C. R. Acad. Sci. Paris, Ser. I 354 (2016) 874–878

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Number theory

On the Hausdorff dimension faithfulness of continued fraction expansion

Sur la fidélité du développement en fraction continue pour la dimension de Hausdorff

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ARTICLE INFO

Article history:
Received 21 February 2016
Accepted 20 July 2016
Available online 1 August 2016

Presented by Jean-Pierre Kahane

ABSTRACT

In this note, we show that the family of all possible unions of finite consecutive cylinders of the same rank of continued fraction expansion is faithful for the Hausdorff dimension calculation on the unit interval.

RÉSUMÉ

Nous montrons dans cette Note que la famille de toutes les unions finies de cylindres consécutifs de même rang n (une telle union est la clôture de l'ensemble des nombres réels dans l'intervalle unité dont les $n-1$ premiers quotients partiels du développement en fraction continue sont fixés et le n^{e} est astreint à parcourir un ensemble donné d'entiers consécutifs) est fidèle pour la dimension de Hausdorff de l'intervalle unité.

1. Introduction

The notion of the Hausdorff dimension is now well known and plays an important role in fractal geometry. The Hausdorff dimension has the advantage of being defined for any set, as it is based on measures. However, a major disadvantage is that in many cases it is a rather non-trivial problem to give the exact Hausdorff dimension. Over the recent years, there has been a large interest in determining the Hausdorff dimension by restricting the family of admissible coverings. In [3,4], the notion of the faithfulness and non-faithfulness of families for Hausdorff dimension calculation is introduced. The faithfulness of family leads us to calculate the Hausdorff dimension by considering a small family of admissible coverings.

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http://dx.doi.org/10.1016/j.crma.2016.07.009

To be more precise, let us shortly recall some definitions. Let Φ be a fine family of coverings on $[0,1]$, i.e. a family of subsets of $[0,1]$ such that, for any $\epsilon > 0$, there exists an at most countable ϵ-covering $\{E_j\}$ of $[0,1]$ with $E_j \in \Phi$. The α-dimensional Hausdorff measure of a set $E \subset [0,1]$ with respect to a fine family of coverings Φ is defined by

$$H^{\alpha}(E,\Phi) = \liminf_{\epsilon\to 0}\left\{\sum_j |E_j|^{\alpha} \mid \{E_j\} \text{ is an } \epsilon\text{-covering of E}, E_j \in \Phi\right\}$$

and the nonnegative number

$$\dim_H(E,\Phi) = \inf\left\{\alpha | H^{\alpha}(E,\Phi) = 0\right\}$$

is called the Hausdorff dimension of the set $E \subset [0,1]$ with respect to the family Φ. If we take Φ as the family of all subsets of $[0,1]$, we denote $\dim_H(E,\Phi)$ by $\dim_H(E)$, which is equal to the classical Hausdorff dimension of the set $E \subset [0,1]$. For more properties of the Hausdorff dimension, one is referred to [6,8].

A fine covering family Φ is said to be a faithful family of coverings (non-faithful family of coverings) for the Hausdorff dimension calculation if

$$\dim_H(E,\Phi) = \dim_H(E), \text{ for any } E \in [0,1]$$
$$(\text{respect to } \exists E \subset [0,1] : \dim_H(E,\Phi) \neq \dim_H(E)).$$

The first result concerning the problem of faithful coverings is due to A. Besicovitch, who proved that the faithfulness of the family of cylinders of a binary expansion. This result was extended to the family of s-adic cylinders by P. Billingsley [5]. S. Albeverio et al. have done a series of works in this direction, e.g., [1–4,7]. Especially, in [3], S. Albeverio et al. gave a general sufficient and necessary condition for the family of cylinders of the famous Cantor series expansion to be faithful. The faithfulness of the families of cylinders generated by infinite linear iterated function systems, which covers the generalized Lüroth expansions, was studied in [4]. Although the convergents of continued fraction expansion are the best approximation of real numbers, the family of cylinders of continued fraction expansion is non-faithful, which is proved by Y. Peres and G. Torbin, see [3]. In this paper, we focus on the family of the finite union of cylinders of continued fraction expansion, and prove that the family of all possible unions of finite consecutive cylinders of the same rank of continued fraction expansion is faithful for the Hausdorff dimension calculation. To some extent, this give an explanation for why one usually uses the (countable) union of cylinders instead of the cylinders themselves for obtaining the lower bound of Hausdorff dimension of some sets in Diophantine approximation.

