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Lelong numbers, complex singularity exponents, and Siu's semicontinuity theorem ☆

Nombres de Lelong, exposants de singularités complexes et théorème de semi-continuité de Siu

Qi'an Guan [a], Xiangyu Zhou [b]

[a] *School of Mathematical Sciences, and Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China*
[b] *Institute of Mathematics, AMSS, and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China*

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ABSTRACT

In this note, we describe a relation between Lelong numbers and complex singularity exponents. As an application, we obtain a new proof of Siu's semicontinuity theorem for Lelong numbers.

RÉSUMÉ

Dans cette note, nous décrivons une relation entre les nombres de Lelong et les exposants de singularités complexes. Comme application, nous obtenons une nouvelle preuve du théorème de semi-continuité de Siu pour les nombres de Lelong.

1. Introduction

Let φ be a plurisubharmonic function near the origin $o \in \mathbb{C}^n$. The Lelong number is classically defined as

Definition 1.1. $\nu(\varphi, o) := \sup\{c \geq 0 : \varphi \leq c \log|z| + O(1)\}$.

In [22], Siu established the semicontinuity theorem for Lelong numbers, namely, that the upper level sets of Lelong numbers of any closed positive current are analytic sets. A few years later, Kiselman [17] (see also [18]) generalized Siu's semicontinuity theorem to directional Lelong numbers. In [3], Demailly introduced generalized Lelong numbers and ex-

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E-mail addresses: guanqian@amss.ac.cn (Q. Guan), xyzhou@math.ac.cn (X. Zhou).

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tended the above result of Kiselman in this context. Later, Demailly (see [6]) gave a simple and completely new proof of Siu's theorem by using the Ohsawa–Takegoshi L^2 extension theorem.

In this note, relying on our previous works [12–16], we present a new proof of Siu's theorem by establishing a relation between Lelong numbers and complex singularity exponents.

Let us first recall the definition of complex singularity exponents (also called log canonical thresholds by algebraic geometers, see [21,19] et al.); for this, it is convenient to use the concept of multiplier ideal sheaf

$$\mathcal{I}(\varphi)_{z_0} := \left\{ f \in \mathcal{O}_{\mathbb{C}^n, z_0} \mid \exists V \text{ open} \ni z_0, \int_V |f(z)|^2 e^{-2\varphi(z)} d\lambda(z) < +\infty \right\},$$

introduced by Nadel [20] (see [4,6,23,25] et al.).

Definition 1.2. The complex singularity exponent of φ at z_0 is defined to be

$$c_{z_0}(\varphi) := \sup\{c \ge 0 : \mathcal{I}(c\varphi)_{z_0} = \mathcal{O}_{\mathbb{C}^n, z_0}\}.$$

Our main result can be stated as follows:

Theorem 1.3. *Let φ be a plurisubharmonic function on an open set $D \subset \mathbb{C}^n$. Then for any $k \in \mathbb{N}$, $k \ge 1$, there exists a plurisubharmonic function φ_k defined on a neighborhood of $D \times \{o\} \subset \mathbb{C}^n \times \mathbb{C}^k$ with coordinates (z, w), such that*

(1) $\varphi_k(z, o) = \varphi(z)$,
(2) $\nu(\varphi_k, (z, o)) = \nu(\varphi, z)$,
(3) $\frac{k}{\nu(\varphi,z)} \le c_{(z,o)}(\varphi_k) \le \frac{n+k}{\nu(\varphi,z)}$

for any $z \in D$, where o is the origin in $\mathbb{C}^k$. One can take for instance

(4) $\varphi_k(z, w) = \sup_{\zeta \in B(z,|w|)} \varphi(\zeta)$.

The reader will observe that the second inequality in (3) can be directly deduced by Skoda's well-known estimate (see [24]) $c_z(\varphi) \le \frac{n}{\nu(\varphi,z)}$ applied to φ_k at (z, o), and combined with (2).

