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Mathematical problems in mechanics

Symmetric solutions to the Leray problem

Solutions symétriques du problème de Leray

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A B S T R A C T

A stationary boundary-value problem for the Navier–Stokes equations of an incompressible fluid in a domain of a spherical layer type is considered. The velocity vector on the boundary is given. The solvability of this problem was proven by Jean Leray (1933) under an additional condition of a zero flux through each connected component of the flow domain boundary. The following problem is open up to now: does a solution to the flux problem exist if only the necessary condition of a zero total flux is satisfied? The present communication is devoted to the consideration of the Leray problem in a spherical-layer-type domain. An a priori estimate of the solution under the condition of flow symmetry with respect to a plane is obtained. This estimate implies the solvability of the problem.

R É S U M É

On considère le problème avec conditions au bord pour les équations de Navier–Stokes stationnaires régissant l'écoulement d'un fluide incompressible dans une couche sphérique. On donne la vitesse au bord. Jean Leray (1933) a démontré la solvabilité de ce problème sous la condition d'un flux nul à travers chacune des composantes connexes du bord. Le problème suivant est à présent ouvert : est-ce qu'une solution du problème avec flux existe sous la seule condition d'un flux total nul ? La note ci-dessous considère le problème de Leray dans une couche sphérique. On obtient une estimation a priori de la solution, sous la condition de symétrie par rapport à un plan. Cette estimation implique la solvabilité du problème.

1. Statement of the problem

Let us consider the stationary motion of an incompressible viscous fluid in a domain Ω of a spherical-layer type with an interior boundary Γ_1 and an exterior boundary Γ_2 of class C^2. For simplicity, we assume that the surfaces Γ_1 and Γ_2 are star-like with respect to the origin. Let us denote the cylindrical coordinates in $\mathbb{R}^3$ as r, φ, and z and the corresponding

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components of the velocity vector $\mathbf{u}$ by u, v, and w. The Navier–Stokes equations in these cylindrical coordinates have the form

$$
\begin{aligned}
& u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \varphi}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 q}{\partial r^2}+\frac{1}{r}\frac{\partial q}{\partial r}+\frac{1}{r^2}\frac{\partial q}{\partial \varphi^2}+\frac{\partial^2 q}{\partial z^2}-\frac{2}{r^2}\frac{\partial v}{\partial \varphi}-\frac{u}{r^2}\right), \\
& u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \varphi}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=-\frac{1}{r}\frac{\partial p}{\partial \varphi}+\nu\left(\frac{\partial^2 q}{\partial r^2}+\frac{1}{r}\frac{\partial q}{\partial r}+\frac{1}{r^2}\frac{\partial q}{\partial \varphi^2}+\frac{\partial^2 q}{\partial z^2}+\frac{2}{r^2}\frac{\partial u}{\partial \varphi}-\frac{v}{r^2}\right), \\
& u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \varphi}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\nu\frac{\partial^2 q}{\partial r^2}+\frac{1}{r}\frac{\partial q}{\partial r}+\frac{1}{r^2}\frac{\partial q}{\partial \varphi^2}+\frac{\partial^2 q}{\partial z^2}, \\
& \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \varphi}+\frac{\partial w}{\partial z}=0.
\end{aligned}
\tag{1}
$$

Here p is the ratio of the pressure to the liquid density and $\nu = const > 0$ is coefficient of viscosity.

The function $\mathbf{u}$ satisfies the boundary conditions

$$\mathbf{u}=\mathbf{a}_i(x),\ \ x\in\Gamma_i,\ \ i=1,\ 2, \tag{2}$$

where the functions $\mathbf{a}_i \in W^{1/2,2}(\Gamma_i)$ satisfy the flux condition

$$\int_{\Gamma_1}\mathbf{a}_1\cdot\mathbf{n}_1\,\mathrm{d}\Gamma_1=-\int_{\Gamma_2}\mathbf{a}_2\cdot\mathbf{n}_2\,\mathrm{d}\Gamma_2=F \tag{3}$$

($\mathbf{n}_i$ is the unit exterior normal vector to the surface Γ_i). Problem (1)–(3) is a particular case of the flux problem for the Navier–Stokes equations. The solvability of this problem in the case where $F = 0$ follows from the classical results of Jean Leray [1]. In this paper, we prove the solvability of problem (1)–(3) in the class of symmetric flows with respect to a plane.