2. Statement of main result

Firstly, we briefly recall some basic properties and known results of continued fraction expansion.

Any irrational number $x \in [0,1)$ has a simple infinite continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}} = [a_1(x), a_2(x), a_3(x), \cdots], \tag{2.1}$$

where $\{a_n(x)\}_{n\geq 1}$ are positive integers and are called the partial quotients of x.

Let $x \in [0,1)$ and $[a_1(x), a_2(x), a_3(x), \cdots]$ be its continued fraction expansion. For any $n \geq 1$, we denote by $p_n(x)/q_n(x) = [a_1(x), a_2(x), \cdots a_n(x)]$ the nth convergent of x. With the conventions $p_{-1}(x) = 1, q_{-1}(x) = 0, p_0(x) = 0, q_0(x) = 1$, we have

$$p_{n+1}(x) = a_{n+1}p_n(x) + p_{n-1}(x), \;\; n \geq 0, \tag{2.2}$$
$$q_{n+1}(x) = a_{n+1}q_n(x) + q_{n-1}(x), \;\; n \geq 0.$$

For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, let $q_n(a_1, a_2, \cdots, a_n)$ be the dominator of the finite continued fraction $[a_1, a_2, \cdots, a_n]$. If there is no confusion, we write q_n instead of $q_n(a_1, a_2, \cdots, a_n)$ for simplicity. For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, let

$$I_n(a_1, a_2, \cdots, a_n) = cl\{x \in [0,1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\},$$

which is called a cylinder of rank n and where 'cl' denotes the closure.

It is well known that

$$\mathcal{L}(I(a_1, a_2, \cdots, a_n)) = \frac{1}{q_n(q_n + q_{n-1})},$$

where $\mathcal{L}$ denotes the Lebesgue measure. By using the equalities (2.2), it is easy to get the following lemma.

Lemma 2.1. *Let $a_1, a_2, \cdots, a_n, s$ be any $n+1$ positive integers and $I_n(s) := I(a_1, a_2, \cdots, a_n, s)$, then*

$$|I_n(s+1)| \le |I_n(s)| \le 3|I_n(s+1)|$$

where $|A|$ denotes the length of the set A.

Let $\mathcal{A}_n$ be the family of all possible unions of finite consecutive intervals of rank n, i.e.

$$\mathcal{A}_n = \left\{ \bigcup_{i=s}^{s+m} I(a_1, a_2, \cdots, a_{n-1}, i) : s \in \mathbb{N}^+,\ m \in \mathbb{N},\ a_j \in \mathbb{N}^+, 1 \le j \le n-1 \right\}.$$

Define

$$\mathcal{A} = \bigcup_{k \ge 1} \mathcal{A}_k.$$

In this paper, we consider the covering family $\mathcal{A}$ of all possible unions of finite consecutive cylinders, and show that $\mathcal{A}$ is faithful.

Theorem 2.2. *The family $\mathcal{A}$ is faithful for Hausdorff dimension calculation on the unit interval.*

3. Proofs of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. For a given set $E \subset (0,1)$, $\alpha > 0$, and let $E_j = (a_j, b_j)$, $j \ge 1$ be an open ϵ-covering of E.

For any $j \in \mathbb{N}$, there exists an interval $I_{n_j}(a_1, a_2, \cdots, a_{n_j})$ of rank n_j such that

(1) $E_j \subset I_{n_j}(a_1, a_2, \cdots, a_{n_j})$,
(2) any interval of rank $n_j + 1$ does not contain E_j.

