C. R. Acad. Sci. Paris, Ser. I 354 (2016) 562–565

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory

# On the lower bound of the discrepancy of $(t, s)$ sequences: I

## *Sur la limite inférieure de la discrépance de $(t, s)$ suites : I*

Mordechay B. Levin

*Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel*

ARTICLE INFO

*Article history:*
Received 25 May 2015
Accepted 10 February 2016
Available online 11 April 2016

Presented by the Editorial Board

ABSTRACT

We find the exact lower bound of the discrepancy of shifted Niederreiter's sequences.



RÉSUMÉ

Nous trouvons une limite inférieure pour la discrépance de suites décalées de Niederreiter.



### 1. Introduction

Let $((\mathbf{x}_n)_{n\geq 1})$ be an $s$-dimensional sequence in the unit cube $[0, 1)^s$, $J_{\boldsymbol{\gamma}} = [0, \gamma_1) \times \cdots \times [0, \gamma_s) \subseteq [0, 1)^s$,

$$\Delta((\mathbf{x}_n)_{n=1}^N, J_{\boldsymbol{\gamma}}) = \sum_{0\leq n<N} \mathbf{1}(\mathbf{x}_n, J_{\boldsymbol{\gamma}}) - N\gamma_1 \dots \gamma_s, \tag{1}$$

where $\mathbf{1}(\mathbf{x}, J) = 1$, if $\mathbf{x} \in J$ and $\mathbf{1}(\mathbf{x}, J) = 0$, if $\mathbf{x} \notin J$. We define the star *discrepancy* of a $N$-point set $(\mathbf{x}_n)_{n=1}^N$ as

$$D^*((\mathbf{x}_n)_{n=1}^N) = \sup_{0<\gamma_1,\dots,\gamma_s\leq 1} |\Delta((\mathbf{x}_n)_{n=1}^N, J_{\boldsymbol{\gamma}})|/N.$$

Let $((\mathbf{x}_n)_{n\geq 1})$ be an arbitrary sequence in $[0, 1)^s$. According to the well-known conjecture (see, e.g., [2, p. 67], [6, p. 32])

$$\overline{\lim}_{N\to\infty} N(\ln N)^{-s} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0. \tag{2}$$

In 1972, W. Schmidt [2, ref. 237] proved this conjecture for $s = 1$. For $s = 2$, Faure and Chaix [2, ref. 75] proved (2) for a class of $(t, s)$-sequences. For a review of research on this conjecture, see for example [1]. About the application of the concept of discrepancy see [2,3,6].

**Definition 1.** Let $b \geq 2, s \geq 1$, and $0 \leq u \leq m$ be integers and let $\mathbf{e} = (e_1, \cdots, e_s) \in \mathbf{N}^s$. A $(u, m, \mathbf{e}, s)$-net in base $b$ is a point set $\mathcal{P}$ of $b^m$ points in $[0, 1)^s$ such that every subinterval $J \subseteq [0, 1)^s$ of volume $\mathrm{Vol}(J) \geq b^{u-m}$ which has the form

*E-mail address:* mlevin@math.biu.ac.il.

http://dx.doi.org/10.1016/j.crma.2016.02.011

$J=\prod_{1\le i\le s}[a_i b^{-d_i},(a_i+1)b^{-d_i})$, with integers $d_i\ge 0$, $0\le a_i<b^{d_i}$ and $e_i|d_i$ for $1\le i\le s$, contains exactly $b^m\mathrm{Vol}(J)$ points of $\mathcal{P}$.

If $e=(e_1,\dots,e_s)=(1,\dots,1)$, we obtain a classical $(u,m,s)$-net. For $x=\sum_{j\ge 1}x_j p_i^{-j}$, where $x_i\in Z_b=\{0,\dots,b-1\}$ and $m\in\mathbf{N}$, we define the truncation $[x]_m=\sum_{1\le j\le m}x_j b^{-j}$. If $\mathbf{x}=(x^{(1)},\dots,x^{(s)})\in[0,1)^s$, then the truncation $[\mathbf{x}]_m$ is defined coordinatewise, that is, $[\mathbf{x}]_m=([x^{(1)}]_m,\dots,[x^{(s)}]_m)$.