2. General case of the flux problem

Let us consider a more general situation. We assume that the boundary $\partial\Omega \in C^2$ of the bounded domain $\Omega \in \mathbb{R}^n$ ($n = 2, 3$) consists of N connected components Γ_i. The task is to find the solution $\mathbf{u}$, p of the boundary-value problem

$$-\nu\,\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=0,\ \ \mathrm{div}\,\mathbf{u}=0,\ \ x\in\Omega, \tag{4}$$

$$\mathbf{u}=\mathbf{a},\ \ x\in\partial\Omega. \tag{5}$$

In view of the continuity equation (5), the function $\mathbf{a}$ satisfies the condition

$$\int_{\partial\Omega}\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S=0, \tag{6}$$

where $\mathbf{n}$ is the unit exterior normal vector to the surface $\partial\Omega$. Equality (7) means that the total flux of an incompressible fluid through the boundary of the flow domain equals zero.

Let F_i be the flux of vector $\mathbf{a}$ through the surface Γ_i. Let us assume that a stronger condition than condition (2) is satisfied:

$$\int_{\Gamma_i}\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S\equiv F_i=0,\ \ i=1,\dots,N. \tag{7}$$

Then problem (4)–(6) has at least one solution [1]. We are interested in the case where $F_i \neq 0$. Problem (4)–(6) is also called the *Leray problem* because it actually goes back to his paper [1]. Fujita [2] and Finn [3] proved the solvability of the three-dimensional problem (4)–(5) for small values of F_i. Fujita and Morimoto [4] established the existence theorem for flows that are close to potential ones. Korobkov, Pileckas, and Russo obtained a positive solution to the flux problem for planar and axially symmetric flows without restrictions on the flux values (see [5] and references therein).

According to [6], we define the space $\mathbf{H}(\Omega)$ as the closure of the set of vector-functions $\boldsymbol{\xi} \in C_0^\infty(\Omega)$, $\mathrm{div}\,\boldsymbol{\xi} = 0$ in the norm of the Dirichlet integral

$$\|\nabla\mathbf{w}\|^2_{L^2(\Omega)}=\int_\Omega\sum_{i,k=1}^{3}(\partial w_i/\partial x_k)^2\mathrm{d}x. \tag{8}$$

Lemma 2.1. *[7,6]. Let the vector field* **a** *belongs to the class* $W^{1/2,2}(\partial\Omega)$*. If condition (7) is satisfied, then a solenoidal continuation* $\mathbf{b}(x,\varepsilon) \in W^{1,2}(\Omega)$ *of the vector* **a** *into the domain* Ω *exists so that for any* $\varepsilon > 0$ *we have*

$$\left| \int_\Omega \mathbf{b}\cdot\mathbf{w}\cdot\nabla\mathbf{w}\,\mathrm{d}x \right| \le \varepsilon\, \|\mathbf{w}\|^2_{W^{1,2}(\Omega)}, \quad \forall \mathbf{w} \in \mathbf{H}(\Omega). \tag{9}$$

3. Symmetric solutions to the planar flux problem

Let the domain $\Omega \in \mathbb{R}^2$ have an axis of symmetry $x_2 = 0$ and let the vector $\mathbf{a} = (a_1, a_2)$ specified on $\partial\Omega$ possess a symmetry property in following sense: a_1 is an even function in x_1 and a_2 is an odd function in x_1. We also assume that the symmetry axis intersects each of the connected components Γ_i of $\partial\Omega$. If $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ then problem (4), (5) has a solution $\mathbf{u} \in W^{1,2}(\Omega)$, $\nabla p \in L^2(\Omega)$ for any value of F_i. Moreover, the functions u_1, p are even functions in x_1, while u_2 is an odd function in x_1. At first, this result was obtained by Amick [8] and independently by Sazonov [9] with arguments from contradiction. The flow domain Ω in those papers was a curvilinear ring. Probably Sazonov did not know about Amick's work, but he proved the existence theorem by a simpler method using the notion of a *virtual drain*. This term was introduced by Fujita [10], who obtained an a priori estimate of the norm $\|\mathbf{u}\|_{W^{1,2}(\Omega)}$ for the planar symmetric flux problem in a multiply connected domain providing the solvability of the problem.