Without loss of generality, we assume that n_j is odd, the other case can be handled in the same way. We divide the proof into two cases.

Case 1: E_j contains at least one interval of rank $n_j + 1$.

(1) If there are only finite intervals contained in E_j, denoted by

$$I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s), \cdots, I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s+k)$$

(for some positive integers s, k) of rank $n_j + 1$ for which

$$E_j \cap I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i) \ne \emptyset, s \le i \le s+k,$$

then we have

$$I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i) \subset E_j, s+1 \le i \le s+k-1$$

and

$$E_j \subset \bigcup_{s \le i \le s+k} I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i).$$

Take

$$J_0 = I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s),$$
$$J_1 = \bigcup_{i=s+1}^{s+k-1} I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i)$$

and

$$J_2 = I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s+k).$$

Thus $\mathcal{F}_{E_j} = \{J_0, J_1, J_2\} \subset \mathcal{A}$ is a covering of E_j. By Lemma 2.1, we have

$$|E_j|^\alpha \ge \frac{1}{2+3^\alpha} \sum_{J \in \mathcal{F}_{E_j}} |J|^\alpha \text{ and } |J_k| \le 3|E_j|, k = 0, 1, 2. \tag{3.1}$$

(2) If there are infinite many intervals of rank n_j+1 that have a non-empty intersection with E_j, then there exists a positive integer d such that

$$I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i) \subset E_j, i \ge d+1$$

and

$$E_j \subset \bigcup_{i \ge d} I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i).$$

For the given α, we can take a sequence $\{l_0 = 0, l_k\}_{k \ge 1}$ of integers such that

$$\sum_{k \ge 0} \left| \bigcup_{i=1}^{l_{k+1}} I_{n_j+1}(d + \sum_{m=0}^{k} l_m + i) \right|^\alpha < +\infty$$

where $I_{n_j+1}(m) = I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, m)$. What is more, we make l_1 sufficiently large so that

$$\left| \bigcup_{i=1}^{l_1} I_{n_j+1}(d+i) \right|^\alpha \ge \sum_{k \ge 1} \left| \bigcup_{i=1}^{l_{k+1}} I_{n_j+1}(d + \sum_{m=0}^{k} l_m + i) \right|^\alpha .$$

Now, take $J_0 = I_{n_j+1}(d)$, $J_k = \bigcup_{i=1}^{l_k} I_{n_j+1}(d + \sum_{m=0}^{k-1} l_m + i)$, $k \ge 1$. Then we have $J_i \in \mathcal{A}$, and

$$E_j \subset \bigcup_{i \ge 0} J_i \text{ and } \bigcup_{i \ge 1} J_i \subset E_j.$$

Note that $|J_0| = |I_{n_j+1}(d)| \le 3|I_{n_j+1}(d+1)| \le 3|J_1|$ by Lemma 2.1. Let $\mathcal{F}_{E_j} = \{J_i\}_{i \ge 0} \subset \mathcal{A}$, then we have

$$\sum_{J \in \mathcal{F}_{E_j}} |J|^\alpha = |J_0|^\alpha + |J_1|^\alpha + \sum_{i \ge 2} |J_i|^\alpha \le (3^\alpha + 2)|E_j|^\alpha \text{ and } |J_k| \le 3|E_j|, k \in \mathbb{N}. \tag{3.2}$$

Case 2: E_j contains no interval of rank n_j+1.

In this case there are only two intervals of rank n_j+1, denoted by

$$I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s), I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s+1)$$

for which

$$E_j \cap I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, i) \ne \emptyset, i = s, s+1.$$

Recall that $E_j = (a_j, b_j)$, then

$$a_j \in I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s) \text{ and } b_j \in I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s+1).$$

Let c be the right endpoint of interval $I_{n_j+1}(a_1, a_2, \cdots, a_{n_j}, s)$, we consider the coverings of (a, c) and $[c, b)$ separately.

