C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1060–1064

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis

Periods of L^2-forms in an infinite-connected planar domain

Périodes de formes L^2 dans un domaine plan infiniment connexe

Mikhail Dubashinskiy [1,2]

Chebyshev Laboratory, St. Petersburg State University, 14th Line 29b, Vasilyevsky Island, Saint Petersburg 199178, Russia

A R T I C L E I N F O

Article history:
Received 29 January 2016
Accepted 14 September 2016
Available online 28 September 2016

Presented by Gilles Pisier

A B S T R A C T

Let $\Omega \subset \mathbb{R}^2$ be a countably-connected domain. In Ω, consider closed differential forms of degree 1 with components in $L^2(\Omega)$. Further, consider sequences of periods of such forms around holes in Ω, i.e. around bounded connected components of $\mathbb{R}^2 \setminus \Omega$. *For which domains Ω the collection of such a period sequences coincides with ℓ^2?* We give an answer in terms of metric properties of holes in Ω.

R É S U M É

Soit $\Omega \subset \mathbb{R}^2$ un domaine infiniment connexe. Considérons des formes différentielles fermées dans Ω de degré 1 et à composantes dans $L^2(\Omega)$. Considérons de plus les suites de périodes de formes telles autour de trous dans le domaine Ω, c'est-à-dire autour des composantes connexes bornées de $\mathbb{R}^2 \setminus \Omega$. *Quels sont les domaines Ω tels que l'ensemble de ces suites de periodes coïncide avec ℓ^2 ?* On obtient un critère en termes de propriétés métriques des trous dans Ω.

1. Statement of the problem

In this paper, we announce a result to be published later [2]. Let us start with some definitions.

1.1. Interpolation of forms by their periods

Let $\mathbb{D}$ be the unit disk in the plane $\mathbb{C} \simeq \mathbb{R}^2$. Suppose that connected compact sets $B_1, B_2, \dots \subset \mathbb{D}$ are pairwise disjoint and accumulate only to unit circle $\mathbb{T}$. Also we assume that each B_j does not separate the plane. Consider a planar countably

E-mail address: mikhail.dubashinskiy@gmail.com.
[1] Fax: +7 (812) 363 68 71.
[2] The work is supported by the Russian Science Foundation grant 14-21-00035.

http://dx.doi.org/10.1016/j.crma.2016.09.007
1631-073X/

connected domain $\Omega := \mathbb{D} \setminus \bigcup_{j=1}^{\infty} B_j$; sets $B_1, B_2, \dots$ are called *holes* in Ω (unbounded connected component of $\mathbb{R}^2 \setminus \Omega$ will have a special status).

Consider the following Hilbert space of real differential forms of degree 1 in Ω:

$$L_c^{2,1}(\Omega) = \{\omega \text{ – 1-form in } \Omega,\ \|\omega\|^2_{L_c^{2,1}(\Omega)} := \int_\Omega |\omega|^2 \, \mathrm{d}\lambda_2 < +\infty, \mathrm{d}\omega = 0 \text{ in the sense of distributions}\}.$$

Here, if $\omega = \omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y$, then $|\omega| := \sqrt{\omega_x^2 + \omega_y^2}$; λ_2 is the Lebesgue measure in $\mathbb{R}^2$.

For any $j = 1, 2, \dots$, pick a closed oriented curve γ_j in Ω such that γ_j winds around hole B_j once in the positive direction and does not wind around other holes $B_{j'}$, $j' \neq j$. *Period functional* $\mathrm{Per}_j \colon L_c^{2,1}(\Omega) \to \mathbb{R}$, $\mathrm{Per}_j(\omega) := \int_{\gamma_j} \omega$ *is well defined and continuous in* $L_c^{2,1}(\Omega)$ (see, e.g., [3]). Now, define the *period operator*: for $\omega \in L_c^{2,1}(\Omega)$ put $\mathrm{Per}\,\omega := \{\mathrm{Per}_j(\omega)\}_{j=1}^{\infty}$.

