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A Riemann–Roch–Grothendieck theorem for flat fibrations with complex fibers

Un théorème de Riemann–Roch–Grothendieck pour une fibration plate de fibre complexe

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ABSTRACT

We consider a proper flat fibration with real base and complex fibers. First we construct odd characteristic classes for such fibrations by a method that generalizes constructions of Bismut–Lott [5]. Then we consider the direct image of a fiberwise holomorphic vector bundle, which is a flat vector bundle on the base. We give a Riemann–Roch–Grothendieck theorem calculating the odd real characteristic classes of this flat vector bundle.

RÉSUMÉ

On considère une fibration propre plate de base réelle et de fibre complexe. On construit d'abord des classes caractéristiques impaires [5] associées qui généralisent des constructions de Bismut–Lott [5]. Puis on considère l'image directe d'un fibré vectoriel holomorphe dans la fibre, qui est un fibré vectoriel plat sur la base. On donne un théorème de Riemann–Roch–Grothendieck calculant les classes caractéristiques impaires de ce fibré plat.

1. Introduction

For a family of Dirac operators, the Chern class of its index bundle can be calculated by the family index theorem [1]. For complex manifolds, the corresponding theorem is the proper Riemann–Roch–Grothendieck theorem.

For a fibration of real manifolds equipped with a flat vector bundle, its direct image gives a flat vector bundle on the base. In this case, the family index theorem gives a trivial result. Bismut and Lott [5] constructed odd real characteristic classes associated with flat vector bundles and they gave a Riemann–Roch–Grothendieck formula, which calculates the odd characteristic classes of the direct image.

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In this note, we consider a similar setting. Let M be a real compact manifold, let N be a Kähler manifold equipped with a holomorphic vector bundle E_0. Given an action of $\pi_1(M)$ on N that lifts to E_0, this action induces in a natural way a flat fibration over M with fiber N. The direct image $H^{\cdot}(N, E_0)$ is a flat vector bundle over M. We will calculate its odd characteristic classes in term of characteristic classes associated with the fibration. We use techniques inspired from [3–5].

In the first section of this note, we construct certain characteristic classes associated with our flat fibration. These classes will appear on the right hand side of our Riemann–Roch–Grothendieck formula. In the second section, we state our main results.

2. Characteristic classes of a flat fibration

2.1. A flat fibration with complex fibers

Let G be a Lie group. Let N be a compact complex manifold of complex dimension n. We assume that G acts holomorphically on N. Let M be a real manifold. Let $p : P \to M$ be a principal G-bundle, that is equipped with a flat connection. Set

$$\mathcal{N} = P \times_G N \,. \tag{1}$$

We denote by q the projection $\mathcal{N} \to M$. The map q defines a fibration with fiber N. Let $T^H\mathcal{N} \subseteq T\mathcal{N}$ be the subbundle induced by the flat connection on P. Then

$$T^H\mathcal{N} \simeq q^*TM \,, \tag{2}$$

and

$$T\mathcal{N} = T^H\mathcal{N} \oplus T_{\mathbb{R}}N \,. \tag{3}$$

Let $d^{\mathcal{N}}$ be the de Rham operator on $\mathcal{N}$. The above splitting induces the decomposition

$$d^{\mathcal{N}} = d^M + d^N \,, \tag{4}$$

where d^N is the fiberwise de Rham operator.

Let ∂^N, $\overline{\partial}^N$ be the fiberwise Dolbeault operators along N, so that

$$d^N = \partial^N + \overline{\partial}^N \,. \tag{5}$$

Let E_0 be a holomorphic vector bundle on N of rank r. We assume that the action of G on N lifts holomorphically to E_0. Then E_0 defines a vector bundle

$$E = P \times_G E_0 \tag{6}$$

on $\mathcal{N}$. This vector bundle is holomorphic along the fiber N, and its holomorphic structure is flat.

