C. R. Acad. Sci. Paris, Ser. I 355 (2017) 128–132

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Lie algebras

A remark on boundary level admissible representations

Une remarque sur les représentations admissibles de niveau limite

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ARTICLE INFO

Article history:
Received 16 January 2017
Accepted 17 January 2017
Available online 1 February 2017

Presented by Michèle Vergne

ABSTRACT

We point out that it is immediate by our character formula that in the case of a *boundary level* the characters of admissible representations of affine Kac–Moody algebras and the corresponding W-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

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RÉSUMÉ

Nous remarquons la conséquence suivante de notre formule de caractères. Pour un niveau limite, les caractères d'une représentation admissible d'une algèbre de Kac–Moody affine ainsi que de la W-algèbre correspondante s'expriment comme des produits de formes de Jacobi $\vartheta_{11}(\tau, z)$.

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Recently a remarkable map between 4-dimensional superconformal field theories and vertex algebras has been constructed [1]. This has led to new insights in the theory of characters of vertex algebras. In particular it was observed that in some cases these characters decompose in nice products [10,8].

The purpose of this note is to explain the latter phenomena. Namely, we point out that it is immediate by our character formula [5,6] that in the case of a *boundary level* the characters of admissible representations of affine Kac–Moody algebras and the corresponding W-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

We would like to thank Wenbin Yan for drawing our attention to this question.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Let $Q = \mathbb{Z}\Delta$ be the root lattice and let $Q^* = \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ be the dual lattice. Let $\Delta_+ \subset \Delta$ be a subset of positive roots, let $\{\alpha_1, \dots, \alpha_\ell\}$ be the set of simple roots and let ρ be half of the sum of positive roots. Let W be the Weyl group. Let $(.|.)$ be the invariant symmetric bilinear form on $\mathfrak{g}$, normalized by the condition $(\alpha|\alpha) = 2$ for a long root α, and let $h^\vee$ be the dual Coxeter number ($= \frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$). We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(.|.)$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the associated with $\mathfrak{g}$ affine Kac–Moody algebra (see [3] for details), let $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$ be its Cartan subalgebra. We extend the symmetric bilinear form $(.|.)$ from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}d) = 0$, $(K|K) = 0$,

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http://dx.doi.org/10.1016/j.crma.2017.01.008
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$(d|d)=0$, $(d|K)=1$, and we identify $\widehat{\mathfrak{h}}^*$ with $\widehat{\mathfrak{h}}$ using this form. Then d is identified with the 0th fundamental weight $\Lambda_0\in\widehat{\mathfrak{h}}^*$, such that $\Lambda_0|_{\mathfrak{g}[t,t^{-1}]+\mathbb{C}d}=0$, $\Lambda_0(K)=1$, and K is identified with the imaginary root $\delta\in\widehat{\mathfrak{h}}^*$. Then the set of real roots of $\widehat{\mathfrak{g}}$ is $\hat{\Delta}^{\mathrm{re}}=\{\alpha+n\delta|\,\alpha\in\Delta, n\in\mathbb{Z}\}$ and the subset of positive real roots is $\hat{\Delta}^{\mathrm{re}}_+=\Delta_+\cup\{\alpha+n\delta|\,\alpha\in\Delta, n\in\mathbb{Z}_{\geq 1}\}$. Let $\hat{\rho}=h^\vee\Lambda_0+\rho$. Let

$$\hat{\Pi}_u=\{u\delta-\theta,\alpha_1,\dots,\alpha_\ell\},$$

where $\theta\in\Delta_+$ is the highest root, so that $\hat{\Pi}_1$ is the set of simple roots of $\widehat{\mathfrak{g}}$. For $\alpha\in\hat{\Delta}^{\mathrm{re}}$ one lets $\alpha^\vee=2\alpha/(\alpha|\alpha)$. Finally, for $\beta\in Q^*$ define the translation $t_\beta\in\operatorname{End}\widehat{\mathfrak{h}}^*$ by

$$t_\beta(\lambda)=\lambda+\lambda(K)\beta-((\lambda|\beta)+\frac{1}{2}\lambda(K)|\beta|^2)\delta.$$

Given $\Lambda\in\widehat{\mathfrak{h}}^*$ let $\hat{\Delta}^\Lambda=\{\alpha\in\hat{\Delta}^{\mathrm{re}}\,|\,(\Lambda|\alpha^\vee)\in\mathbb{Z}\}$. Then Λ is called an *admissible* weight if the following two properties hold:

(i) $(\Lambda+\widehat{\rho}|\alpha^\vee)\notin\mathbb{Z}_{\leq 0}$ for all $\alpha\in\hat{\Delta}_+$,
(ii) $\mathbb{Q}\hat{\Delta}^\Lambda=\mathbb{Q}\hat{\Delta}^{\mathrm{re}}$.