Definition 1. We say that domain Ω has *complete interpolation property* if operator $\mathrm{Per} \colon L_c^{2,1}(\Omega) \to \ell^2$ is bounded and surjective.

The problem of interpolation by periods is to describe domains Ω possessing the complete interpolation property in terms of the metric characteristics of the layout of holes B_j in Ω. By change of variable, we ensure that our problem is invariant under the action of a conformal mapping that does not turn Ω inside out.

1.2. Equilibrium currents

The question of interpolation by periods is motivated by the following higher-dimensional problem on the equilibrium current (see [6]). Consider a multiply-connected compact subset K, say, in $\mathbb{R}^3$. Let $S_1, S_2, \dots$ be some sequence of, say, smooth compact surfaces (sections) with boundaries with $\partial S_j \cap K = \varnothing$, $S_j \cap K \neq \varnothing$, $j = 1, 2, \dots$. Let us search for an electric current $\vec{I}$ supported on K such that $\operatorname{div} \vec{I} = 0$, the flows $\int_{S_j} \vec{I}_n$ have prescribed values, and the current $\vec{I}$ minimizes the energy $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\langle \vec{I}(x), \vec{I}(y) \rangle \, \mathrm{d}x \, \mathrm{d}y}{|x-y|}$ among all such currents. This is a certain analog of a classical problem on the equilibrium charge on a compact set, but the condition $\operatorname{div} \vec{I} = 0$ makes these two problems non-equivalent. We would like to work with such a statement for arbitrarily non-smooth compact subsets K (one of the questions is, for example, in what amount the minimum of energy depends on the choice of the sections).

With an electric current $\vec{I}$, one associates its *Biot–Savart magnetic field* $\mathrm{BS}^{\vec{I}} := \operatorname{curl}\left(\vec{I} \star 1/|x|\right)$, where $\star$ is the convolution. If $\vec{f} = \mathrm{BS}^{\vec{I}}$, then, in terms of such $\vec{f}$, we have the following problem of interpolation in the exterior domain: to find a field $\vec{f}$ in $\mathbb{R}^3 \setminus K$ such that $\operatorname{curl} \vec{f} = 0$, circulations $\int_{\partial S_j} \vec{f}_\tau$ have prescribed values and $\|\vec{f}\|_{L^2(\mathbb{R}^3 \setminus K)}$ is minimal under these conditions. To the author's knowledge, the planar version of the interpolation by periods problem was not studied before.

Before we state a metric criterion for complete interpolation property, let us give some of its equivalent reformulations.

1.3. Interpolation in the Bergman space

Let $\mathscr{A}^2(\Omega)$ be the usual (unweighted) Bergman space in Ω. If $f \in \mathscr{A}^2(\Omega)$ and curves γ_j are as in the above, then we may define the *complex period operator* $\mathrm{Per}^{\mathbb{C}}$ as $\mathrm{Per}^{\mathbb{C}} f = \{\oint_{\gamma_j} f(\zeta) \, d\zeta\}_{j=1}^{\infty}$. The interpolation problem in the Bergman space is stated as in $L_c^{2,1}(\Omega)$ (just replace Per by $\mathrm{Per}^{\mathbb{C}}$). *Domain Ω has the complete interpolation property for forms if and only if it has this property for Bergman functions.* This follows from the fact that minimizers of $\|\omega\|_{L_c^{2,1}(\Omega)}$ under given periods are harmonic forms in Ω and can be understood as analytic functions. Recall that a form ω is called *harmonic* if $\mathrm{d}\omega = 0$, $d * \omega = 0$, where $*$ is the Hodge star operator.