Let $\Omega^{\cdot}(\mathcal{N}, E)$ be the vector space of differential forms on $\mathcal{N}$ with values in E. Let $\Omega^{\cdot}(N, E)$ be the vector space of fiberwise differential forms with values in E, which may be seen as an inifinite dimensional vector bundle on M. We have the identification

$$\Omega^{\cdot}(\mathcal{N}, E) = \Omega^{\cdot}(M, \Omega^{\cdot}(N, E)) \,. \tag{7}$$

Then d^M can be viewed as a flat connection on $\Omega^{\cdot}(\mathcal{N}, E)$.

Let g^E be a Hermitian metric on E. Let $\nabla^{E,N}$ be the fiberwise Chern connection with respect to g^E.

Let $d^{M,*}$ be the horizontal adjoint connection on E in the following sense: for any $\alpha, \beta \in \mathscr{C}^\infty(\mathcal{N}, E)$, we have

$$d^M g^E(\alpha, \beta) = g^E(d^M\alpha, \beta) + g^E(\alpha, d^{M,*}\beta) \,. \tag{8}$$

Set

$$d^{M,u} = \frac{1}{2}\left(d^M + d^{M,*}\right) , \qquad \omega^E = d^{M,*} - d^M \,. \tag{9}$$

Let

$$A^E = \nabla^{E,N} + d^{M,u} \,. \tag{10}$$

Then A^E is a Hermitian connection on E, and its curvature is $A^{E,2}$.

2.2. The odd forms

If N is reduced to a point, our constructions are the same as in Bismut–Lott [5, Definition 1.7].

Let $N^{\Lambda^{\cdot}(T^*\mathcal{N})}$ be the number operator on $\Lambda^{\cdot}(T^*\mathcal{N})$. Set $\varphi = (2\pi i)^{-\frac{1}{2}N^{\Lambda^{\cdot}(T^*\mathcal{N})}}$.

Let Q be an invariant polynomial on $\mathfrak{gl}(r,\mathbb{C})$.

Definition 2.1. Set

$$Q(E,g^E) = \varphi\, Q\left((A^E)^2\right),$$
$$\widetilde{Q}(E,g^E) = \sqrt{2\pi i}\varphi \left\langle Q'\left((A^E)^2\right), -\frac{\omega^E}{2}\right\rangle . \tag{11}$$

Proposition 2.2. *The even differential form $q_*[Q(E,g^E)]$ is concentrated in degree zero. The odd differential form $q_*[\widetilde{Q}(E,g^E)]$ on M is closed, and its cohomology class does not depend on g^E.*

Proof. Let $N^{\Lambda^{\cdot}(T^*_{\mathbb{R}}N)}$ be the number operator on $\Lambda^{\cdot}(T^*_{\mathbb{R}}N)$. Set $U = (-1)^{N^{\Lambda^{\cdot}(T^*_{\mathbb{R}}N)}}$. Using the flatness of the fibration in the same way as [5], we can show that

$$(A^E)^2 = -U^{-1}\left(\nabla^{E,N} + \frac{\omega^E}{2}\right)^2 U . \tag{12}$$

Trivially, $q_*[Q((\nabla^{E,N})^2)]$ is concentrated in degree zero. By using Chern–Weil theory along the fiber N for de Rham cohomology of $\Omega^{\cdot}\left(N, q^*\Lambda^{\cdot}(T^*M)\right)$, we can show that

$$\int_N Q\left((A^E)^2\right) = \int_N Q\left(\left(\nabla^{E,N} + \frac{\omega^E}{2}\right)^2\right) = \int_N Q\left((\nabla^{E,N})^2\right), \tag{13}$$

which proves the first part of our proposition.

To establish the second part of our proposition, we construct a one parameter deformation of A^E,

$$A^E_t = \nabla^{E,N} + td^M + (1-t)d^{M,*} . \tag{14}$$

By using the same procedure as before, we can show that $q_*[Q((A^E_t)^2)]$ is concentrated in degree zero. As a consequence, $q_*[Q((A^E_t)^2)]$ is a constant which does not depend on t. Furthermore, we can show that

$$d^{\mathcal{N}}\widetilde{Q}(E,g^E) = \sqrt{2\pi i}\frac{\partial}{\partial t} Q\left((A^E_t)^2\right)\Big|_{t=1/2}, \tag{15}$$

from which deduce that the form $q_*[\widetilde{Q}(E,g^E)]$ is closed. By the functoriality of our construction, the cohomology class of $q_*[\widetilde{Q}(E,g^E)]$ does not depend on g^E. □

3. A Riemann–Roch–Grothendieck formula

From now on, we assume that the fiber N is a compact complex Kähler manifold.

