volume element, which gives the chirality decomposition of the spinor bundle. The boundary condition associated with this operator is defined by: $P_{\mathrm{CHI}} = \frac{1}{2}(Id - \gamma(\nu)G)$. As proved in [8], this condition is self-adjoint and elliptic and so, under this boundary condition, the Dirac operator has a real discrete spectrum.

MIT boundary conditions: the MIT boundary condition is also a local boundary condition. It is defined as follows: for any spinor field φ on ∂M, $P_{\mathrm{MIT}}\varphi = \frac{1}{2}(\varphi - \mathrm{i}\gamma(\nu)\varphi)$. It is an elliptic condition and the spectrum of the Dirac operator is discrete. However, the Dirac operator is not self-adjoint anymore and its spectrum consists of complex eigenvalues whose imaginary part is strictly positive.

Lemma 2.1. *Let $b \leqslant 0$. We denote by D the Dirac operator on the boundary ∂M of M. Then,*

$$\int_{\partial M} \langle D\varphi, \varphi\rangle \begin{cases} \leqslant b \int_{\partial M} |\varphi|^2 & \text{under the gAPS condition, } P_{\geq b}\varphi = 0, \\ = 0 & \text{under the mgAPS condition, } P^m_{\geq b}\varphi = 0. \end{cases}$$

Under the CHI and MIT conditions, then, pointwise on ∂M we have $\langle D\varphi, \varphi\rangle = 0$.

Proof. Let $(\varphi_k, \lambda_k)_{k\in\mathbb{Z}}$ be a spectral resolution of D on the boundary ∂M. Any spinor φ of $\Gamma(\Sigma\partial M)$ expresses as follows $\varphi = \sum_k a_k\varphi_k$ with $a_k = \int_{\partial M} < \varphi, \varphi_k > \mathrm{d}s$. Under the gAPS condition, we have $P_{\geqslant b}\varphi = 0$, that is, $\varphi = \sum_{\lambda_k < b} a_k\varphi_k$. Then, we have

$$\int_{\partial M} < D\varphi, \varphi > \mathrm{d}s = \sum_{\lambda_k < b} \lambda_k |a_k|^2 \leqslant b \sum_{\lambda_k < b} |a_k|^2 = b \int_{\partial M} |\varphi|^2 \mathrm{d}s.$$

Let φ such that $P^m_{\geqslant b}\varphi = 0$, for $b \leqslant 0$, that is, $P_{\geqslant b}(\varphi + \gamma(\nu)\varphi) = 0$. From this, we can see easily that $P_{>-b}(\varphi + \gamma(\nu)\varphi) = 0$. Moreover, from the relation $D(\gamma(\nu)) = -\gamma(\nu)D$, we see that for any b and any spinor ψ, $P_{<b}\gamma(\nu)\psi = \gamma(\nu)P_{>-b}\psi$. Hence, we have

$$\begin{aligned} P_{>-b}(\gamma(\nu)\varphi - \varphi) &= \gamma(\nu)P_{<b}\varphi - P_{>-b}(\varphi) = \gamma(\nu)\Big(P_{<b}\varphi + \gamma(\nu)P_{>-b}(\varphi)\Big) \\ &= \gamma(\nu)P_{<b}(\varphi + \gamma(\nu)\varphi) = \gamma(\nu)\Big[\varphi + \gamma(\nu)\varphi - \underbrace{P_{\geqslant b}(\varphi + \gamma(\nu)\varphi}_{=0}\Big] \\ &= \gamma(\nu)\varphi - \varphi. \end{aligned}$$

Now, using that $< D\varphi, \varphi >= \frac{1}{2} < D(\varphi + \gamma(\nu)\varphi), \varphi - \gamma(\nu)\varphi >$, the fact that $\gamma(\nu)\varphi - \varphi = P_{>-b}(\gamma(\nu)\varphi - \varphi)$ and $P_{>-b}(\varphi + \gamma(\nu)\varphi) = 0$, we deduce that

$$\int_{\partial M} < D\varphi, \varphi >= \int_{\partial M} \frac{1}{2} < D(\varphi + \gamma(\nu)\varphi), \varphi - \gamma(\nu)\varphi >= 0.$$