1.4. Estimates of harmonic functions

In this paragraph, we assume for simplicity that any B_j is a closure of a domain with C^∞-smooth boundary. Let $\mathring{W}^{1,2}(\mathbb{D})$ be the Sobolev space of functions $u \colon \Omega \to \mathbb{R}$ with $\|u\|_{\mathring{W}^{1,2}(\mathbb{D})} := \left(\int_{\mathbb{D}} |\nabla u|^2 \, \mathrm{d}\lambda_2\right)^{1/2} < +\infty$ and $u = 0$ on $\mathbb{T}$. The complete interpolation property of Ω turns out to be equivalent to the following condition: *for any* $\{a_j\}_{j=1}^{\infty} \in \ell^2$, *there exists a function* $u \in \mathring{W}^{1,2}(\mathbb{D})$ *with* $\Delta u = 0$ *in* Ω, $u|_{B_j} = a_j$ *almost everywhere for any* $j = 1, 2, \dots$, *and*

$$C_1 \cdot \|\{a_j\}_{j=1}^{\infty}\|_{\ell^2} \leq \|u\|_{\mathring{W}^{1,2}(\mathbb{D})} \leq C_2 \cdot \|\{a_j\}_{j=1}^{\infty}\|_{\ell^2} \tag{1}$$

with some $C_1, C_2 \in (0, +\infty)$ *not depending on* $\{a_j\}_{j=1}^{\infty}$. This is clear from the explicit form of reproducing kernels (see below) and Riesz basis condition for these kernels.

1.5. Riesz basis in Hilbert homologies

We may define the L^2-cohomology space in Ω as

$$H^1_{L^2}(\Omega) := L^{2,1}_c(\Omega)/\{\omega \in L^{2,1}_c(\Omega) \colon \omega = \mathrm{d}u \text{ for some } u \in W^{1,2}_{\mathrm{loc}}(\Omega)\},$$

and let $H_{1,L^2}(\Omega)$ be the dual of its space; this is the space of *Hilbert homologies* in Ω. Any curve γ_j, $j = 1, 2, \ldots$, can be understood as an element of $H_{1,L^2}(\Omega)$. In this language, the complete interpolation property is equivalent to the following: *system* $\{\gamma_j\}_{j=1}^\infty$ *is a Riesz basis* (see, e.g., [4]) *in* $H_{1,L^2}(\Omega)$.

2. A criterion

To state a necessary and sufficient condition of complete interpolation, let us give some definitions.

We say that holes B_j are *separated* if, for any $j, j' = 1, 2, \ldots$, $j \neq j'$, we have $\mathrm{dist}(B_j, B_{j'}) \geq \varepsilon \cdot \min\{\mathrm{diam}\, B_j, \mathrm{diam}\, B_{j'}\}$ with some $\varepsilon > 0$ not depending on j and j' (symbols dist and diam denote distance and diameter in Euclidean metric).

Denote by $\mathcal{B}_H(z, r)$ the open disk of radius $r > 0$ in hyperbolic metric $\frac{2|\mathrm{d}z|}{1-|z|^2}$ in $\mathbb{D}$ and centered in some $z \in \mathbb{D}$. Let us say that the *holes in domain* Ω *are uniformly locally finite* if there exists $N = N(\Omega) < +\infty$ such that any disk of the form $\mathcal{B}_H(z, 1)$ ($z \in \mathbb{D}$) intersects no more than N of holes B_j, $j = 1, 2, \ldots$.

Now, for $S < +\infty$, define a graph $G(\Omega, S)$. Its vertices are sets B_j, $j = 1, 2, \ldots$, and also set $\mathbb{R}^2 \setminus \mathbb{D}$. If E_1, E_2 are two of such sets, then join them with an edge in $G(\Omega, S)$ if $\mathrm{dist}(E_1, E_2) \leq S \cdot \min\{\mathrm{diam}\, E_1, \mathrm{diam}\, E_2\}$. (One may also use the condenser capacity to define this graph.) The distance between two vertices in $G(\Omega, S)$ in the graph metric is the number of edges of the shortest path connecting these vertices.