If instead of (ii) a stronger condition holds:

(ii)′ $\varphi(\hat{\Delta}^\Lambda)=\hat{\Delta}^{\mathrm{re}}$ for a linear isomorphism $\varphi:\widehat{\mathfrak{h}}^*\to\widehat{\mathfrak{h}}^*$,

then Λ is called a *principal* admissible weight. In [6] the classification and character formulas for admissible weights are reduced to that for principal admissible weights. The latter are described by the following proposition.

Proposition 1. [6] *Let Λ be a principal admissible weight and let $k=\Lambda(K)$ be its level. Then*

(a) k is a rational number with denominator $u\in\mathbb{Z}_{\geq 1}$, such that

$$k+h^\vee\geq\frac{h^\vee}{u}\text{ and }\gcd(u,h^\vee)=\gcd(u,r^\vee)=1,\tag{1}$$

where $r^\vee=1$ for $\mathfrak{g}$ of type A-D-E, $=2$ for $\mathfrak{g}$ of type B, C, F, and $=3$ for $\mathfrak{g}=G_2$.
(b) All principal admissible weights are of the form

$$\Lambda=(t_\beta y).(\Lambda^0-(u-1)(k+h^\vee)\Lambda_0),\tag{2}$$

where $\beta\in Q^*$, $y\in W$ are such that $(t_\beta y)\hat{\Pi}_u\subset\hat{\Delta}_+$, Λ^0 is an integrable weight of level $u(k+h^\vee)-h^\vee$, and dot denotes the shifted action: $w.\Lambda=w(\Lambda+\widehat{\rho})-\widehat{\rho}$.
(c) For $\mathfrak{g}=s\ell_N$ all admissible weights are principal admissible.

Recall that the normalized character of an irreducible highest weight $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level $k\neq -h^\vee$ is defined by

$$\mathrm{ch}_\Lambda(\tau,z,t)=q^{m_\Lambda}\mathrm{tr}_{L(\Lambda)}\mathrm{e}^{2\pi\mathrm{i}h}$$

where

$$h=-\tau d+z+tK,\ z\in\mathfrak{h},\ \tau,t\in\mathbb{C},\ \mathrm{Im}\,\tau>0,\ q=\mathrm{e}^{2\pi\mathrm{i}\tau},\tag{3}$$

and $m_\Lambda=\frac{|\Lambda+\widehat{\rho}|^2}{2(k+h^\vee)}-\frac{\dim\mathfrak{g}}{24}$ (the normalization factor q^{m_Λ} "improves" the modular invariance of the character).

In [6], the characters of the $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ for arbitrary admissible Λ were computed, see Theorem 3.1, or formula (3.3) there for another version in case of a principal admissible Λ. In order to write down the latter formula, recall the normalized affine denominator for $\widehat{\mathfrak{g}}$:

$$\hat{R}(h)=q^{\frac{\dim\mathfrak{g}}{24}}e^{\widehat{\rho}(h)}\prod_{n=1}^{\infty}(1-q^n)^\ell\prod_{\alpha\in\Delta_+}(1-\mathrm{e}^{\alpha(z)}q^n)(1-\mathrm{e}^{-\alpha(z)}q^{n-1}).$$

In coordinates (3) this becomes:

$$\hat{R}(\tau,z,t)=(-\mathrm{i})^{|\Delta_+|}\mathrm{e}^{2\pi\mathrm{i}h^\vee t}\eta(\tau)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\vartheta_{11}(\tau,\alpha(z)),\tag{4}$$

where

$$\vartheta_{11}(\tau,z) = -\mathrm{i}q^{\frac{1}{12}}\mathrm{e}^{-\pi\mathrm{i}z}\eta(\tau)\prod_{n=1}^{\infty}(1-\mathrm{e}^{-2\pi\mathrm{i}z}q^n)(1-\mathrm{e}^{2\pi\mathrm{i}z}q^{n-1})$$

is one of the standard Jacobi forms ϑ_{ab}, $a, b = 0$ or 1 (see, e.g., Appendix to [7]), and $\eta(\tau)$ is the Dedekind eta function.