First of all, we consider the coverings of (a, c). Without loss of generality, we assume that point a belongs to some interval $I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, l)$ of rank n_j+2.

(i) If $l \ge 2$, then

$$(a, c) \subset \bigcup_{1 \le i \le l} I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, i)$$

and

$$I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, i) \subset (a, c), 1 \le i \le l-1.$$

Take $J = \bigcup_{1 \le i \le l} I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, i)$, then $J \in \mathcal{A}$ and $|J| \le 2|E_j|$, hence

$$|J|^\alpha \le 2^\alpha |E_j|^\alpha. \tag{3.3}$$

(ii) If $l = 1$, the point a must fall in some cylinder of rank n_j+3 contained in $I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, 1)$. Suppose the cylinder of rank n_j+3 is $I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, 1, m_0)$ for some integers $m_0 > 0$. Then we have

$$I_{n_j+2}(a_1, a_2, \cdots, a_{n_j}, s, 1, k) \subset (a, c), k \ge m_0 + 1$$

and

$$(a,c) \subset \bigcup_{k\geq m_0} I_{n_j+2}(a_1,a_2,\cdots,a_{n_j},s,1,k).$$

By a similar analysis as in the case 1, we can take a sequence $\{l_k\}_{k\geq 0}$ of integers such that

$$\left|\bigcup_{i=1}^{l_1} I_{n_j+3}(m_0+i)\right|^\alpha \geq \sum_{k\geq 1}\left|\bigcup_{i=1}^{l_{k+1}} I_{n_j+3}(m_0+\sum_{n=0}^{k} l_n+i)\right|^\alpha$$

where $I_{n_j+3}(m) = I_{n_j+3}(a_1,a_2,\cdots,a_{n_j},s,1,m)$. Take

$$J_0 = I_{n_j+3}(m_0),\;\; J_k = \bigcup_{i=1}^{l_k} I_{n_j+3}(m_0+\sum_{m=0}^{k-1} l_m+i)), k\geq 1.$$

Let $\mathcal{F}_{E_j} = \{J_i\}_{i\geq 0}$, then

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha \leq (3^\alpha+2)|E_j|^\alpha \text{ and } |J| \leq 3|E_j|,\, J\in\mathcal{F}_{E_j}. \tag{3.4}$$

Secondly, we consider the coverings of $[c,b)$. Since $b \in I_{n_j+1}(a_1,\cdots,a_{n_j},s+1)$, we assume that the point b in cylinder $I_{n_j+2}(a_1,\cdots,a_{n_j},s+1,s_0)$ is of rank n_j+2 for some integer s_0. Thus $I_{n_j+2}(a_1,\cdots,a_{n_j},s+1,s_0+i) \subset [c,b)$ for all $i\geq 1$ and $[c,b) \subset \bigcup_{i\geq 0} I_{n_j+2}(a_1,\cdots,a_{n_j},s+1,s_0+i)$. Do the same analysis as above, there exists a subfamily $\mathcal{F}_{E_j}$ of $\mathcal{A}$ that covers $[c,b)$ such that

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha \leq (3^\alpha+2)|E_j|^\alpha \text{ and } |J| \leq 3|E_j|,\, J\in\mathcal{F}_{E_j}. \tag{3.5}$$

So, for a given open interval E_j, there exists a countable subfamily $\mathcal{F}_{E_j}$ of $\mathcal{A}$ that covers E_j such that

$$\sum_{J\in\mathcal{F}_{E_j}} |J|^\alpha \leq K(a)|E_j|^\alpha,$$

where $K(\alpha)$ is a constant and independent of j. Therefore, for any $\alpha > 0$, $E \subset [0,1]$, we have

$$H^\alpha(E) \leq H^\alpha(E,\mathcal{A}) \leq K(\alpha)H^\alpha(E).$$

Thus $\dim_H(E,\mathcal{A}) = \dim_H(E)$.

Acknowledgements

This work is supported by NSFC (Grant No. 11501168).

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