Theorem 2 *(Complete interpolation criterion). Domain* Ω *possesses the complete interpolation property if and only if the following conditions are satisfied.*

1. *Family of holes* $\{B_j\}_{j=1}^\infty$ *is uniformly locally finite.*
2. *Holes* B_j *are separated; also,* $\sup\{\mathrm{diam}_H(B_j) \mid j \in \mathbb{N}\} < +\infty$*, where* diam_H *is the hyperbolic diameter.*
3. *For some* $S < +\infty$*, the graph* $G(\Omega, S)$ *is connected and its diameter in the graph metric is finite.*

3. About the proofs

In this section, we assume that *each* B_j *is a closure of a domain with* C^∞*-smooth boundary*. Theorem 2 in the case of non-smooth holes is obtained by approximation of such holes by smooth ones.

3.1. Reproducing kernels

Let us point out *period reproducing kernels* (see also [1]).

Proposition 3. *For any* $j = 1, 2, \ldots$*, there exists a function* $\mathfrak{v}_j \in \mathring{W}^{1,2}(\mathbb{D})$ *for which* $\Delta \mathfrak{v}_j = 0$ *in* Ω*,* $\mathfrak{v}_j = 1$ *almost everywhere in* B_j *and* $\mathfrak{v}_j = 0$ *almost everywhere on* $B_{j'}$ *for any* $j' \neq j$.

The form $\kappa_j = -(*\mathrm{d}\mathfrak{v}_j)$ *is a* period reproducing kernel*, i.e.* $\langle \kappa_j, \omega \rangle_{L^{2,1}_c(\Omega)} = \mathrm{Per}_j(\omega)$ *for any* $\omega \in L^{2,1}_c(\Omega)$.

It turns out that $\langle \kappa_j, \kappa_{j'} \rangle_{L^{2,1}_c(\Omega)} < 0$ if $j, j' = 1, 2, \ldots$, $j \neq j'$. This implies, in particular, that the operator Per is bounded provided that $\sup_{j\in\mathbb{N}} \|\mathrm{Per}_j\|_{\left(L^{2,1}_c(\Omega)\right)^*} < +\infty$. We make essential use of the latter inequality; note that this estimate is equivalent to the following one:

$$\sup_{j\in\mathbb{N}} \mathrm{Cap}_2\left(B_j, \mathbb{R}^2 \setminus (\Omega \cup B_j)\right) < +\infty. \tag{2}$$

Here $\mathrm{Cap}_2(\cdot, \cdot)$ is the capacity of a condenser with two plates defined as, e.g., in [5].

3.2. Uniform local finiteness

The most difficult part of Theorem 2 is to derive the uniform local finiteness of holes from the complete interpolation property of Ω. We make use of inequality (1) by constructing a function $u \in \mathring{W}^{1,2}(\mathbb{D})$ with the following properties: for each $j = 1, 2, \ldots$, the function u is constant almost everywhere in B_j, $\|u\|_{\mathring{W}^{1,2}(\mathbb{D})}$ is not large, whereas the values $u|_{B_j}$ are not very small; then it remains to put into (1) a function $v \in \mathring{W}^{1,2}(\mathbb{D})$, which is harmonic in Ω and coincides with u in $\mathbb{D} \setminus \Omega$.

In order to construct such u, we consider a (degenerated) metric $1_\Omega|dz|$ in $\mathbb{R}^2$. Let $\rho(\cdot,\cdot)$ be the inner metric generated by $1_\Omega|dz|$. Define $u(z)$, $z \in \mathbb{R}^2$, as the distance in the metric ρ from z to $\mathbb{T}$. Then $u \in \mathring{W}^{1,2}(\mathbb{D})$ since $|\nabla u| \le 1$ almost everywhere, and $u = 0$ on $\mathbb{T}$. Also, all the holes B_j collapse into points in metric ρ and hence u is constant on any hole.

It remains to estimate $u|_{B_j}$ from below. The following inequality easily provides uniform local finiteness:

$$u|_{B_j} \ge c_1 \cdot \operatorname{dist}(B_j, \mathbb{T}) \tag{3}$$

for any $j = 1, 2, \ldots$, and some $c_1 > 0$ not depending on j. To prove this, we have, according to the definition of u, to estimate $\mathcal{H}^1(\Gamma \cap \Omega)$ from below for any parameterized curve Γ starting in B_j and ending on $\mathbb{T}$ ($\mathcal{H}^1$ is Hausdorff measure).