4. Modification of the cutting off function

We define a family of functions $\varsigma_\kappa(t)$ depending on the parameter $\kappa > 0$ by virtue of the relations

$$\varsigma_\kappa \in C_0^{2,\,\mathrm{Lip}(\mathbb{R})};\ \varsigma_\kappa \ge 0,\ \varsigma_\kappa(-t) = \varsigma_\kappa(t),\ t \in \mathbb{R};\ \varsigma_\kappa \le \frac{1}{t}\ (0 < t < \infty), \tag{10}$$

$$\left|\varsigma'_\kappa\right| \le \frac{2}{t^2}\ (0 < t < \infty),\ \varsigma_\kappa = 0\ (1 \le t < \infty),\ \varsigma_\kappa = \frac{1}{t}\ (\kappa \le t \le \frac{1}{2}),\ \varsigma'_\kappa = 0\ (0 \le t \le \frac{\kappa}{2}).$$

In comparison with the construction of the cutting-off function proposed by Hopf [7] and Fujita [10], we added here a restriction on the value of $\left|\varsigma'_\kappa\right|$ and a requirement $\varsigma_\kappa = const$ if $|t| < \kappa/2$.

Setting $\gamma_\kappa = \int_{-\infty}^{\infty}\varsigma_\kappa(t)\,\mathrm{d}t = \int_{-\infty}^{\infty}\varsigma_\kappa(t)\,\mathrm{d}t$, we see that $\gamma_\kappa \ge 2\int_\kappa^{1/2}\frac{\mathrm{d}t}{t} \to \infty$ as $\kappa \to 0$. Now we introduce an auxiliary function $\eta(t) = \eta(t;\,\delta,\kappa)$ by means of the equality

$$\eta(t) = \frac{1}{\gamma_\kappa}\frac{1}{\delta}\varsigma_\kappa\left(\frac{t}{\delta}\right),\ t \in \mathbb{R}, \tag{11}$$

where $\delta = const > 0$ is small enough, but has a fixed value. From Eq. (10), (11), we derive the estimates

$$0 \le \eta(t) \le \frac{1}{\gamma_\kappa}\frac{1}{\delta}\frac{\delta}{t} = \frac{1}{\gamma_\kappa}\frac{1}{t},\ 0 \le \left|\eta'(t)\right| \le \frac{1}{\gamma_\kappa}\frac{2}{\delta^2}\frac{\delta^2}{t^2} = \frac{1}{\gamma_\kappa}\frac{2}{t^2},\ (t \ne 0),$$

which imply the relations

$$\sup_t |t|\,\eta(t) \to 0,\ \sup_t t^2\left|\eta'(t)\right| \to 0 \text{ if } \kappa \to +0. \tag{12}$$

5. A priori estimate of the Dirichlet integral

Let us return to problem (1)–(3). Assume that there exists a function **b** with the properties $\mathbf{b} \in W^{1,2}(\Omega)$,

$$\operatorname{div}\mathbf{b} = 0,\ x \in \Omega, \tag{13}$$

$$\mathbf{b} = \mathbf{a_i},\ x \in \Gamma_i,\ i = 1, 2. \tag{14}$$

Further we consider that the domain Ω has a symmetry plane $z = 0$. In addition, we assume that the projections of the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ onto the axes r and φ of the cylindrical coordinate system are even functions of the variable z, while their projections onto the z axis are odd functions of z. In this situation, we expect that problem (1)–(3) has a symmetric solution where the projections of the velocity vector **u** onto the axes r and φ and the pressure p are even functions of the variable z and the projection of **u** onto the axis z is an odd function of z.