3.1. Hermitian metrics on TN, E

By partition of unity, there exists a smooth fiberwise Kähler metric g^{TN} on TN. Let ω be the associated fiberwise Kähler form. Let $g^{\Lambda^{\cdot}(T^*_{\mathbb{C}}N)}$ be the induced Hermitian metric on $\Lambda^{\cdot}(T^*_{\mathbb{C}}N)$. The fiberwise volume form induced by g^{TN} is denoted dv_N.

Let $\mathscr{E} = \Omega^{0,\cdot}(N,E)$ be the vector space of antiholomorphic differential forms on N with values in E, which is equipped with a Hermitian metric $g^{\mathscr{E}}$, such that for $\alpha,\beta \in \mathscr{E}$,

$$g^{\mathscr{E}}(\alpha,\beta) = \frac{1}{(2\pi)^n}\int_N \left(g^{\Lambda^{\cdot}(T^*_{\mathbb{C}}N)} \otimes g^E\right)(\alpha,\beta)dv_N . \tag{16}$$

We set

$$\omega^{\mathscr{E}} = (g^{\mathscr{E}})^{-1}d^M g^{\mathscr{E}} \in \mathscr{C}^\infty(M, T^*M \otimes_{\mathbb{R}} \mathrm{End}(\mathscr{E})) . \tag{17}$$

3.2. The Levi-Civita superconnection

Let $\overline{\partial}^{E,*}$ be the adjoint of $\overline{\partial}^{E}$ with respect to $g^{\mathscr{E}}$. Set

$$C^{\mathscr{E}} = \overline{\partial}^{E} + \overline{\partial}^{E,*} + d^M + \frac{1}{2}\omega^{\mathscr{E}} ,$$
$$D^{\mathscr{E}} = -\overline{\partial}^{E} + \overline{\partial}^{E,*} + \frac{1}{2}\omega^{\mathscr{E}} . \tag{18}$$

Then $C^{\mathscr{E}}$, $D^{\mathscr{E}}$ act on $\Omega^{\cdot}(M,\mathscr{E})$. And the following identity holds

$$C^{\mathscr{E},2} = -D^{\mathscr{E},2} . \tag{19}$$

We may view $\mathscr{E}$ as an infinite dimensional vector bundle on M equipped with a flat connection d^M. Then $C^{\mathscr{E}}$ is a superconnection on $\mathscr{E}$. Its degree zero part $\overline{\partial}^{E} + \overline{\partial}^{E,*}$ is the fiberwise Dirac operator. Its degree one part is $d^M + \frac{1}{2}\omega^{\mathscr{E}} = \frac{1}{2}(d^M + d^{M,*})$, where $d^{M,*}$ is the adjoint connection with respect to $g^{\mathscr{E}}$. Thus $C^{\mathscr{E}}$ is the Levi-Civita superconnection by definition [2]. (In general, Levi-Civita superconnection has a degree-two part, which vanishes if the fibration in question is flat.)

For $t > 0$, when replacing g^{TN} by $\frac{1}{t}g^{TN}$, the above operators are denoted $C_t^{\mathscr{E}}$, $D_t^{\mathscr{E}}$.

