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# The Euler class of an umbilic foliation

## *La class d'Euler d'un feuilletage totalement ombilical*

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ABSTRACT

Given a foliation on a manifold with suitable curvature form, the Euler class of its tangent bundle is explicitly computed whenever it admits an umbilic leaf. If the leaf is compact, then topological obstructions arise by considering foliated manifolds with certain trivial cohomology group. The results fully generalize to distributions tangent to at least one compact umbilic submanifold.

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RÉSUMÉ

Étant donné une variété feuilletée munie de formes de courbure convenables, nous calculons explicitement sa classe d'Euler dans le cas totalement ombilical. Si la feuille est compacte, nous obtenons des obstructions topologiques dans le cas où certains groupes d'homologie de la variété feuilletée sont triviaux. Les résultats se généralisent aux distributions tangentes à une sous-variété ombilicale.

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## 1. Introduction

The main idea of this manuscript is to compute the Euler class of a foliation $\mathcal{F}$ of even dimension, assuming that it admits a compact and umbilic leaf. Besides the umbilicity of the leaf, the geometrical assumptions considered are the sectional curvatures of the ambient manifold restricted to the leaves of $\mathcal{F}$, and they are the key to write explicitly this class. Translating geometrical hypothesis into topological ones implies obstructions to the existence of these foliations by looking at the cohomology of the ambient manifold as well as by asking for positiveness of sectional curvatures of $M$ along $\mathcal{F}$.

**Theorem 1.1.** *Let $\mathcal{D}^{2k}$ be a distribution on a Riemannian manifold $M^{2k+p}$ with pure curvature form. Let $L$ be a compact umbilic submanifold of $M$, with dimension $2k$, and suppose the sectional curvatures of $M$ are nonnegative along $L$. If $\mathcal{D}$ is tangent to $L$, then $\epsilon(\mathcal{D}) \neq 0$.*

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Some consequences of Theorem 1.1 are related to conditions on the class where $\epsilon(\mathcal{D})$ lives. If the ambient manifold is a homology sphere, then it can not be foliated by a foliation satisfying the above hypothesis.

**Theorem 1.2.** *Let $\mathcal{F}^4$ be a foliation on a Riemannian manifold $M^{4+p}$. Let $L$ be a compact umbilic leaf of $M$, with dimension 4, and suppose that the sectional curvatures of $M$ are positive along $L$. If $\mathcal{D}$ is tangent to $L$, then $\epsilon(\mathcal{D}) \neq 0$.*

There is a theorem of Milnor (which was not published, but can be found in [8]) which asserts that the fourth sectional curvature of $M$ is positive when its sectional curvatures are positive, and in this case the only hypothesis on $M$ is the positiveness of its sectional curvatures along $L$. On the other hand, Alain Connes introduced the Euler characteristic $\chi(\mathcal{F}, \nu)$ for a foliation endowed with a transverse measure, see [9]. The particular case where $\mathcal{F}$ is determined by a closed and global form $\nu$ of its normal distribution, which is called $SL$-foliation ([18]), $\chi(\mathcal{F}, \nu)$ is shown to be nonnegative, provided the scalar curvatures of the leaves of $\mathcal{F}$ are nonnegative, by means of Milnor's result. The next theorem can be viewed as direct consequence of Connes' "Gauss–Bonnet type" theorem for foliations of dimension 4. It reads:

**Theorem 1.3.** *Let $\mathcal{F}$ be a $SL$-foliation of dimension 4 on a closed Riemannian manifold $M^{4+p}$. If the scalar curvatures of the leaves are nonnegative, then $\chi(\mathcal{F}, \nu) = \int_M \epsilon(\mathcal{F}) \wedge \nu \geq 0$.*

Foliations are integrable subbundles of the tangent bundle, and although in the literature (see [7] and references therein) characteristic classes are constructed on the normal bundle, there are interesting consequences when they are computed on the tangent distributions themselves. In this context, geometrical and topological hypotheses on the foliations and on the ambient manifold are assumed in order to explicitly determine the properties of the classes.

For example, if the foliation is totally geodesic and of odd dimension $n$, a theorem of [12] asserts that the $(n+p)$-th Pontryagin class of $\mathcal{F}$ vanishes. If the leaves are surfaces, and the normal distribution is a minimal foliation, then from [3], the Euler class of $\mathcal{F}$ is different from zero when $\mathrm{Ric}(M) > 0$.