Now, we observe that from (5), we have the following pointwise equality $< D\varphi, \varphi >=< \gamma(\nu)GD\varphi, \gamma(\nu)G\varphi >$. Moreover, we have $DG = GD$ and since $P_{\mathrm{CHI}}\varphi = 0$, then $\gamma(\nu)G\varphi = \varphi$. So, we get

$$< D\varphi, \varphi >=< \gamma(\nu)GD\varphi, \varphi >=< \gamma(\nu)DG\varphi, \varphi >= - < D\gamma(\nu)G\varphi, \varphi >= - < D\varphi, \varphi > .$$

Finally, we have $< D\varphi, \varphi >= 0$. □

The proof for the MIT condition is similar.

3. Eigenvalue estimates for manifolds with boundary

From now on and for simplicity, we denote the Clifford multiplication γ by $\cdot$. First, as in [7] and for any real functions a and u, we consider the following modified connection $\nabla^{a,u}$ on M

$$\nabla^{a,u}_X \varphi = \nabla_X \varphi + a\nabla_X u \cdot \varphi + \frac{a}{n} X \cdot \nabla u \cdot \varphi + \frac{\lambda}{n} X \cdot \varphi,$$

where $X \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(\Sigma M)$. Assume that φ is an eigenspinor associated with an eigenvalue λ of the Dirac operator D^M on M. A simple calculation, using the Spin^c Reilly identity (see [7,11]), we get

$$\begin{aligned} \int_M |\nabla^{a,u}\varphi|^2 = \int_M \left[\left(1 - \frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4}\right]|\varphi|^2 \mathrm{d}v_g - \int_M < \frac{\mathrm{i}}{2}\Omega \cdot \varphi, \varphi > \mathrm{d}v_g \\ + \int_{\partial M} \left(< D\varphi, \varphi > + \left[a\, \mathrm{d}u(\nu) - \frac{n-1}{2}H\right]|\varphi|^2\right) \mathrm{d}s, \end{aligned} \tag{6}$$

where $R_{a,u}$ is defined by

$$R_{a,u} = R - 4a\Delta u + 4 < \nabla a, \nabla u > -4\left(1 - \frac{1}{n}\right)a^2|\nabla u|^2.$$

We have [6] that $< \mathrm{i}\Omega \cdot \varphi, \varphi > \geqslant -\frac{c_n}{2}|\Omega|_g|\varphi|^2$, where $|\Omega|_g$ is the norm of Ω with respect to the metric g given by $|\Omega|^2_g = \sum_{i<j} \Omega^2_{ij}$ in any orthonormal frame and $c_n = 2[\frac{n}{2}]^{1/2}$. Moreover, equality occurs if and only if $\Omega \cdot \varphi = \mathrm{i}\frac{c_n}{2}|\Omega|_g\varphi$. Using this and the fact that $|\nabla^{a,u}\varphi|^2 \geqslant 0$, Identity (6) becomes

$$\begin{aligned} \int_M \left[\left(1 - \frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4} + \frac{c_n}{4}|\Omega|_g\right]|\varphi|^2 \mathrm{d}v_g \\ \geqslant - \int_{\partial M} \left(< D\varphi, \varphi > + \left[a\, \mathrm{d}u(\nu) - \frac{n-1}{2}H\right]|\varphi|^2\right) \mathrm{d}s. \end{aligned} \tag{7}$$

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. From Inequality (7), Lemma 2.1 and the assumption $b + a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H$ or $a\,\mathrm{d}u(\nu) \leqslant \frac{n-1}{2}H$ respectively, we get immediately that

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \sup_{a,u} \inf_M \left(R_{a,u} - c_n|\Omega|\right).$$

We just have to prove that equality cannot occur under the gAPS boundary condition. For this, we need the following lemma, generalizing Lemma 3 in [7] to the case of Spin^c manifolds.