For a principal admissible Λ, given by (2), formula (3.3) from [6] becomes in coordinates (3):

$$(\hat{R}\mathrm{ch}_\Lambda)(\tau,z,t) = (\hat{R}\mathrm{ch}_{\Lambda^0})\left(u\tau, y^{-1}(z+\tau\beta), \frac{1}{u}(t+(z|\beta)+\frac{\tau|\beta|^2}{2})\right). \tag{5}$$

It follows from (5) that if $\Lambda^0 = 0$ in (2) (so that $\mathrm{ch}_{\Lambda^0} = 1$), which is equivalent to

$$k + h^\vee = \frac{h^\vee}{u} \text{ and } \gcd(u, h^\vee) = \gcd(u, r^\vee) = 1, \tag{6}$$

the (normalized) character ch_Λ turns into a product. The level k, defined by (6), is naturally called the *boundary principal admissible* level in [4], see formula (3.5) there. We obtain from Proposition 1, (4) and (5)

Proposition 2.

(a) All boundary principal admissible weights are of level k, given by (6), and are of the form

$$\Lambda = (t_\beta y).(k\Lambda_0), \tag{7}$$

where $\beta \in Q^*$, $y \in W$ are such that $(t_\beta y)\hat{\Pi}_u \subset \hat{\Delta}_+$. In particular, $k\Lambda_0$ is a principal admissible weight of level (6).

(b) If Λ is of the form (7), then

$$\mathrm{ch}_\Lambda(\tau,z,t) = \mathrm{e}^{2\pi\mathrm{i}(kt+\frac{h^\vee}{u}(z|\beta))}q^{\frac{h^\vee}{2u}|\beta|^2}\left(\frac{\eta(u\tau)}{\eta(\tau)}\right)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\frac{\vartheta_{11}(u\tau, y(\alpha)(z+\tau\beta))}{\vartheta_{11}(\tau,\alpha(z))}.$$

Remark 1. For the vacuum module $L(k\Lambda_0)$ of the boundary principal admissible level k the character formula from Proposition 2(b) becomes

$$\mathrm{ch}_{k\Lambda_0}(\tau,z,t) = \mathrm{e}^{2\pi\mathrm{i}kt}\left(\frac{\eta(u\tau)}{\eta(\tau)}\right)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\frac{\vartheta_{11}(u\tau,\alpha(z))}{\vartheta_{11}(\tau,\alpha(z))}.$$

Example 1. Let $\mathfrak{g} = s\ell_2$, so that $h^\vee = 2$. Then the boundary levels are $k = \frac{2}{u} - 2$, where u is a positive odd integer, and all admissible weights are

$$\Lambda_{k,j} := t_{-\frac{j}{2}\alpha_1}.(k\Lambda_0) = (k+\frac{2j}{u})\Lambda_0 - \frac{2j}{u}\Lambda_1,\ j = 0, 1, \ldots, u-1,$$

and the character formula from Proposition 2(b) becomes:

$$\mathrm{ch}_{\Lambda_{u,j}} = \mathrm{e}^{2\pi\mathrm{i}(kt-\frac{j}{u}z)}q^{\frac{j^2}{2u}}\frac{\vartheta_{11}(u\tau, z-j\tau)}{\vartheta_{11}(\tau,z)}. \tag{8}$$

For $u = 3$ and 5 some of these formulas is conjectured in [8].

Example 2. Let $\mathfrak{g} = s\ell_N$, so that $h^\vee = N$, let $N > 1$ be odd, and let $u = 2$. Then the boundary admissible level is $k = -\frac{N}{2}$, and the boundary admissible weights of the form $t_\beta.(k\Lambda_0)$ are:

$$\Lambda_{N,p} = -\frac{N}{2}\Lambda_p,\ p = 0, 1, \ldots, , N-1,$$

where Λ_p are the fundamental weights of $\hat{\mathfrak{g}}$. Letting $z = \sum_{i=1}^{N-1} z_i\bar{\Lambda}_i$, where $\bar{\Lambda}_i$ are the fundamental weights of $\mathfrak{g}$, the character formula from Proposition 2 (b) becomes:

$$\mathrm{ch}_{\Lambda_{N,p}}(\tau,z,t) = \mathrm{i}^{p(N-p)}\mathrm{e}^{-\pi\mathrm{i}Nt}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-\frac{(N-1)(N-2)}{2}}$$
$$\times\frac{\prod\limits_{\substack{1\le i\le j<p\\ \text{or } p<i\le j<N}}\vartheta_{11}(2\tau, z_i+\ldots+z_j)\prod\limits_{1\le i\le p\le j<N}\vartheta_{01}(2\tau, z_i+\ldots+z_j)}{\prod\limits_{1\le i\le j<N}\vartheta_{11}(\tau, z_i+\ldots+z_j)},$$

where

$$\vartheta_{01}(\tau,z)=\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^{n-\frac{1}{2}})(1-e^{-2\pi i z}q^{n-\frac{1}{2}}).$$

This follows from Proposition 2(b) by applying to ϑ_{11} an elliptic transformation (see, e.g., [7], Appendix). In particular,

$$\mathrm{ch}_{-\frac{N}{2}\Lambda_0}=e^{-\pi i N t}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-\frac{(N-1)(N-2)}{2}}\prod_{1\le i\le j<N}\frac{\vartheta_{11}(2\tau,z_i+\ldots+z_j)}{\vartheta_{11}(\tau,z_i+\ldots+z_j)}.$$

The latter formula was conjectured in [10].

Remark 2. For principal admissible weights $\Lambda=(t_\beta y).(k\Lambda_0)$ and $\Lambda'=(t_{\beta'}y').(k\Lambda_0)$ of boundary level $k=\frac{h^\vee}{u}-h^\vee$ the S-transformation matrix $(a(\Lambda,\Lambda'))$, given by [6], Theorem 3.6, simplifies to

$$a(\Lambda,\Lambda')=|Q/uh^\vee Q^*|^{-\frac{1}{2}}\varepsilon(yy')\prod_{\alpha\in\Delta_+}2\sin\frac{\pi i u(\rho|\alpha)}{h^\vee}e^{-2\pi i\left((\rho|\beta+\beta')+\frac{h^\vee(\beta|\beta')}{u}\right)}.$$

Remark 3. If $\mathfrak{g}=s\ell_2$ and k is as in Example 1, then

$$a(\Lambda_{k,j},\Lambda_{k,j'})=(-1)^{j+j'}e^{-\frac{2\pi i jj'}{u}}\frac{1}{\sqrt{u}}\sin\frac{u\pi}{2}.$$

One can compute fusion coefficients [9] by Verlinde's formula:

$$N_{\Lambda_{k,j_1},\Lambda_{k,j_2},\Lambda_{k,j_3}}=(-1)^{j_1+j_2+j_3}\text{ if }j_1+j_2+j_3\in u\mathbb{Z},\text{ and }=0\text{ otherwise.}$$

Example 3. Let $\mathfrak{g}=sl_3$, so that $h^\vee=3$, and let u be a positive integer, coprime to 3. Then all (principal) admissible weights have level $k=\frac{3}{u}-3$ and are of the form (7), where

$$\beta=-(-1)^p(k_1\bar\Lambda_1+k_2\bar\Lambda_2),\ y=r_\theta^p,\ p=0\text{ or }1,\ k_i\in\mathbb{Z},\ k_i\ge\delta_{p,1},\ k_1+k_2\le u-\delta_{p,0}.$$

Denote this weight by $\Lambda^{(p)}_{u;k_1,k_2}=(t_\beta y).(k\Lambda_0)$. Using Remark 2, one computes the fusion coefficients by Verlinde's formula:

$$N_{\Lambda^{(p)}_{u;k_1,k_2}\Lambda^{(p')}_{u;k'_1,k'_2}\Lambda^{(p'')}_{u;k''_1,k''_2}}=(-1)^{p+p'+p''}\text{ if }(-1)^pk_i+(-1)^{p'}k'_i+(-1)^{p''}k''_i\in u\mathbb{Z}\text{ for }i=1,2,$$

and $=0$ otherwise.