**Definition 2.** Let $b\ge 2$, $s\ge 1$, and $0\le u\le m$ be integers and let $\mathbf{e}=(e_1,\dots,e_s)\in\mathbf{N}^s$. A sequence $\mathbf{x}_0,\mathbf{x}_1,\dots$ of points in $[0,1)^s$ is a $(u,\mathbf{e},s)$-sequence in base $b$ if for all integers $k\ge 0$ and $m>u$ the points $[\mathbf{x}_n]_m$ with $kb^m\le n<(k+1)b^m$ form a $(u,m,\mathbf{e},s)$-net in base $b$.

If $e=(e_1,\dots,e_s)=(1,\cdots,1)$, we obtain a classical $(u,s)$-sequence. For $x=\sum_{j\ge 1}x_i p_i^{-i}$, and $\gamma=\sum_{j\ge 1}\gamma_i p_i^{-i}$ where $x_i,\gamma_i\in Z_b$, we define the ($b$-adic) digitally shifted point $v$ by $v=x\oplus\gamma:=\sum_{j\ge 1}v_i p_i^{-i}$, where $v_i\equiv x_i+\gamma_i(\bmod\ b)$ and $v_i\in Z_b$. For higher dimensions $s>1$ let $\boldsymbol{\gamma}=(\gamma^{(1)},\dots,\gamma^{(s)})\in[0,1)^s$. For $\mathbf{x}=(x^{(1)},\dots,x^{(s)})\in[0,1)^s$, we define the ($b$-adic) digital shifted point $\mathbf{v}$ by $\mathbf{v}=\mathbf{x}\oplus\boldsymbol{\gamma}=(x^{(1)}\oplus\gamma^{(1)},\dots,x^{(s)}\oplus\gamma^{(s)})$. For $n_1,n_2\in[0,b^m)$, we define $n_1\oplus n_2:=(n_1/b^m\oplus n_2/b^m)b^m$.

For $x=\sum_{j\ge 1}x_i p_i^{-i}$, where $x_i\in Z_b$, $x_i=0$ $(i=1,\dots,k)$ and $x_{k+1}\neq 0$, we define the absolute valuation $\|.\|_b$ of $x$ by $\|x\|_b=b^{-k-1}$. Let $\|n\|_b=b^k$ for $n\in[b^k,b^{k+1})$.

**Definition 3.** A digital point set $(\mathbf{x}_n)_{0\le n<b^m}$ in $[0,1)^s$ is $d$-admissible in base $b$ if

$$\min_{0\le k<n<b^m}\|\mathbf{x}_n\ominus\mathbf{x}_k\|_b>b^{-m-d}\quad\text{where}\quad\|\mathbf{x}\|_b:=\prod_{i=1}^{s}\left\|x^{(i)}\right\|_b. \tag{3}$$

A sequence $(\mathbf{x}_n)_{n\ge 0}$ in $[0,1)^s$ is $d$-admissible in base $b$ if $\inf_{n>k\ge 0}\|n\ominus k\|_b\,\|\mathbf{x}_n\ominus\mathbf{x}_k\|_b\ge b^{-d}$.

The theory of $(t,m,s)$-nets and $(t,s)$-sequences is significant for quasi-Monte Carlo methods in scientific computing (see [2–4,6]). By [6, p. 60] $ND^*((\beta_n)_{n=0}^{N-1})=O((\ln N)^s)$ for every $(t,s)$-sequence $(\beta_n)_{n\ge 0}$. In this paper we prove that this estimate is exact for digitally shifted $d$-admissible $(t,s)$ sequences and in particular for digitally shifted Niederreiter's sequence (see, e.g., [2–7]). This result supports the conjecture (2). In [5], we prove that $(t,s)$ sequences from [2, Section 8] are $d$-admissible.