Lemma 3.1. *Suppose that there exists a spinor field φ satisfying*

$$\nabla^{a,u}\varphi = 0 \quad \textit{and} \quad \Omega\cdot\varphi = \mathrm{i}\frac{c_n}{2}|\Omega|\varphi, \tag{8}$$

for some real number λ and real functions a and u. Then $a = 0$ or $\mathrm{d}u = 0$, that is, φ is a Killing spinor.

Assuming this lemma, then φ is a non-trivial real Killing spinor, and so $|\varphi|$ is a positive constant. Let $(\varphi_k)_{k\in\mathbb{Z}}$ be an Hilbertian basis of eigenspinors for the Dirac operator of the boundary, associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{Z}}$. Under the gAPS condition, we have $P_{\geqslant b}\varphi = 0$, that is, $\varphi = \sum_{\lambda_k<b} a_k\varphi_k$, where $a_k = \int_{\partial M} <\varphi, \varphi_k> \mathrm{d}s$. Of course, not all a_k vanish, since otherwise φ would vanish on all M because it has constant length. Then, we have

$$\begin{aligned}
0 &= \int_{\partial M} <D\varphi,\varphi> \mathrm{d}s - \int_{\partial M} \frac{n-1}{2}H|\varphi|^2\mathrm{d}s \\
&= \sum_{\lambda_k<b} \lambda_k|a_k|^2 - \frac{n-1}{2}\sum_{\lambda_j,\lambda_k<b} \bar{a}_j a_k \int_{\partial M} H <\varphi_j,\varphi_k> \mathrm{d}s \\
&\leqslant \sum_{\lambda_k<b} (\lambda_k - b)|a_k|^2 < 0,
\end{aligned}$$

since $(n-1)H \geqslant 2b$. This is a contradiction and so equality cannot occur.

Proof of Lemma 3.1. First, we observe that (8) implies that $D\varphi = \lambda\varphi$. Now, we use the Ricci identity and we get

$$\frac{1}{2}e_k\cdot Ric(e_k)\cdot\varphi = \frac{i}{2}e_k\cdot(e_k\lrcorner\Omega)\cdot\varphi + e_k\cdot\sum_{j=1}^n e_j\cdot\mathcal{R}(e_j,e_k)\varphi.$$

Hence, by summing on k from 1 to n, we have:

$$\begin{aligned}
&\sum_{j,k}\frac{1}{2}R_{kj}e_k\cdot e_j\cdot\varphi \\
&\quad= \frac{\mathrm{i}}{2}\Omega\cdot\varphi - D^2\varphi + \sum_k e_k\cdot D(\nabla_{e_k}\varphi) \\
&\quad= \Bigg(-\frac{c_n}{4}|\Omega| + \frac{2(1-n)}{n}\lambda^2 + \frac{2a(1-n)}{n}\Delta u - \frac{2(2-n)}{n} <\nabla u, \nabla a> \\
&\qquad + \frac{a^2(1-n)(n-2)}{n^2}|\nabla u|^2\Bigg)\varphi - \frac{2}{n}\nabla u\cdot\nabla a\cdot\varphi + \frac{4a\lambda(1-n)}{n^2}\nabla u\cdot\varphi.
\end{aligned}$$

From this, we deduce that the term $\frac{4a\lambda(1-n)}{n^2}\nabla u\cdot\varphi$ necessarily vanishes and so $a = 0$ or $\nabla u = 0$, which implies that φ is a Killing spinor with Killing constant $-\frac{\lambda}{n}$. □

For conditions mgAPS and CHI, if equality occurs in (3), then, $\nabla^{a,u}\varphi = 0$ and $\frac{n-1}{2}H = a\,\mathrm{d}u(\nu)$. From Lemma 3.1 again, φ is a Killing spinor and $a\nabla u = 0$, hence $H = 0$. Conversely, we can check immediately that if φ is a Killing spinor and $H = 0$, then, equality occurs in (3). This concludes the proof of Theorem 1.1. □