Umbilic foliations were studied from the perspective of conformal geometry in [14]. Their approach includes the properties of local and global invariants, the question whether a Riemannian manifold admits an umbilic or a foliation with weaker conditions, such as Dupin foliations, as well as asking how far from umbilic a foliation is by defining a conformal invariant quantity. In dimension 3, they were classified in the light of transversely holomorphic fields in [5].

Euclidean spheres do not admit totally geodesic nor umbilic foliations of codimension one. However, for codimensions greater than one, they are far from being geometrically classified. The geometrical abundance is made explicit already in the codimension-2 case of $S^3$,

**Theorem 1.4.** *(See [10].) A submanifold of $\widetilde{G}_2(\mathbb{R}) \cong S^2 \times S^2$ corresponds to a fibration of $S^3$ by oriented great circles if and only if it is the graph of a certain distance decreasing map $f : S^2 \to S^2$.*

Umbilic foliations of $S^3$ and other odd spheres $S^{2k+1}$ are obtained by taking a smooth positive function $f$ constant on the leaves of a totally geodesic foliation and making a conformal change of the induced metric, $\langle \cdot, \cdot \rangle \mapsto f \langle \cdot, \cdot \rangle$, or by considering small deformations of all planes which give great circle fibrations, in order to obtain affine nonlinear planes intersecting the sphere (see [1,5] and [6] for details).

## 2. Preliminaries

Let $\mathcal{F}$ be an oriented and transversely oriented foliation on a Riemannian manifold $M$, $\dim(M) = 2k + p$ and $\dim(\mathcal{F}) = 2k$. Let $U$ be a neighborhood of $x \in M$, $\{e_A\}$ an orthonormal frame defined on $U$, such that its dual frame, curvature and connection forms are related by the structural equations of $M$

$$\omega_A(e_B) = \delta_{AB}, \quad \delta_{AB} = 0 \text{ if } A \neq B, \quad \delta_{AA} = 0, \quad \nabla e_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$\mathrm{d}\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \mathrm{d}\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \Omega_{AB}, \tag{2}$$

$$\Omega_{AB} = \frac{1}{2} \sum_{C,D} R_{ABCD} \omega_C \wedge \omega_D, \quad R_{ABCD} + R_{ABDC} = 0, \tag{3}$$

where $1 \leq \ldots A, B, C, D, \ldots \leq 2k + p$. The orthonormal frame is assumed to be an adapted frame, which means that the range of indexes vary according to the following notation: $1 \leq \ldots i, j, \ldots \leq 2k < \ldots \alpha, \beta, \ldots \leq 2k + p$. A given leaf $L$ of $\mathcal{F}$ is an immersed submanifold of $M$, and $\theta_A$ and $\theta_{AB}$ denote the pullback of $\omega_A$ and $\omega_{AB}$ via the immersion, respectively. For every section $X \in \Gamma L$, $\theta_\alpha(X) = 0$, then by the previous structure equations, $0 = \mathrm{d}\theta_\alpha = \sum_i \theta_{\alpha i} \wedge \theta_i$, $\mathrm{d}\theta_i = \sum_j \theta_{ij} \wedge \theta_j$, $\theta_{ij} + \theta_{ji} = 0$. By Cartan's Lemma, $\theta_{i\alpha} = \sum_j h^\alpha_{ij} \theta_j$, where $h^\alpha_{ij} = h^\alpha_{ji}$ are the entries of the second fundamental form matrix of $L$ in the normal direction $e_\alpha$. Thus, the curvature forms of $M$ and of a leaf $L$ of $\mathcal{F}$ are related by

$$\Omega^L_{ij} = \sum_\alpha \theta_{i\alpha} \wedge \theta_{j\alpha} + \Omega_{ij}. \tag{4}$$

If $L$ is an umbilic submanifold, $\theta_{\alpha i} = \lambda_\alpha \theta_i$, with $\lambda_\alpha : M \to \mathbb{R}$ a continuous function. Unless otherwise stated, totally geodesic submanifolds will also be called umbilic, since all $\lambda_\alpha$ are identically zero in this case.