Below $\mathbf{H}_s(\Omega)$ denotes a subspace of the space $\mathbf{H}(\Omega)$ generated by the vector-functions that are symmetric in the above-mentioned sense. The function **u** is called a weak solution to problem (1)–(3) if $\mathbf{u} = \mathbf{U} + \mathbf{b}$, where $\mathbf{U} \in \mathbf{H}_s(\Omega)$, and the following integral identity is satisfied:

$$\nu \int_\Omega \nabla \mathbf{U} \cdot \nabla \boldsymbol{\eta}\, \mathrm{d}x - \int_\Omega ((\mathbf{U}+\mathbf{b})\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{U}\,\mathrm{d}x - \int_\Omega (\mathbf{U}\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{b}\,\mathrm{d}x = \tag{15}$$
$$= -\nu \int_\Omega \nabla \mathbf{b}\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla)\,\boldsymbol{\eta}\cdot\mathbf{b}\,\mathrm{d}x \;\; \forall\, \boldsymbol{\eta} \in \mathbf{H}_s(\Omega).$$

Lemma 5.1. *Let us assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is a surface of the class C^2 and $\mathbf{a_1}$, $\mathbf{a_2} \in W^{1/2,2}(\partial\Omega)$. Let the domain Ω and the functions $\mathbf{a_1}$, $\mathbf{a_2}$ be symmetric in the above-mentioned sense. Then for any weak solution to problem (1)–(3), the following estimate is valid:*

$$\|\nabla \mathbf{U}\|^2_{L^2(\Omega)} \le C_1. \tag{16}$$

6. Scheme for proving Lemma 5.1

Let us set $\boldsymbol{\eta} = \mathbf{U}$ in Eq. (15). We obtain

$$\nu \int_\Omega \nabla \mathbf{U}\cdot\nabla\mathbf{U}\,\mathrm{d}x - \int_\Omega \mathbf{b}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\mathrm{d}x = -\nu\int_\Omega \nabla\mathbf{b}\cdot\nabla\mathbf{U}\,\mathrm{d}x + \int_\Omega \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\,\cdot\mathbf{U}\,\mathrm{d}x. \tag{17}$$

For our purpose, it is sufficient to derive the inequality

$$\left|\int_\Omega \mathbf{b}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\,\mathrm{d}x\right| \le \frac{\nu}{2}\int_\Omega \nabla\mathbf{U}\cdot\nabla\mathbf{U}\,\mathrm{d}x. \tag{18}$$

We construct the vector-function $\mathbf{b}$ satisfying relations (13), (14) in the form $\mathbf{b} = \mathbf{c} + \mathbf{d}$, where

$$\mathbf{c} = (c_r,\,0,\,0),\;\; c_r = \frac{F}{2\pi r}\eta(z) \tag{19}$$

(virtual drain) and $\mathbf{d}$ satisfies the condition of zero partial fluxes (7) with $N = 2$. The second term in the left part of equality (17) can be written as

$$\int_\Omega \mathbf{b}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\,\mathrm{d}x = \frac{F}{2\pi}\int_\Omega \eta\,(U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial\varphi} + W\frac{\partial U}{\partial z})\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z - \frac{F}{2\pi}\int_\Omega \frac{\eta V^2}{r}\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z + \int_\Omega \mathbf{d}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\,\mathrm{d}x.$$

(Here U, V, and W are projections of the vector $\mathbf{U}$ onto the corresponding coordinate axis.) Integrating by parts and taking into account the continuity equation, we can rewrite the last identity as

$$\int_\Omega \mathbf{b}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\,\mathrm{d}x = \frac{F}{2\pi}\int_\Omega \eta'\,UW\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z + \frac{F}{2\pi}\int_\Omega \frac{\eta(U^2-V^2)}{r}\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z + \int_\Omega \mathbf{d}\cdot(\mathbf{U}\cdot\nabla)\,\mathbf{U}\,\mathrm{d}x. \tag{20}$$

On the basis of Lemma 2.1, we are able to choose $\mathbf{d}$ in order to guarantee the fulfillment of the inequality

$$|I_3| = \left|\int_\Omega \mathbf{d}\cdot\mathbf{U}\cdot\nabla\mathbf{U}\,\mathrm{d}x\right| \le \frac{\nu}{6}\,\|\nabla\mathbf{U}\|^2_{L^2(\Omega)},\;\; \forall\mathbf{U}\in\mathbf{H}_s(\Omega). \tag{21}$$