**Theorem 1.** *Let $s\ge 2$, $d\ge 1$, $E_m=\{[y]_m\mid y\in[0,1)\}$, $(\mathbf{x}_n)_{0\le n<b^m}$ be a d-admissible $(t,m,s)$ net in base b, $m\ge 9(d+t)(s-1)^2$. Then*

$$\max_{\mathbf{w}\in E_m^s}b^m D^*((\mathbf{x}_n\oplus\mathbf{w})_{0\le n<b^m})\ge b^{-d}K_{d,t,s}^{-s+1}m^{s-1}\quad\text{with}\quad K_{d,t,s}=4(d+t)(s-1)^2.$$

**Theorem 2.** *Let $s\ge 1$, $d\ge 1$, $(\mathbf{x}_n)_{n\ge 0}$ be a d-admissible $(t,s)$ sequence in base b. Then*

$$1+\min_{0\le Q<b^m}\ \max_{1\le N\le b^m,\mathbf{w}\in E_m^s}ND^*((\mathbf{x}_{n\oplus Q}\oplus\mathbf{w})_{0\le n<N})\ge b^{-d}K_{d,t,s+1}^{-s}m^s\quad\textit{for}\quad m\ge 9(d+t)s^2. \tag{4}$$

**Theorem 3.** *Let $s\ge 1$, $(\mathbf{x}_n)_{n\ge 0}$ be a generalized Niederreiter sequence with generating polynomials $p_1,\dots,p_s$ (see [2, p. 266], [7, p. 242]), $e_i=\deg(p_i)$ $1\le i\le s$, $e_0=e_1+\dots+e_s$, $d=e_0$, $t=e_0-s$. Then (4) holds.*

## 2. Proof

**Lemma 1.** *Let $\dot{s}\ge 2$, $d\ge 1$, $(\mathbf{x}_n)_{0\le n<b^m}$ be a d-admissible $(t,m,\dot{s})$ net in base b, $d_0=d+t$, $\hat{e}\in\mathbf{N}$, $0<\epsilon\le(2d_0\hat{e}(\dot{s}-1))^{-1}$, $\dot{m}=[m\epsilon]$, $\ddot{m}_i=0$, $\dot{m}_i=d_0\hat{e}\dot{m}$ $(1\le i\le\dot{s}-1)$, $\dot{m}_{\dot{s}}=m-(\dot{s}-1)\dot{m}_1-t\ge 1$, $\ddot{m}_{\dot{s}}=\dot{m}_{\dot{s}}+\dot{m}_1$, $B_i\subset\{0,\dots,\dot{m}-1\}$ $(1\le i\le\dot{s})$, $\mathbf{w}\in E_m^{\dot{s}}$ and let $\gamma^{(i)}=\gamma_1^{(i)}/b+\dots+\gamma_{\dot{m}_i}^{(i)}/b^{\dot{m}_i}$,*

$$\gamma^{(i)}_{\ddot{m}_i+d_0(\hat{j}_i\hat{e}+\check{j}_i)+\check{j}_i}=0\quad\textit{for}\quad 1\le\check{j}_i<d_0,\qquad\gamma^{(i)}_{\ddot{m}_i+d_0(\hat{j}_i\hat{e}+\check{j}_i)+\check{j}_i}=1\quad\textit{for}\quad\check{j}_i=d_0, \tag{5}$$

*and $\hat{j}_i\in\{0,\dots,\dot{m}-1\}\setminus B_i$, $0\le\check{j}_i<\hat{e}$, $1\le i\le\dot{s}$, $\boldsymbol{\gamma}=(\gamma^{(1)},\dots,\gamma^{(\dot{s})})$, $B=\#B_1+\dots+\#B_{\dot{s}}$. Let us assume that there exists $n_0\in[0,b^m)$ such that $[(\mathbf{x}_{n_0}\oplus\mathbf{w})^{(i)}]_{\dot{m}_i}=\gamma^{(i)}$, $1\le i\le\dot{s}$, and $m\ge 4\epsilon^{-1}(\dot{s}-1)(1+\dot{s}B)+2t$. Then*

$$\tilde{\Delta}:=\Delta((\mathbf{x}_n\oplus\mathbf{w})_{0\le n<b^m},J_{\boldsymbol{\gamma}})\le -b^{-d}\left(\hat{e}\epsilon(2(\dot{s}-1))^{-1}\right)^{\dot{s}-1}m^{\dot{s}-1}+b^{t+s}d_0\hat{e}Bm^{\dot{s}-2}.$$