Let $H^*(M, \mathbb{R})$ be the cohomology ring of $M$. Taking the aforementioned adapted frame and according to [13], page 318, $\epsilon(\mathcal{D}) \in H^{2k}(M, \mathbb{R})$ can be written as

$$\epsilon(\mathcal{D}) = \frac{(-1)^k}{(4\pi)^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sgn}(\sigma) \Omega^L_{\sigma(1)\sigma(2)} \wedge \cdots \wedge \Omega^L_{\sigma(2k-1)\sigma(2k)}, \tag{5}$$

where $\mathcal{D}$ is the distribution corresponding to $\mathcal{F}$, and $\mathfrak{S}_n$ stands for the permutation group of $n$ elements. The first step towards an explicit computation of [5] is to find a manifold where $\Omega_{AB}$ can be simply written. The simplest examples are space forms, where $\Omega_{AB} = c\,\omega_A \wedge \omega_B$. Following [2] and [15], there is a much larger class for such manifolds,

**Definition 2.1.** A Riemannian manifold $M$ is said to have pure curvature form if there exists an orthonormal frame $\{e_A\}$ such that $\Omega_{AB} = c_{AB}\,\omega_A \wedge \omega_B$, on every point of $M$.

**Example 1.** Let $N^n$ be an immersed submanifold of $M^{n+p}(c)$ and suppose it has flat normal bundle. For every $x \in N$, there exists a basis of $T_x N$ which diagonalizes $(h^\alpha_{ij})$ for simultaneous normal directions $e_\alpha$ and $e_\beta$. By [4], $N$ has pure curvature form. In particular, every immersed hypersurface of a space form $M^{n+1}(c)$ has pure curvature form.

**Example 2.** By [2], conformally flat manifolds are examples of Riemannian manifolds with pure curvature form. In particular, every manifold with dimension three is an example, and they are characterized by the fact that their Weyl tensor is identically zero.

**Remark 1.** If $M$ and $N$ are Riemannian manifolds with pure curvature form, then the Riemannian product $M \times N$ also satisfies the pure form condition.

## 3. $SL$-foliations and higher order sectional curvatures

Given any Riemannian manifold, let $X$, $Y$ smooth fields on $M$, and define $X^\top$ and $X^\perp$ as natural tangent and normal projections of $X$, respectively. Also, define $A(X, Y) = (\nabla_{Y^\perp} X^\perp)^\top$. The application $A$ is a mimetic second fundamental form of the normal bundle $\nu\mathcal{F}$. Nevertheless, according to [11], $\mathcal{F}$ is a Riemannian foliation if and only if $A$ is antisymmetric, and it is possible to verify that $\nu\mathcal{F}$ is integrable if and only if $A$ is symmetric. Consider $H^\perp = trA = \sum_{i,\alpha} A^i_{\alpha\alpha} e_i$; it is also similar to the mean curvature vector field.

**Theorem 3.1.** *(See [18].) Let $\mathcal{F}$ be a codimension $p$ foliation on a Riemannian manifold $M$. Then the following are equivalent:*

(a) *$\mathcal{F}$ is a $SL$-foliation, that is, the Haefliger cocycles representing $\mathcal{F}$ preserve the volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p$ in $\mathbb{R}^p$;*
(b) *$\mathcal{F}$ admits a global and closed $p$-form $\nu$;*
(c) *$H^\perp \equiv 0$.*

In addition to Riemannian foliations, it is straightforward to see that flows of nonsingular solenoidal vector fields (divergence free) are also examples of $SL$-foliations.

Following [17], for each even integer $2 \le j \le 2k$, define a smooth function $\gamma_j$, called the $j$-th sectional curvature of $M$, by

$$\gamma_j = \Theta_j(e_1, \ldots, e_j) = \sum_{\sigma \in \mathcal{S}_j} \mathrm{sgn}(\sigma) \Omega_{\sigma(1)\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(j-1)\sigma(j)}(e_1, \ldots, e_j), \tag{6}$$

and $\gamma_0 = 1$ is the constant function. It measures the Lipschitz–Killing curvature of a small $j$-dimensional submanifold of $M$, under the exponential image of a $j$-plane. The second sectional curvature $\gamma_2$ is precisely the usual Riemannian sectional curvature. For more details on higher sectional curvatures, see [17] and references therein. In the case where $M$ has pure curvature form, $\gamma_j$ is simply written as

$$\gamma_j = \sum_{i_1<\cdots<i_j} c_{i_1 i_2}\cdots c_{i_{j-1}i_j}. \tag{7}$$

## 4. Proofs

**Lemma 4.1.** *Under the assumptions of Theorem 1.1, the Euler class can be written as*

$$\epsilon(\mathcal{D}) = \frac{(-1)^k}{(4\pi)^k k!}\sum_{l=0}^{k}\binom{k}{l}(2l)!(2k-2l)!||H||^{2l}\gamma_{2k-2l}\theta_1\wedge\cdots\wedge\theta_{2k}. \tag{8}$$