Examples. Let $M=(\mathbb{S}^3, g_{\kappa,\tau})$ be the sphere endowed with the Berger metric. For $\kappa>0$ and $\tau\neq 0$, this metric is defined by $g_{(\kappa,\tau)}(X,Y)=\frac{\kappa}{4}\Big(g(X,Y)+(\frac{4\tau^2}{\kappa}-1)g(X,\xi)g(Y,\xi)\Big)$, where g is the round metric and ξ the Killing vector tangent to the fibers of the Hopf fibration of $\mathbb{S}^3$. For $\kappa=4\tau^2$, we found the round sphere of curvature κ. Berger spheres are also embedded spheres of constant mean curvature in the complex space forms of constant holomorphic sectional curvature $1-\tau^2$. In [10], we proved that M has a canonical Spin^c structure carrying a Killing spinor of Killing constant $-\frac{\tau}{2}$. The curvature of the line bundle associated with this canonical Spin^c structure is given in a local orthonormal frame $\{e_1, e_2, e_3=\xi\}$ by

$$\Omega(e_1,e_2)=(\kappa-4\tau^2) \text{ and } \Omega(e_i,e_j)=0 \text{ if not.} \tag{9}$$

It is straightforward that $D^M\varphi=\frac{3\tau}{2}\varphi$. Moreover, the scalar curvature of M is given by $2\kappa-2\tau^2$. By definition of the canonical Spin^c structure, we have $|\Omega|=\kappa-4\tau^2$. Finally, since $c_3=2$, we get $\frac{3}{8}(R-c_3|\Omega|)=\frac{9\tau^2}{4}$. Let now consider a domain of M bounded by a minimal surface. It remains to prove that φ satisfies the condition $P^m_{\mathrm{APS}}\varphi=0$. The restriction of φ to the boundary satisfies $D\varphi=H\varphi-\tau\gamma(\nu)\varphi$. Because the boundary is minimal, we have $D\varphi=-\tau\gamma(\nu)\varphi$. Using the super-symmetry property $D(\gamma(\nu)\varphi)=-\gamma(\nu)D\varphi$, we have $D(\varphi+\gamma(\nu)\varphi)=-\tau(\varphi+\gamma(\nu)\varphi)$. For $\tau>0$, this implies that $P^m_{\mathrm{APS}}\varphi=0$. For Berger spheres with $\tau<0$, we have to take the anti-canonical Spin^c structure that has a Killing spinor of opposite Killing constant. To summarize, we proved that every domain of the Berger sphere bounded by a minimal surface is an example of the limiting case of Inequality (3) for the condition mAPS. Such domains exist because we know examples of compact minimal surfaces embedded into Berger spheres (for example, the equator of the Berger spheres and the minimal Clifford tori). In [13], Torralbo constructed a family of minimal unduloids and some of them are embedded.

The case of MIT bag condition. Under the MIT bag condition, the spectrum of the Dirac operator is an unbounded sequence of complex numbers with positive imaginary part. Equality in (3) cannot hold. Following the ideas of Raulot [12], we will derive an optimal inequality for the eigenvalues of the Dirac operator for the boundary condition P_{MIT}.

Lemma 3.2. *Let μ be a complex number and ψ a non-trivial* Spin^c *Ω-Killing spinor field of Killing constant μ, i.e., for any $X\in\Gamma(TM)$,*

$$\begin{cases}\nabla_X\psi=\mu X\cdot\psi,\\ \Omega\cdot\psi=\mathrm{i}\frac{c_n}{2}|\Omega|\psi.\end{cases}$$

Then, μ is a real or an imaginary number and ψ has no zeros.