Under some technical assumptions on Γ, a simple stepwise process leads to the following lemma.

Lemma 4. *There exist a sequence of points $z_0, w_0, \xi_0, z_1, w_1, \xi_1, \ldots$ on curve Γ ordered in the direction of increase of parameter of Γ and also a sequence of distinct indices $j_0, j_1, \ldots \in \mathbb{N}$ such that:*

1. *for $m = 0, 1, \ldots$, an arc of curve Γ starting in z_m and ending in w_m lies entirely in hole B_{j_m}; point w_m is the point of exit of Γ from B_{j_m}; Γ does not return to B_{j_m} after w_m;*
2. *$|\xi_m - w_m| = |z_m - w_m|$ for $m = 0, 1, \ldots$;*
3. *if Γ_m, $m = 0, 1, \ldots$, is the arc of Γ from z_m to ξ_m then $\Gamma \cap (\mathbb{R}^2 \setminus \Omega) \subset \bigcup_{m=0,1,\ldots} \Gamma_m$.*

If Ω has the complete interpolation property, then estimate (2) for $j = j_m$ and the Cauchy–Schwartz inequality imply that $\mathcal{H}^1(\Gamma_m \cap \Omega) \ge c_2 \cdot |\xi_m - z_m|$, with some $c_2 > 0$ depending only on $\|\operatorname{Per}\|_{L^{2,1}_c(\Omega)\to\ell^2}$. This leads to (3), what was desired.

3.3. Boundedness of Per: poset structure

There is a partial order structure on the set of holes, which expresses the essence of the continuity of operator Per. Denote by $U_t(E)$ the open Euclidean t-neighbourhood ($t > 0$) of a set $E \subset \mathbb{R}^2$.

Lemma 5. *Under the first and second conditions of Theorem 2, it is possible to define a partial order relation $\succeq$ on the set of holes B_j and also to associate a set $A_j \subset \Omega$ with each hole B_j, such that:*

1. *for each $j = 1, 2, \ldots$, the set A_j is of the form $U_{s_j}(B_j) \setminus U_{t_j}(B_j)$ for some t_j, s_j ($s_j > t_j$). Also, $s_j - t_j \ge c_2 \cdot \operatorname{diam} B_j$, $s_j \le c_1 \cdot \operatorname{diam} B_j$ where $c_1, c_2 > 0$ do not depend on j. The overlapness multiplicity of sets A_j is bounded from above;*
2. *$B_{j'} \preceq B_j$ if and only if $B_{j'} \subset U_{t_j}(B_j)$. For a fixed j_0, the number of indices j for which $B_j \preceq B_{j_0}$ does not exceed some constant C. In particular, lengths of chains in order $\succeq$ are bounded uniformly. If $B_{j_1}, B_{j_2} \succeq B_j$, then either $B_{j_1} \succeq B_{j_2}$ or $B_{j_2} \succeq B_{j_1}$.*

We may force c_1 to be small; thence, roughly speaking, $B_k \prec B_j$ ($k \ne j$) if $\operatorname{diam} B_k \ll \operatorname{diam} B_j$ and $\operatorname{dist}(B_j, B_k) \ll \operatorname{diam} B_j$.

Now suppose that all A_j are annular domains (this may not, in general, be true). By the first assertion of Lemma 5, A_j is wide enough in the sense of extremal length. In this case, for any $j = 1, 2, \ldots$, we have estimates

$$\int_{A_j} |\omega|^2 \, d\lambda_2 \ge c \cdot \left(\sum_{j' :\ B_{j'} \preceq B_j} \operatorname{Per}_{j'} \omega \right)^2$$

with some $c > 0$ not depending on $\omega \in L^{2,1}_c(\Omega)$ and j. Consecutive application of these estimates starting from the minimal holes in sense of order $\preceq$ up to maximal ones gives us the continuity of operator Per.