**Proof.** By equations (4) and (5),

$$\epsilon(\mathcal{D}) = \frac{(-1)^k}{(4\pi)^k k!}\sum_{\sigma\in\mathfrak{S}_{2k}}\operatorname{sgn}(\sigma)\left(\sum_\alpha \theta_{\sigma(1)\alpha}\wedge\theta_{\alpha\sigma(2)} + \Omega_{\sigma(1)\sigma(2)}\right)\wedge\cdots$$
$$\cdots\wedge\left(\sum_\alpha \theta_{\sigma(2k-1)\alpha}\wedge\theta_{\alpha\sigma(2k)} + \Omega_{\sigma(2k-1)\sigma(2k)}\right).$$

The mean curvature field of $L$ is $H=\sum_\alpha \lambda_\alpha e_\alpha$, where $\lambda_\alpha$ is the principal curvature in the normal direction $e_\alpha$. Thus,

$$\epsilon(\mathcal{D}) = \frac{(-1)^k}{(4\pi)^k k!}\sum_{l=0}^{k}\binom{k}{l}||H||^{2l}\sum_{\sigma\in\mathfrak{S}_{2k}}\operatorname{sgn}(\sigma)\theta_{\sigma(1)}\wedge\cdots\wedge\theta_{\sigma(2l)}\wedge\Omega_{\sigma(2l+1)\sigma(2l+2)}\wedge\cdots\wedge\Omega_{\sigma(2k-1)\sigma(2k)}, \tag{9}$$

where $||H||^2=\sum_\alpha \lambda_\alpha^2$ and the index $l$ counts the factors $\lambda_\alpha^2\theta_i\wedge\theta_j$ which appear when the wedge product is distributed over the sum. In terms of the permutations $\mathfrak{S}_{2k}$, it is possible to verify that the terms appear precisely $\binom{k}{l}$ times.

Taking into account the pureness of the curvature form of $M$,

$$\epsilon(\mathcal{D}) = \frac{(-1)^k}{(4\pi)^k k!}\sum_{l=0}^{k}\binom{k}{l}||H||^{2l}\sum_{\sigma\in\mathfrak{S}_{2k}}\operatorname{sgn}(\sigma)c_{\sigma(1)\sigma(2)}\cdots c_{\sigma(2k-2l-1)\sigma(2k-2l)}\theta_{\sigma(1)}\wedge\cdots\wedge\theta_{\sigma(2k)}. \tag{10}$$

Therefore, the proof is finished by counting the factors and using Eq. (7). □

**Proof of Theorem 1.1.** The immersion $i: L\to M$ naturally induces $i^*: H^{2k}(M,\mathbb{R})\to H^{2k}(L,\mathbb{R})$ and one is able to compute the integral $\int_L i^*(\epsilon(\mathcal{D}))$. In addition, $L$ is an umbilic submanifold, then equation (8) holds. If $\epsilon(\mathcal{D})$ is an exact form, then the above integral is zero by Stokes' theorem, and combined with (8) and the fact that the integrand is positive leads to a contradiction. Therefore, $\epsilon(\mathcal{D})\neq 0$ in $H^{2k}(M,\mathbb{R})$. □

This theorem generalizes the results in [19] and [4]. The first paper considers some obstructions to the dimension of leaves of an umbilic foliation, but with additional hypotheses; the foliation is assumed to be Riemannian. In the second article, the authors take into account a totally geodesic foliation, an hypothesis on the curvature of the ambient manifold; however, the normal bundle is integrable.