Remark 1.4. It is clear that property (3) in Theorem 1.3 is equivalent to

$$\frac{k}{n+k}\nu^{-1}(\varphi, z) \le c_{(z,o)}((n+k)\varphi_k) \le \nu^{-1}(\varphi, z),$$

which implies

$$\left\{\nu(\varphi, z) \ge \frac{k}{n+k}c\right\} \supseteq \left\{c_{(z,o)}((n+k)\varphi_k) \le \frac{1}{c}\right\} \supseteq \{\nu(\varphi, z) \ge c\}.$$

Since $\lim_{k\to+\infty} \frac{k}{n+k} = 1$, we obtain

$$\left\{z|\nu(\varphi, z) \ge c\right\} = \bigcap_k \left\{z|c_{(z,o)}((n+k)\varphi_k) \le \frac{1}{c}\right\}.$$

It is however well known that the sublevel sets $\{z|c_z(\psi) \le a\}$ of complex singularity exponents of any plurisubharmonic function ψ are analytic. This follows, e.g., from Berndtsson's solution [1] of the openness conjecture (the conjecture was posed by Demailly and Kollár [7]; for a proof of the two-dimensional case, see [10,9,8]). In fact, this had been known since a long time as a consequence of the Hörmander–Bombieri theorem [11,2]. We conclude that the set $\{z|c_{(z,o)}((n+k)\varphi_k) \le \frac{1}{c}\}$ is analytic for any $k \in \mathbb{N}$ and $c > 0$, whence Siu's semicontinuity theorem for Lelong numbers [22]:

Corollary 1.5. *(See [22].) $\{z|\nu(\varphi, z) \ge c > 0\}$ is an analytic set.*

We refer to [6] and [5] for alternative proofs by Demailly. We would like to thank the referee for pointing out the Hörmander–Bombieri theorem.

2. Preparation

2.1. Restriction formula for complex singularity exponents and Lelong numbers

Let φ be a plurisubharmonic function on a neighborhood of the origin $o \in \mathbb{C}^n$. In [7], the following restriction formula ("important monotonicity result") about complex singularity exponents is obtained by using the Ohsawa–Takegoshi L^2 extension theorem.

Proposition 2.1. *(See [7].) For any regular complex submanifold $(H, o) \subset (\mathbb{C}^n, o)$, the inequality*

$$c_o(\varphi|_H) \le c_o(\varphi) \tag{2.1}$$

holds whenever $\varphi|_H \not\equiv -\infty$.

We recall the following (much easier to prove) restriction property of Lelong numbers.

Lemma 2.2. *(See [6].) For any regular complex submanifold $(H, o) \subset (\mathbb{C}^n, o)$, the inequality*

$$\nu(\varphi|_H, o) \ge \nu(\varphi, o) \tag{2.2}$$

holds whenever $\varphi|_H \not\equiv -\infty$.

2.2. Lelong number and complex singularity exponent for $U(n)$ invariant plurisubharmonic functions on $\mathbb{C}^n$

We recall the following characterization of $U(n)$ invariant plurisubharmonic function (see, e.g., Lemma III.7.10 in [6]).

Lemma 2.3. *Let φ be a plurisubharmonic function on a ball $B(0, r) \subset \mathbb{C}^n$ which is $U(n)$ invariant. Then $\varphi(z) = \chi(\log|z|)$, where $\chi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a convex increasing function.*

The following remark is a direct consequence of Lemma 2.3.

Remark 2.4. Let φ be a plurisubharmonic function on a ball $B(0, r) \subset \mathbb{B}^n$ that is $U(n)$ invariant. Then $c_o(\varphi) = \frac{n}{\nu(\varphi, o)}$.

Proof. By Definition 1.1, it is clear that

$$\nu(\varphi, o) = \lim_{t \to -\infty} \frac{\chi(t)}{t},$$

i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|z| < \delta$,

$$(\nu(\varphi, o) + \varepsilon) \log|z| \le \varphi(z) \le (\nu(\varphi, o) - \varepsilon) \log|z|,$$

and the present remark is deduced by looking at the integrability of $e^{-c\varphi}$. □

2.3. A holomorphic change of variables for Lelong numbers

Take a surjective linear map $\ell : \mathbb{C}^k \to \mathbb{C}^n$ $(k \ge n)$, and define a holomorphic map p_k from $\mathbb{C}^n \times \mathbb{C}^k$ such that

$$p_k(z, w) = z + \ell(w).$$

Let φ be a plurisubharmonic function on $D \subset \mathbb{C}^n$. Then the pull-back function $p_k^* \varphi = \varphi \circ p_k$ is well defined on the open set $p_k^{-1}(D) \subset \mathbb{C}^n \times \mathbb{C}^k$.