Proof. The fact that ψ has no zeros is well known (see [4]). The Schrödinger–Lichnerowicz formula applied for the spinor ψ, gives

$$D^2\psi=n^2\mu^2\psi=\nabla^*\nabla\psi+\frac{1}{4}R\psi-\frac{1}{4}c_n|\Omega|\psi=n\mu^2\psi+\frac{1}{4}R\psi-\frac{1}{4}c_n|\Omega|\psi.$$

Since ψ has no zeros, we deduce that $n(n-1)\mu^2=\frac{1}{4}(R-c_n|\Omega|)$. Thus μ^2 is real and hence μ is real or pure imaginary. □

Proof of Theorem 1.2. Proceeding as in [12], we obtain from the Spin^c Reilly formula, for an eigenspinor φ:

$$\int_M\left(\frac{n-1}{n}|\lambda-\frac{n}{2}\mathrm{i}H_0|^2-\frac{R}{4}+\frac{c_n}{4}|\Omega|-\frac{n(n-1)}{4}H_0^2\right)|\varphi|^2\mathrm{d}v_g\geqslant 0, \tag{10}$$

where H_0 is the infimum of H on ∂M, with equality if and only if φ is a Killing spinor of Killing constant $-\frac{\lambda}{n}$, $\Omega\cdot\varphi=\mathrm{i}\frac{c_n}{2}|\Omega|_g\varphi$, and H is constant (equals to H_0). From (10), we obtain immediately the desired lower bound. If equality holds in this lower bound, from Lemma 3.2, λ is either real or imaginary, but as an eigenvalue of the Dirac operator for the MIT boundary condition, λ has positive imaginary part. Hence, lambda is imaginary and φ is an imaginary Ω-Killing spinor. The fact that the boundary ∂M is umbilical is similar to the spin case (see [12]). □

Examples. Riemannian manifolds with imaginary Spin^c Killing spinors of Killing number $\mathrm{i}\mu$ have been classified in [5]. Such manifolds are the hyperbolic space endowed with its unique Spin structure and the warped product with $\mathbb{R}$ of a Riemannian Spin^c manifold carrying a parallel spinor, i.e. $(F^{n-1}\times\mathbb{R}, \mathrm{e}^{4\mu t}h\oplus \mathrm{d}t^2)$ where (F^{n-1},h) is a complete Spin^c manifold with a parallel spinor field. As examples that are not Spin, we can state $(\mathbb{C}P^2\times\mathbb{R}, \mathrm{e}^{4\mu t}g_{FS}\oplus\mathrm{d}t^2)$ or $(F^{2m}\times\mathbb{R}, \mathrm{e}^{4\mu t}h\oplus\mathrm{d}t^2)$, where g_{FS} is the Fubini Study metric and (F^{2m},h) is a Kähler manifold endowed with the canonical or the anti-canonical Spin^c structure. Totally umbilical embedded hypersurfaces of constant mean curvature in $(F^{2m}\times\mathbb{R}, \mathrm{e}^{4\mu t}h\oplus\mathrm{d}t^2)$ exist. For example, Montiel [9] proved that such a hypersurface is a leaf of a foliation generated by a non-trivial conformal closed vector field (such a vector exists for $(F^{2m}\times\mathbb{R}, \mathrm{e}^{4\mu t}h\oplus\mathrm{d}t^2)$) or is locally a Riemannian product $\mathbb{R}\times\mathcal{Q}^{2m-1}$ immersed into $\mathbb{R}^2\times\mathcal{Q}^{2m-1}$ as $\gamma\times I_{\mathcal{Q}^{2m-1}}$, where γ is a line in $\mathbb{R}^2$ and $\mathcal{Q}$ a manifold of dimension $2m-1$.

Moreover, Montiel also proved in [9] that the only compact hypersurfaces with constant mean curvature embedded into a pseudo-hyperbolic space $\mathbb{R} \times_f P^{n-1}$, $n \geqslant 2$, where P^{n-1}, is a compact Riemannian manifold with positive Ricci curvature, are the slices $\{t\} \times P^{n-1}$, for each t. These slices are embedded and totally umbilical. Therefore, a domain $[a, b] \times \mathbb{C}P^m$ into $\mathbb{R} \times_f \mathbb{C}P^m$ is an example of the limiting case in Theorem 1.2. Note that the boundary is not connected (it is the reunion of two homothetic $\mathbb{C}P^m$ with inverse orientations). Note also hat other examples can be derived by taking P^{n-1} as an other compact Spin^c manifold with positive Ricci curvature and carrying a parallel spinor.

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