**Proof.** Let $\mathbf{r}=(r_1,\dots,r_{\dot{s}})\in\mathbf{N}^{\dot{s}}$, $r_0=r_1+\dots+r_{\dot{s}}$, $A=\{\mathbf{r}\mid 1\le r_i\le \dot{m}_i,\ i=1,\dots,\dot{s}$ and $\gamma^{(1)}_{r_1}\dots\gamma^{(\dot{s})}_{r_{\dot{s}}}\ne 0\}$, $\dot{A}=\{\mathbf{r}\in A\mid \exists i\in[1,\dot{s}]: [(r_i-\ddot{m}_i-1)/(d_0\hat{e})]\in B_i\}$, $A_1=\{\mathbf{r}\in A\mid r_0\le m-t\}$, $A_2=\{\mathbf{r}\in A\cap\dot{A}\mid r_0>m-t\}$, $A_3=\{\mathbf{r}\in A\setminus\dot{A}\mid m-t<r_0<m+d\}$ and $A_4=\{\mathbf{r}\in A\setminus\dot{A}\mid r_0\ge m+d\}$. We have $A=A_1\cup A_2\cup A_3\cup A_4$. Let

$$J_{\boldsymbol{\gamma}}=\prod_{1\le j\le\dot{s}}[0,\gamma^{(i)})\qquad\text{and}\qquad J_{\mathbf{r},\boldsymbol{\gamma},\mathbf{g}}=\prod_{1\le j\le\dot{s}}\big[[\gamma^{(i)}]_{r_i-1}+g_ib^{-r_i},[\gamma^{(i)}]_{r_i-1}+(g_i+1)b^{-r_i}\big).$$

Similarly to [6, p. 37,38], from (1) we have that $\tilde{\Delta}=\Delta_1+\Delta_2+\Delta_3+\Delta_4$, where

$$\Delta_i=\sum_{\mathbf{r}\in A_i}\Psi^{(\mathbf{r})}_{\boldsymbol{\gamma}},\qquad \Psi_{\mathbf{r},\boldsymbol{\gamma}}=\sum_{0\le g_i<\gamma^{(i)}_{r_i},\ 1\le i\le\dot{s}}\Psi_{\mathbf{r},\boldsymbol{\gamma},\mathbf{g}}\quad\text{and}\quad \Psi_{\mathbf{r},\boldsymbol{\gamma},\mathbf{g}}=\sum_{0\le n<b^m}\big(\mathbf{1}(\mathbf{x}_n\oplus\mathbf{w},J_{\mathbf{r},\boldsymbol{\gamma},\mathbf{g}})-b^{-r_0}\big).$$

Consider $\Delta_1$. Bearing in mind that $(\mathbf{x}_n\oplus\mathbf{w})_{0\le n<b^m}$ is a $(t,m,\dot{s})$ net, we obtain $\Psi_{\mathbf{r},\boldsymbol{\gamma},\mathbf{g}}=0$. Hence $\Delta_1=0$.
Consider $\Delta_2$. It is easy to verify that $\Delta_2\le b^{t+\dot{s}-1}d_0\hat{e}Bm^{\dot{s}-2}$.
Consider $\Delta_3$. We see that $r_0\in(m-t,m+d)$. Hence $r_{\dot{s}}=r_0-r_1-\dots-r_{\dot{s}-1}>m-t-(\dot{s}-1)\dot{m}_1=\ddot{m}_{\dot{s}}$. Taking into account that $\gamma^{(i)}_{r_i}\ne0$ and $[(r_i-\ddot{m}_i-1)/(d_0\hat{e})]\notin B_i$, we get $r_i=\ddot{m}_i+d_0j_i$ with some $j_i\ge1$, $1\le i\le\dot{s}$. Hence

$$r_0=r_1+\dots+r_{\dot{s}}=\ddot{m}_{\dot{s}}+d_0(j_1+\dots+j_{\dot{s}})=m-t+d_0(j_1+\dots+j_{\dot{s}}-(\dot{s}-1)\hat{e}\dot{m})>m-t.$$