Lemma 2.5. *For any $z_0 \in D$, the pull-back function $p_k^* \varphi$ satisfies by construction the following properties:*

(1) $p_k^* \varphi(z_0, o) = \varphi(z_0)$;
(2) $\nu(p_k^* \varphi, (z_0, o)) = \nu(\varphi, z_0) = \nu(p_k^* \varphi|_{z=z_0}, (z_0, o))$,

where the notation $\nu(p_k^* \varphi|_{z=z_0}, (z_0, o))$ indicates that the Lelong number is computed along the submanifold $\{z = z_0\} = \{z_0\} \times \mathbb{C}^k$.

Proof. (1) is obvious by definition of p_k. In order to prove (2), it suffices to consider the case $z_0 = (0, \cdots, 0) \in D$. It is clear that $p_k^* \log|z| = \log|z + \ell(w)| \leq \log(|z| + |w|) + O(1)$. Therefore, for any $c > 0$ satisfying $\varphi(z) \leq c\log|z| + O(1)$ when $z \to 0$, we have

$$p_k^*\varphi(z, w) \leq c\log(|z| + |w|) + O(1),$$

which implies $\nu(p_k^*\varphi, (z_0, o)) \geq \nu(\varphi, z_0)$. Conversely, if $H \subset \mathbb{C}^k$ is taken to be a linear subspace on which $\ell : H \to \mathbb{C}^n$ is bijective, Lemma 2.2 yields the sequence of inequalities

$$\nu(p_k^*\varphi, (z_0, o)) \leq \nu(p_k^*\varphi|_{\{z_0\}\times\mathbb{C}^k}, (z_0, o)) \leq \nu(p_k^*\varphi|_{\{z_0\}\times H}, (z_0, o)) = \nu(\varphi, z_0).$$

The last equality comes from the invariance of Lelong numbers by linear changes of variable, which is obvious from Definition 1.1. This implies (2), and the lemma is proved. □

2.4. An invariance property of Lelong numbers

Let φ be a plurisubharmonic function on a product domain $\Omega \subset \mathbb{C}^n \times \mathbb{C}^k$ containing the origin, and let (z, w) denote the coordinates. One can define

$$\tilde{\varphi}(z, w) := \sup_{g\in U(k)} \varphi(z, gw)$$

on a $U(k)$ invariant neighborhood of $\{w = o\}$ in $\mathbb{C}^k$. It is a plurisubharmonic function. By Definition 1.1, we immediately get

Lemma 2.6. *The equalities*

$$\nu(\tilde{\varphi}, (z_0, o)) = \nu(\varphi, (z_0, o)) \quad \text{and} \quad \nu(\tilde{\varphi}|_{z=z_0}, (z_0, o)) = \nu(\varphi|_{z=z_0}, (z_0, o))$$

hold for any $(z_0, o) \in (\Omega \cap \{w = o\})$.

3. Proof of Theorem 1.3

Let φ be a plurisubharmonic function on an open set $D \subset\subset \mathbb{C}^n$. With the same notation as above, let us consider the plurisubharmonic function

$$\varphi_k(z, w) := \sup_{g\in U(k)} p_k^*\varphi(z, gw),$$

which is defined on a neighborhood of $D \times \{o\}$ in $\mathbb{C}^n \times \mathbb{C}^k$. By Lemmas 2.6 and 2.5 (2), second equality, we have

$$\nu(\varphi_k|_{z=z_0}, (z_0, o)) = \nu(p_k^*\varphi|_{z=z_0}, (z_0, o)) = \nu(\varphi, z_0). \tag{3.1}$$

However, by Remark 2.4, since $w \mapsto \varphi_k(z_0, w)$ is $U(k)$-invariant, it follows that

$$\nu(\varphi_k|_{z=z_0}, (z_0, o)) = \frac{k}{c_{(z_0,o)}(\varphi_k|_{z=z_0})}. \tag{3.2}$$

From Proposition 2.1 and equalities (3.2), (3.1) respectively, one gets

$$c_{(z_0,o)}(\varphi_k) \geq c_{(z_0,o)}(\varphi_k|_{z=z_0}) = \frac{k}{\nu(\varphi_k|_{z=z_0}, (z_0, o))} = \frac{k}{\nu(\varphi, z_0)}.$$

This completes the proof of Theorem 1.3.

Remark 3.1. By taking $\ell : \mathbb{C}^k \to \mathbb{C}^n$ to be the projection onto the first n coordinates, one simply gets

$$\varphi_k(z, w) = \sup_{\zeta\in B(z,|w|)} \varphi(\zeta), \qquad w \in \mathbb{C}^k. \tag{3.3}$$

In fact with such a choice of φ_k, Theorem 1.3 even holds for any $k \geq 1$, as one can easily see.

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