3.3. The index bundle and its characteristic classes

Let $H^{\cdot}(N, E_0)$ be the Dolbeault cohomology of $E_0 \to N$. Let $\chi(N, E_0)$ be its Euler characteristic. The action of G on $E_0 \to N$ induces an action of G on $H^{\cdot}(N, E_0)$. Set

$$H^{\cdot}(N, E) = P_G \times_G H^{\cdot}(N, E_0) . \tag{20}$$

Let $\nabla^{H^{\cdot}(N,E)}$ be the connection on $H^{\cdot}(N, E)$ induced by the flat connection on P_G. Let $s \in \mathscr{C}^\infty(M, \mathscr{E})$ such that $\nabla^{E,N''}s = 0$. We have

$$\nabla^{H^{\cdot}(N,E)}[s] = [d^M s] . \tag{21}$$

By Hodge theory, there is an identification $H^{\cdot}(N, E) \simeq \ker D^E \subseteq \mathscr{E}$. Thus $H^{\cdot}(N, E)$ inherits a metric from $h^{\mathscr{E}}$, denoted $g^{H^{\cdot}(N,E)}$.

Let $\nabla^{H^{\cdot}(N,E),*}$ be the adjoint connection of $\nabla^{H^{\cdot}(N,E)}$ with respect to $g^{H^{\cdot}(N,E)}$. Set

$$\nabla^{H^{\cdot}(N,E),u} = \frac{1}{2}\big(\nabla^{H^{\cdot}(N,E)} + \nabla^{H^{\cdot}(N,E),*}\big) . \tag{22}$$

Proposition 3.1. *For any $t > 0$, we have*

$$\varphi \mathrm{Tr_s}\big[\exp(D_t^{\mathscr{E},2})\big] = \chi(N, E_0) . \tag{23}$$

Proof. By the local families index theorem [2], as $t \to 0$,

$$\varphi \mathrm{Tr_s}\big[\exp(D_t^{\mathscr{E},2})\big] = q_*\big[\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\big] + \mathscr{O}(\sqrt{t}) . \tag{24}$$

Furthermore,

$$\frac{\partial}{\partial t}\mathrm{Tr_s}\big[\exp(D_t^{\mathscr{E},2})\big] = \mathrm{Tr_s}\big[[D_t^{\mathscr{E}}, \frac{\partial}{\partial t}D_t^{\mathscr{E}}]\exp(D_t^{\mathscr{E},2})\big] = \mathrm{Tr_s}\big[[D_t^{\mathscr{E}}, (\frac{\partial}{\partial t}D_t^{\mathscr{E}})\exp(D_t^{\mathscr{E},2})]\big] = 0 . \tag{25}$$

By Proposition 2.2 and by the Riemann–Roch–Hirzebruch formula, we have

$$q_*\big[\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\big] = \chi(N, E_0) . \tag{26}$$

Then (23) follows from (24), (25) and (26). □

3.4. The Riemann–Roch–Grothendieck formula

The following constructions generalize [5, equation (2.22) and (2.23)]. Let $N^{\Lambda^{\cdot}(\overline{T^*N})}$ be the number operator on $\Lambda^{\cdot}(\overline{T^*N})$. For any $t > 0$, set

$$\alpha_t = \sqrt{2\pi}\,\mathrm{i}\varphi\mathrm{Tr_s}\left[D_t^{\mathscr{E}}\exp\left(D_t^{\mathscr{E},2}\right)\right] ,$$
$$\beta_t = \varphi\mathrm{Tr_s}\left[\frac{N^{\Lambda^{\cdot}(\overline{T^*N})}}{2}\left(1 + 2D_t^{\mathscr{E},2}\right)\exp\left(D_t^{\mathscr{E},2}\right)\right] . \tag{27}$$

Proposition 3.2. *For any $t > 0$, α_t is a closed odd form on M, whose cohomology class does not depend on the metric. In particular, this class does not depend on t. Also β_t is an even form on M. For any $t > 0$, the following identity holds,*

$$\frac{\partial}{\partial t}\alpha_t = \frac{1}{t} d^M \beta_t \,. \tag{28}$$

Let $f(x) = xe^{x^2}$. As in Bismut–Lott [5, equation (2.41)], set

$$f(H^{\cdot}(N,E), \nabla^{H^{\cdot}(N,E)}, g^{H^{\cdot}(N,E)}) = \sqrt{2\pi \mathrm{i}}\,\varphi \mathrm{Tr_s}\left[f\left(\frac{\omega^{H^{\cdot}(N,E)}}{2} \right)\right], \tag{29}$$

which is an odd closed form on M.