Thus $r_0\ge m-t+d_0=m+d$. We have a contradiction. Hence $A_3=\emptyset$ and $\Delta_3=0$.
Consider $\Delta_4$. Suppose that $\mathbf{1}(\mathbf{x}_k\oplus\mathbf{w},J_{\mathbf{r},\boldsymbol{\gamma},\mathbf{0}})=1$ for some $k\in[0,b^m)$ and $r_0\ge m+d$. Then $[(\mathbf{x}_k\oplus\mathbf{w})^{(i)}]_{r_i}=[\gamma^{(i)}]_{r_i}-b^{-r_i}$, $i=1,\dots,\dot{s}$. Hence $x^{(i)}_{k,j}\ominus x^{(i)}_{n_0,j}=0$ for $j\in[1,r_i)$, $i=1,\dots,\dot{s}$. Therefore

$$\left\|x^{(i)}_k\ominus x^{(i)}_{n_0}\right\|_b\le b^{-r_i}\quad\text{for}\quad i=1,\dots,\dot{s}\quad\text{and}\quad\left\|\mathbf{x}_k\ominus\mathbf{x}_{n_0}\right\|_b\le b^{-r_0}\le b^{-m-d}.$$

By (3) and the conditions of the lemma, we have a contradiction. Thus $\mathbf{1}(\mathbf{x}_k\oplus\mathbf{w},J_{\mathbf{r},\boldsymbol{\gamma},\mathbf{0}})=0$.

We have $\Delta_4\le-\sum_{\mathbf{r}\in A_4}b^{m-r_0}$. We derive $\Delta_4\le -b^{-d}\#A_5$ with $A_5=\{\mathbf{r}\in A_4\mid r_0=m+d\}$.

Let $\hat{j}_i\in\{0,\dots,\dot{m}-1\}\setminus B_i$, $\check{j}_i\in[0,\hat{e}-1]$ and $r_i=\ddot{m}_i+d_0(\hat{e}\hat{j}_i+\check{j}_i+1)$ for $i\in[1,\dot{s}]$. By (5), we get that $\gamma^{(i)}_{r_i}=1$ for $i\in[1,\dot{s}]$. Hence $A_5\supseteq A_6$, where

$$A_6=\{\mathbf{r}\mid r_0=m+d,\ r_i=\ddot{m}_i+d_0(\hat{e}\hat{j}_i+\check{j}_i+1),\ \ \hat{j}_i\in\{0,\dots,\dot{m}-1\}\setminus B_i,\ \check{j}_i\in[0,\hat{e}-1],\ i\in[1,\dot{s}]\}.$$

Let $j_i=\hat{e}\hat{j}_i+\check{j}_i+1$, for $i\in[1,\dot{s}]$. We have:

$$r_0=\ddot{m}_{\dot{s}}+d_0(j_1+\dots+j_{\dot{s}})=m-t+d_0(j_1+\dots+j_{\dot{s}}-(\dot{s}-1)\hat{e}\dot{m})=m+d\ \text{with}\ d_0=d+t.$$

Hence $j_{\dot{s}}=(\dot{s}-1)\hat{e}\dot{m}+1-j_1-\dots-j_{\dot{s}-1}$. It is easy to verify that $j_{\dot{s}}\in[1,\hat{e}\dot{m}]$ for $\hat{j}_i\in[\dot{m}-\dddot{m},\dot{m}-1]$, for $i\in[1,\dot{s}-1]$, with $\dddot{m}=[\dot{m}/(\dot{s}-1)]$. Thus $\#A_6\ge\#A_7$, where

$$\begin{aligned}A_7=\{(j_1,\dots,j_{\dot{s}-1})\mid j_i=\hat{e}\hat{j}_i+\check{j}_i+1,\quad &\hat{j}_i\in\{0,\dots,\dot{m}-1\}\setminus B_i,\quad \check{j}_i\in[0,\hat{e}-1],\quad i\in[1,\dot{s}],\\ \hat{j}_i\in[\dot{m}-\dddot{m},\dot{m}-1],\quad i\in[1,\dot{s}-1]\quad&\text{and}\quad j_{\dot{s}}=(\dot{s}-1)\hat{e}\dot{m}+1-j_1-\dots-j_{\dot{s}-1}\}.\end{aligned}$$