**Corollary 4.2.** *Let $M^{2k+p}(c)$ be a Riemannian manifold of constant curvature $c\geq 0$, endowed with a distribution $\mathcal{D}^{2k}$. Assume that $\mathcal{D}$ is tangent to a compact submanifold $L$. If $H^{2k}(M)=0$ and $L$ is an umbilic submanifold of $M$, then $c=0$ and $L$ is totally geodesic.*

**Corollary 4.3.** *Let $\mathcal{F}^{2k}$ be a foliation of a Riemannian manifold $M^{2k+p}$ of pure curvature form. If there exists a compact umbilic leaf $L$ and the sectional curvatures of $M$ are positive along $L$, then $\epsilon(\mathcal{D})\neq 0$.*

**Corollary 4.4.** *Let $M^{2k+p}$ be a homology sphere immersed with codimension one in the Euclidean space. If its sectional curvatures are positive along a certain umbilic or totally geodesic submanifold $L^{2k}$ of $M$, then there is no distribution $\mathcal{D}^{2k}$ tangent to $L$.*

There is an example in [3], which enlightens the importance of positive sectional curvature, and for the sake of completeness it is reproduced here.

**Example 3.** Take the fibration $\pi = F \circ \pi_1 : S^3 \times S^3 \to S^2$, where $\pi_1 : S^3 \times S^3 \to S^3$ is the projection onto the first factor and $F : S^3 \to S^2$ is a fibration of the sphere by great circles. $\pi$ defines a totally geodesic foliation $\mathcal{F}_\pi$ of $S^3 \times S^3$, where each leaf is $S^3 \times S^1 \subset S^3 \times S^3$. It follows that $\epsilon(\mathcal{F}_\pi) \in H^4(S^3 \times S^3) = 0$, but sectional curvatures of planes generated by a vector tangent to $S^1$ and any other vector on $S^3$ is zero, and it agrees with the aforementioned corollary.

**Remark 2.** The reader should notice that for any given totally geodesic foliation $\mathcal{F}$, $\mathcal{F} \times \mathcal{F}$ is always totally geodesic (and obviously even dimensional), and similar arguments assure analogous examples for a product of a fibration of $S^3$ by great circles, since $H^2(S^3 \times S^3) = 0$.

**Proof of Theorem 1.2.** In this case, equation (4) is written as $\Omega^L_{ij} = (||H||^2 + c)\theta_i \wedge \theta_j$, and

$$\epsilon(\mathcal{D}) = \frac{(-1)^k (2k)!}{(4\pi)^k k!} \sum_{l=0}^{k} \binom{k}{l} ||H||^{2l} c^{k-l} \theta_1 \wedge \cdots \wedge \theta_{2k}. \tag{11}$$

The same argument as Theorem 1.1 completes the proof. □

Consider $S^{2n} \times (0,1)$ endowed with the metric $g = f^2 ds^2_{2n} + dr^2$, such that $ds^2_{2n}$ is the induced canonical metric of $S^{2n}$ and $f$ is a smooth function on $(0,1)$. Let $c > 0$ be real constant. Then, for $f = \frac{1}{c}\sin^2(\sqrt{c}r)$, $(S^{2n} \times (0,1), g)$ has positive constant curvature $c$ and for $f = \frac{1}{c}\sinh^2(\sqrt{-c}r)$, $(S^{2n} \times (0,1), g)$ has negative constant curvature $c$ (see [15]). By Proposition 2.21, page 46 of [16], $\{S^{2n} \times \{r\} : r \in (0,1)\}$ is an umbilic foliation. However, $H^{2n}(S^{2n} \times (0,1))$ is not trivial.

For topological reasons, regular foliations do not exist on even dimensional spheres. Therefore,

**Corollary 4.5.** *Euclidean spheres do not admit umbilic foliations of odd codimension.*

**Proof of Theorem 1.3.** Take the cup product between $\epsilon(\mathcal{F})$ and $\nu$ and integrate it over $M$. From the definition of $\epsilon(\mathcal{F})$,

$$\int_M \epsilon(\mathcal{F}) \wedge \nu = \frac{1}{32\pi^2} \sum_{\sigma \in \mathfrak{S}_4} \operatorname{sgn}(\sigma) \int_M \Omega^L_{\sigma(1)\sigma(2)} \wedge \Omega^L_{\sigma(3)\sigma(4)} \wedge \nu$$
$$= \frac{1}{32\pi^2} \binom{p+4}{4} \int_M \gamma^L_4.$$

Theorem 5 in [8], together with the definition of higher-order sectional curvatures, implies that the fourth sectional curvatures of $L$ are nonnegative. This completes the proof. □

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