Further we represent the functions U and V in the form $U = U_1 + U_2$, $V = V_1 + V_2$, where

$$U_1 = [1-\chi(z)]\,U\,,\;\; U_2 = \chi(z)\,U\,,\quad V_1 = [1-\chi(z)]\,V\,,\;\; V_2 = \chi(z)\,V$$

and the function χ has the following properties: $\chi(z) \in C^\infty(\mathbb{R})$, $\chi(-z) = \chi(z)$, $\chi \ge 0$, $\chi' \le 0$, $z \in \mathbb{R}_+$; $\chi(0) = 1$, $\chi = 0$, $z \ge \kappa^2$; let also $\kappa \le 1/2$. Let us denote the first integral in the right part of Eq. (20) as I_4. We have

$$|I_4| \le C_2\,|F|\,\sup_z\,(z^2\,|\eta'|)\;\|U_1\|_{H^1(\Omega)}\,\|W\|_{H^1(\Omega)} \le \frac{\nu}{6}\;\|\nabla\mathbf{U}\|^2_{L^2(\Omega)} \tag{22}$$

in view of (12), if κ is small enough. Here C_2 depends on the domain Ω. In deriving this estimate, we used the Hardy inequality, the estimate $|\eta'| \le 2/\gamma_\kappa z^2$, and the equalities $U_1(r,\varphi,0) = 0$ and $\eta' U_2 = 0$ in Ω.

Let denote the second integral in the right part of Eq. (20) as I_5 and evaluate its absolute value

$$
\begin{aligned}
|I_5| = \left| \frac{F}{2\pi} \int\limits_{\Omega} \frac{\eta(U^2 - V^2)}{r} \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \right| &\leq C_3 \, |F| \sup_z (|z| \, \eta)(\|U_1\|_{H^1(\Omega)} + \|V_1\|_{H^1(\Omega)}) \, \|\nabla \mathbf{U}\|_{L^2(\Omega)} \\
&+ \frac{|F|}{2\pi} \int\limits_{\Omega'} \frac{1}{r} \eta \, (z) \, (U_2^2 + V_2^2) \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \leq 2C_3 \sup_z (|z| \, \eta) \, \|\nabla \mathbf{U}\|^2_{L^2(\Omega)} + \\
&+ \frac{|F|}{\pi \rho^2} \left(\int\limits_{\Omega'} \eta^{3/2} r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \right)^{2/3} \left(\int\limits_{\Omega'} (U_2^6 + V_2^6) r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \right)^{1/3} \leq \\
&\leq \left(\frac{\nu}{12} + \frac{C_4 \delta^{-1} \, |F| \, \kappa^{1/3}}{\rho^2 \ln(1/\kappa)} \right) \|\nabla \mathbf{U}\|^2_{L^2(\Omega)} \leq \frac{\nu}{6} \, \|\nabla \mathbf{U}\|^2_{L^2(\Omega)} ,
\end{aligned} \tag{23}
$$

if κ is sufficiently small. Here $\rho = \operatorname{dist}(\Gamma_1, \{0\})$ and the domain Ω' is the region of intersection of the domain Ω and the layer $|z| < \kappa^2$; the number δ is fixed; C_3 and C_4 depend on the domain Ω only. To obtain estimate (23), we used the Ladyzhenskaya and Young inequalities together with estimates following from definition (10), (11) of the function η: $\eta \leq a/\kappa$ for any $t \in \mathbb{R}$ and $\gamma_\kappa \geq b \ln(1/\kappa)$ (one can take $a = 2$ and $b = 1$). Inequalities (21)–(23) provide the desired estimate (18) of the norm $\|\nabla \mathbf{U}\|_{L^2(\Omega)}$, which completes the proof of Lemma 5.1.

Theorem 6.1. *Let the conditions of Lemma 5.1 be satisfied. Then problem (1)–(3) has the solution* $\mathbf{v} \in W^{1,2}(\Omega)$, $\nabla p \in L^2(\Omega)$.

The proof of Theorem 6.1 is omitted here. It is based on standard arguments of the theory of the Navier–Stokes equations [6].

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