We obtain $\#A_7\ge\#A_8-\hat{e}\#B_{\dot{s}}m^{\dot{s}-2}$, where

$$A_8=\{(j_1,\dots,j_{\dot{s}-1})\mid j_i=\hat{e}\hat{j}_i+\check{j}_i+1,\ \hat{j}_i\in\{\dot{m}-\dddot{m},\dots,\dot{m}-1\}\setminus B_i,\ \check{j}_i\in[0,\hat{e}-1],\ i\in[1,\dot{s}-1]\}.$$

Therefore

$$\begin{aligned}\#A_8\hat{e}^{-\dot{s}+1}&\ge\#\{(\hat{j}_1,\dots,\hat{j}_{\dot{s}-1})\mid 1\le\hat{j}_i\le\dddot{m}-\#B_i,\ 1\le i\le\dot{s}-1\}\ge(\dddot{m}-B)^{\dot{s}-1}\\ &=\dddot{m}^{\dot{s}-1}(1-B/\dddot{m})^{\dot{s}-1}\ge\dddot{m}^{\dot{s}-1}\big(1-(\dot{s}-1)B/\dddot{m}\big)\ge\big(m\epsilon(2(\dot{s}-1))^{-1}\big)^{\dot{s}-1}-(\dot{s}-1)B\dddot{m}^{\dot{s}-2}\end{aligned}$$

for $m\ge4\epsilon^{-1}(\dot{s}-1)(1+\dot{s}B)+2t$. Therefore $\tilde{\Delta}\le-b^{-d}\big(\hat{e}\epsilon(2(\dot{s}-1))^{-1}\big)^{\dot{s}-1}m^{\dot{s}-1}+b^{t+s}d_0\hat{e}Bm^{\dot{s}-2}$. Thus Lemma 1 is proved. □

**Proof of Theorem 1.** Using Lemma 1 with $\dot{s}=s$, $B_i=\emptyset$ $(1\le i\le s)$, $B=0$, $\hat{e}=1$, $\epsilon=(2(s-1)d_0)^{-1}$, $n_0=0$, and $\mathbf{w}=[\boldsymbol{\gamma}\ominus\mathbf{x}_0]_m$, we obtain the assertion of Theorem 1. □

**Proof of Theorem 2.** According to [6, Lemma 3.7], we have

$$1+\sup_{1\le N\le b^m}ND^*((\mathbf{x}_{n\oplus Q}\oplus\mathbf{w})_{n=0}^{N-1})\ge b^mD^*((\mathbf{x}_{n\oplus Q}\oplus\mathbf{w},n/b^m)_{n=0}^{b^m-1})=b^mD^*((\mathbf{x}_n\oplus\mathbf{w},(n\ominus Q)/b^m)_{n=0}^{b^m-1}).$$

By (3) and [2, Lemma 4.38], we have that $((\mathbf{x}_n,n/b^m)_{0\le n<b^m})$ is a $d$-admissible $(t,m,s+1)$-net in base $b$. Using Lemma 1 with $\dot{s}=s+1$, $x^{(s+1)}_n=n/b^m$, $B_i=\emptyset$ $(1\le i\le s+1)$, $B=0$, $\hat{e}=1$, $\epsilon=(2sd_0)^{-1}$, $n_0=Q\oplus\gamma^{(s+1)}b^m$, and $\mathbf{w}=([(\gamma^{(1)},\dots,\gamma^{(s)})\ominus\mathbf{x}_{n_0}]_m,-Q/b^m)$, we get the assertion of Theorem 2. □

**Lemma 2.** *Let* $(\mathbf{x}_n)_{n\geq 0}$ *be a* $(0,\mathbf{e},s)$ *sequence in base* $b$*. Then* $(\mathbf{x}_n)_{n\geq 0}$ *is* $e_0$*-admissible.*