Put

$$\chi'(N,E) = \sum_{p=0}^{n} (-1)^p \dim H^p(N,E) \,. \tag{30}$$

Theorem 3.3. *As $t \to +\infty$,*

$$\alpha_t = f(H^{\cdot}(N.E), \nabla^{H^{\cdot}(N,E)}, g^{H^{\cdot}(N,E)}) + \mathscr{O}(\frac{1}{\sqrt{t}}) ,$$
$$\beta_t = \frac{1}{2}\chi'(N,E) + \mathscr{O}(\frac{1}{\sqrt{t}}) \,. \tag{31}$$

As $t \to 0$,

$$\alpha_t = q_*\left[\mathrm{Td}(TN, \nabla^{TN})\widetilde{\mathrm{ch}}(E, g^E) + \widetilde{\mathrm{Td}}(TN, g^{TN})\mathrm{ch}(E, \nabla^E)\right] + \frac{1}{2t} d^M q_*\left[\frac{\omega}{2\pi}\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\right] + \mathscr{O}\big(\sqrt{t}\big) ,$$
$$\beta_t = \frac{1}{2} q_*\left[\mathrm{Td}'(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\right] - \frac{1}{2t} q_*\left[\frac{\omega}{2\pi}\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\right] + \mathscr{O}\big(\sqrt{t}\big) \,. \tag{32}$$

Proof. First, we consider α_t.

The $t \to \infty$ part is done in exactly the same way as [5, Theorem 3.16].

We turn to prove the $t \to 0$ part. By [6], the asymptotic expansion of α_t is given by a Laurent series on $\sqrt{t}$. Furthermore, the local index theorem technique [2] implies that

$$\lim_{t\to 0} t\alpha_t = \frac{1}{2} d^M q_*\left[\frac{\omega}{2\pi}\mathrm{Td}(TN, \nabla^{TN})\mathrm{ch}(E, \nabla^E)\right] . \tag{33}$$

Then

$$\alpha_t = b_0 t^{-1} + b_1 t^{-1/2} + b_2 + \mathscr{O}\big(\sqrt{t}\big) , \tag{34}$$

with b_0 given by the right hand side of (33). Same as [3, Theorem 1.20], we apply the following trick

$$\left(1 + t\frac{\partial}{\partial t}\right)\alpha_t = \frac{1}{2} b_1 t^{-1/2} + b_2 + \mathscr{O}\big(\sqrt{t}\big) . \tag{35}$$

Following [3, Theorem 1.21], we construct a Laplacian involving additional Grassman variables $da, d\bar{a}$ (then its heat kernel is a polynomial on $da, d\bar{a}$), such that the $dad\bar{a}$ part of the supertrace of its heat kernel is exactly $\big(1 + t\frac{\partial}{\partial t}\big)\alpha_t$. By applying local index theorem technique [2] to this Laplacian, we get

$$\lim_{t\to 0}\left(1 + t\frac{\partial}{\partial t}\right)\alpha_t = q_*\left[\mathrm{Td}(TN, \nabla^{TN})\widetilde{\mathrm{ch}}(E, g^E) + \widetilde{\mathrm{Td}}(TN, g^{TN})\mathrm{ch}(E, \nabla^E)\right] , \tag{36}$$

which implies that $b_1 = 0$ and b_2 equals the right hand side of (36).

The results for β follows by a transgression argument similar to Proposition 3.2. □

As a consequence of Theorem 3.3, we have the following result, which is an analogue of the Riemann–Roch–Grothendieck theorem.

Corollary 3.4. *We have*

$$\left[f(H^{\cdot}(N,E), \nabla^{H^{\cdot}(N,E)}, g^{H^{\cdot}(N,E)})\right] = \left[q_*\big[\mathrm{Td}(TN, \nabla^{TN})\widetilde{\mathrm{ch}}(E, g^E) + \widetilde{\mathrm{Td}}(TN, g^{TN})\mathrm{ch}(E, \nabla^E)\big]\right] \tag{37}$$

in $H^{\cdot}(M)$.

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