Remark 4. If Λ is an arbitrary admissible weight, then $\hat\Delta^\Lambda$ decomposes in a disjoint union of several affine root systems. Then Λ has *boundary level* if restrictions of it to each of them has boundary level, and formula (3.4) from [6] shows that ch_Λ decomposes in a product of the corresponding boundary level characters. Note also that all the above holds also for twisted affine Kac–Moody algebras [6].

Remark 5. The product character formula for boundary level affine Kac–Moody superalgebras holds as well, see [2], formula (2).

Recall that with any $s\ell_2$-triple $\{f,x,e\}$ in $\mathfrak{g}$, where $[x,f]=-f$, $[x,e]=e$, one associates a W-algebra $W^k(g,f)$, obtained from the vacuum $\widehat{\mathfrak{g}}$-module of level k by quantum Hamiltonian reduction, so that any $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level k produces either an irreducible $W^k(g,f)$-module $H(\Lambda)$ or zero. The characters of $L(\Lambda)$ and $H(\Lambda)$ are related by the following simple formula ([4] or [7]):

$$\left(\overset{W}{R}\,\mathrm{ch}_{H(\Lambda)}\right)(\tau,z)=\left(\hat R\,\mathrm{ch}_\Lambda\right)(\tau,-\tau x+z,\frac{\tau}{2}(x|x)).\tag{9}$$

Here $z\in\mathfrak{h}^f$, the centralizer of f in $\mathfrak{h}$, and

$$\overset{W}{R}(\tau,z)=\eta(\tau)^{\frac{3}{2}l-\frac{1}{2}\dim(\mathfrak{g}_0+\mathfrak{g}_{1/2})}\prod_{\alpha\in\Delta_+^0}\vartheta_{11}(\tau,\alpha(z))\left(\prod_{\alpha\in\Delta_{1/2}}\vartheta_{01}(\tau,\alpha(z))\right)^{1/2},\tag{10}$$

where $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is the eigenspace decomposition for ad x, $\Delta_j \subset \Delta$ is the set of roots of root spaces in $\mathfrak{g}_j$ and $\Delta_+^0 = \Delta_+ \cap \Delta_0$ (we assume that $\Delta_j \subset \Delta_+$ for $j > 0$). If k is a boundary level (6), we obtain from Proposition 2(b) and formulas (9), (10) the following character formula for $H(\Lambda)$ if Λ is a principal admissible weight (7) ($z \in \mathfrak{h}^f$):

$$\mathrm{ch}_{H(\Lambda)}(\tau, z) = (-\mathrm{i})^{|\Delta_+|} q^{\frac{h^\vee}{2u}|\beta - x|^2} e^{\frac{2\pi \mathrm{i} h^\vee}{u}(\beta|z)} \times \frac{\eta(u\tau)^{\frac{3}{2}\ell - \frac{1}{2}\dim \mathfrak{g}}}{\eta(\tau)^{\frac{3}{2}\ell - \frac{1}{2}\dim(\mathfrak{g}_0 + \mathfrak{g}_{1/2})}} \frac{\prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, y(\alpha)(z + \tau\beta - \tau x))}{\prod_{\alpha \in \Delta_+^0} \vartheta_{11}(\tau, \alpha(z)) \left(\prod_{\alpha \in \Delta_{1/2}} \vartheta_{01}(\tau, \alpha(z)) \right)^{1/2}}. \tag{11}$$

Remark 6. A formula, similar to Proposition 2(b) and to formula (11), holds if $\mathfrak{g}$ is a basic Lie superalgebra; one has to replace the character by the supercharacter, dim by sdim, and the factor ϑ_{ab}, corresponding to a root α, by its inverse if this root is odd. Also, the character is obtained from the supercharacter by replacing ϑ_{ab} by $\vartheta_{a,b+1 \bmod 2}$ if the root α is odd.

Remark 7. An example of (11) is the minimal series representations of the Virasoro algebra with central charge $c = 1 - \frac{3(u-2)^2}{u}$, obtained by the quantum Hamiltonian reduction from the boundary admissible $\hat{sl}_2$-modules from Example 1. For $j = u - 1$ one gets 0, for $u = 3$ and $j = 0, 1$ one gets the trivial representation, but for all other j and $u \geq 5$ the characters are the product sides of the Gordon generalizations of the Rogers–Ramanujan identities (the latter correspond to $u = 5$). Another example is the minimal series representations of the $N = 2$ superconformal algebras, see [4], Section 7.

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