**Proof.** Suppose that $(\mathbf{x}_n)_{n\geq 0}$ is not a $e_0$-admissible. Then there exists $n_0 > k_0 \geq 0$ with $\|n_0 \ominus k_0\|_b \times \left\|\mathbf{x}_{n_0} \ominus \mathbf{x}_{k_0}\right\|_b \leq b^{-e_0-1}$. Let $\|n_0 \ominus k_0\|_b = b^{\tilde{d}}$, and let $\left\|x^{(i)}_{n_0} \ominus x^{(i)}_{k_0}\right\|_b = b^{-d_i-1}$ $(i=1,\dots,s)$. Hence $\kappa := \tilde{d} - \sum_{1\leq i\leq s}(d_i+1) + e_0 + 1 \leq 0$. Let $\dot{d}_i = [d_i/e_i]e_i \geq d_i - e_i + 1$, $a_i = [x^{(i)}_{n_0}]_{\dot{d}_i} b^{\dot{d}_i}$ $(i=1,\dots,s)$ and let $J = \prod_{1\leq i\leq s}[a_i b^{-\dot{d}_i}, (a_i+1)b^{-\dot{d}_i})$. We have $x^{(i)}_{n_0,j} = x^{(i)}_{k_0,j}$ for all $j \in [1,d_i]$, $i \in [1,s]$. Hence $\mathbf{x}_{n_0}, \mathbf{x}_{k_0} \in J$. We derive

$$0 \geq \kappa = \tilde{d} + 1 - \sum_{1\leq i\leq s}(d_i - e_i + 1) \geq \tilde{d} + 1 - \sum_{1\leq i\leq s}\dot{d}_i, \quad \text{and} \quad 1 \geq b^{\tilde{d}+1}\mathrm{Vol}(J). \tag{6}$$

Let $n_0 = \dot{n}_0 b^{\tilde{d}+1} + \ddot{n}_0$ where $\ddot{n}_0 \in [0, b^{\tilde{d}+1})$. It is easy to see that $k_0 = \dot{n}_0 b^{\tilde{d}+1} + \ddot{k}_0$, with some $\ddot{k}_0 \in [0, b^{\tilde{d}+1})$. Hence $n_0, k_0 \in [\dot{n}_0 b^{\tilde{d}+1}, (\dot{n}_0+1)b^{\tilde{d}+1}) =: W$. Thus $\sum_{n\in W}\mathbf{1}(\mathbf{x}_n, J) \geq 2$. Bearing in mind (6), we obtain that $(\mathbf{x}_n)_{n\geq 0}$ is not $(0,\mathbf{e},s)$ sequence. We have a contradiction. Hence Lemma 2 is proved. □

**Proof of Theorem 3.** Let $\mathbf{e} = (e_1,\dots,e_s)$. By [7] and [2, p. 266], we have that $(\mathbf{x}_n)_{n\geq 0}$ is a $(0,\mathbf{e},s)$ and $(e_0 - s, s)$ sequence. Applying Lemma 2 and Theorem 2, we obtain the assertion of Theorem 3. □

## References


[1] D. Bilyk, On Roth's orthogonal function method in discrepancy theory, Unif. Distrib. Theory 6 (1) (2011) 143–184.
[2] J. Dick, F. Pillichshammer, Digital Nets and Sequences, Discrepancy Theory and Quasi-Monte Carlo Integration, Cambridge University Press, Cambridge, UK, 2010.
[3] M. Drmota, R. Tichy, Sequences, Discrepancies and Applications, Lecture Notes in Mathematics, vol. 1651, 1997.
[4] C. Lemieux, Monte Carlo and Quasi-Monte Carlo Sampling, Springer Series in Statistics, Springer, New York, 2009.
[5] M.B. Levin, On the lower bound of the discrepancy of $(t,s)$ sequences: II, http://arXiv.org/abs/1505.04975.
[6] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 63, SIAM, 1992.
[7] S. Tezuka, On the discrepancy of generalized Niederreiter sequences, J. Complexity 29 (2013) 240–247.