3.4. Surjectivity of Per

This property is provided by the first and third conditions in Theorem 2. The lower estimate on $\int_\Omega |\nabla u|^2 \, d\lambda_2$ in (1) is responsible for the surjectivity of Per. To prove this inequality, we implement the connectivity of $G(\Omega, S)$ in the plane. Namely, if some holes B_j and B_k are adjacent in $G(\Omega, S)$, then we may construct a wide "road" R_{jk} joining B_j and B_k in $\mathbb{R}^2$. This road is a planar set such that, if some function u is constant on B_j and on B_k, then $\int_{R_{jk}} |\nabla u|^2 \, d\lambda_2 \ge c \cdot \left|(u|_{B_j}) - (u|_{B_k})\right|^2$ with some $c > 0$ not depending on j and k. Also, the overlapness multiplicity of *almost all* of these roads is bounded from above. This allows us, for u as in (1), to estimate $\|u\|_{\mathring{W}^{1,2}(\mathbb{D})}$ from below by passing graph $G(\Omega, S)$ in breadth-first order and starting from its vertices adjacent to $\mathbb{R}^2 \setminus \mathbb{D}$. The estimates of $u|_{B_j}$ for the latter vertices are obtained by use of the boundedness of Hardy's average operator.

4. An open question

Our problem is not completely geometrically invariant. For example, an inversion with a center in one of the holes turns domain Ω inside out and throws the problem out of the studied class. Let us give a statement free of such a disadvantage.

If Ω is some domain in $\mathbb{R}^2$ (or even a Riemann surface), then consider the following property of Ω.

(†) *In the space $H_{1,L^2}(\Omega)$, there exists a Riesz basis consisting of integer homologies.*

Here $H_{1,L^2}(\Omega)$ is defined as above. By an integer homology, we mean a functional of the kind $\omega \mapsto \int_\beta \omega$ (ω is a closed 1-form) delivered by some closed loop $\beta \subset \Omega$. The question is to describe domains (or Riemann surfaces) Ω having property (†).

Let H be an abstract Hilbert space and $\{x_j\}_{j=1}^\infty$ be a countable system of vectors in H. Consider the following property of this system:

(‡) *space H has a Riesz basis whose elements are linear combinations of vectors x_j, $j = 1, 2, \ldots$, with integer coefficients.*

If planar domain Ω and curves γ_j are as in Section 1, then the property (†) of Ω is equivalent to the property (‡) of the system $\{\gamma_j\}_{j=1}^\infty$ in $H_{1,L^2}(\Omega)$.

We do not know any investigation on such *integer Riesz bases theory*. Let us only note that if $\{e_j\}_{j=1}^\infty$ is an orthonormal basis in the Hilbert space H, then it is easy to see that system of the kind $\{a_j e_j\}_{j=1}^\infty$ with $a_j \xrightarrow{j\to\infty} +\infty$ does not possess the property (‡) in H. This observation allows us to construct domains Ω not having property (†). We thus may conclude that the *property* (†) *is a non-trivial quasiconformal invariant of countably-connected Riemann surfaces*.

References

[1] R.D.M. Accola, Differentials and extremal length on Riemann surfaces, Proc. Math. Acad. Sci. USA 46 (4) (1960) 540–543.
[2] M. Dubashinskiy, Interpolation by periods in planar domain, http://arxiv.org/abs/1511.07015, submitted for publication in St. Petersburg Math. J.
[3] B. Fuglede, Extremal Length and Closed Extensions of Partial Differential Operators, Gjellerup, Copenhagen, 1960.
[4] N.K. Nikol'skii, Treatise on the Shift Operator: Spectral Function Theory, Grundlehren Math. Wiss., 1986.
[5] Yu.G. Reshetnyak, Space Mappings with Bounded Distortion, Transl. Math. Monogr., vol. 73, American Mathematical Society, Providence, RI, USA, 1989.
[6] H. Yamaguchi, Equilibrium vector potentials in $\mathbb{R}^3$, Hokkaido Math. J. 25 (1) (1996) 1–53.