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Partial differential equations

On the existence of correctors for the stochastic homogenization of viscous Hamilton–Jacobi equations ☆

Sur l'existence de correcteurs en homogénéisation stochastique d'équations de Hamilton–Jacobi

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ABSTRACT

We prove, under some assumptions, the existence of correctors for the stochastic homogenization of "viscous" possibly degenerate Hamilton–Jacobi equations in stationary ergodic media. The general claim is that, assuming knowledge of homogenization in probability, correctors exist for all extreme points of the convex hull of the sublevel sets of the effective Hamiltonian. Even when homogenization is not a priori known, the arguments imply the existence of correctors and, hence, homogenization in some new settings. These include positively homogeneous Hamiltonians and, hence, geometric-type equations including motion by mean curvature, in radially symmetric environments and for all directions. Correctors also exist and, hence, homogenization holds for many directions for nonconvex Hamiltonians and general stationary ergodic media.

RÉSUMÉ

Nous démontrons l'existence, sous certaines conditions, de correcteurs en homogénéisation stochastique d'équations de Hamilton–Jacobi et d'équations de Hamilton–Jacobi visqueuses. L'énoncé général est que, si l'on sait qu'il y a homogénéisation en probabilité, un correcteur existe pour toute direction étant un point extrémal de l'enveloppe convexe d'un ensemble de niveau du Hamiltonien effectif. Même lorsque que l'homogénéisation n'est pas connue a priori, les arguments développés dans cette note montrent l'existence d'un correcteur, et donc l'homogénéisation, dans certains contextes. Cela inclut les équations de type géométrique dans des environnements dont la loi est à symmétrie radiale. Dans le cas général stationnaire ergodique et sans hypothèse de convexité sur le hamiltonien, on montre que des correcteurs existent pour plusieurs directions.

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1. Introduction

The aim of this note is to show the existence of correctors for the stochastic homogenization of "viscous" Hamilton–Jacobi equations of the form

$$u_t^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) D^2u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter that tends to zero, $H = H(p, y, \omega)$ is the Hamiltonian and $A = A(p, y, \omega)$ is a (possibly) degenerate diffusion matrix. Both A and H depend on a parameter $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We assume that $\mathbb{P}$ is stationary ergodic with respect to translations on $\mathbb{R}^d$ and that A and H are stationary.

The basic question in the stochastic homogenization of (1.1) is the existence of a deterministic effective Hamiltonian $\overline{H}$ such that the solutions u^ε to (1.1) converge, as $\varepsilon \to 0$, locally uniformly and with probability one, to the solution to the effective equation

$$u_t + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1.2}$$

When H is convex with respect to the p variable and coercive, this was first proved independently by Souganidis [17] and Rezakhanlou and Tarver [16] for first-order Hamilton–Jacobi equations, and later extended to the viscous setting by Lions and Souganidis [14] and Kosygina, Rezakhanlou and Varadhan [10]. See also Armstrong and Souganidis [1,2] and Armstrong and Tran [4] for generalizations and alternative arguments.

In periodic homogenization, convergence and, hence, homogenization rely on the existence of correctors (see Lions, Papanicolaou and Varadhan [12]). The random setting is, however, fundamentally different.

Following Lions and Souganidis [13], a corrector associated with a direction $p \in \mathbb{R}^d$ is a solution χ to the corrector equation

$$-\operatorname{tr}(A(D\chi(x) + p, x, \omega)D^2\chi(x)) + H(D\chi(x) + p, x, \omega) = \overline{H}(p) \quad \text{in } \mathbb{R}^d \tag{1.3}$$

which has a sublinear growth at infinity, that is, with probability one,

$$\lim_{|x| \to +\infty} \frac{\chi(x, \omega)}{|x|} = 0. \tag{1.4}$$

It was shown in [13] that in general such solutions do not exist; see also the discussion by Davini and Siconolfi [8] in the 1-d case. Note that the main point is the existence of solutions satisfying (1.4).

Not knowing how to find correctors is the main reason that the theory of homogenization in random media is rather complicated and required the development of new arguments. General qualitative results in the references cited earlier required the quasiconvexity assumption. A more direct approach to prove homogenization (always in the convex setting), which is based on weak convergence methods and yields only convergence in probability, was put forward by Lions and Souganidis [15]. Our approach here is close in spirit to the one of [15]. With the exception of a case with Hamiltonians of a very special form (see Armstrong, Tran and Yu [3,5]), the main results known in nonconvex settings are quantitative. That is it is necessary to make some strong assumptions on the environment (finite-range dependence) and to use sophisticated concentration inequalities to prove directly that the solutions to the oscillatory problems converge; see, for example, Armstrong and Cardaliaguet [3] and Feldman and Souganidis [9]. It should be noted that the counterexamples of Ziliotto [18] and [9] yield that in the setting of nonconvex homogenization in random media is not possible to prove the existence of correctors for all directions.

Our main result states that a corrector in the direction p exists provided p is an extreme point of the convex hull of the sub-level set $\{q \in \mathbb{R}^d : \overline{H}(q) \le \overline{H}(p)\}$. For instance, this is the case if the law of the pair (A, H) under $\mathbb{P}$ is radially symmetric, and A, H satisfy some structure conditions.

This kind of result is already known in the context of first-passage percolation, where the correctors are known as Buseman function; see, for example, Licea and Newman [11]. The techniques we use here are strongly inspired by the arguments of Damron and Hanson [7]. There the authors build a type of weak solutions and prove that, when the time function is strictly convex, they are actually genuine Buseman functions.

2. The assumptions and the main result

The underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a Polish space, $\mathcal{F}$ is the Borel σ-field on Ω, and $\mathbb{P}$ is a Borel probability measure. We assume that there exists a one-parameter group $(\tau_x)_{x \in \mathbb{R}^d}$ of measure preserving transformations on Ω, that is $\tau_x : \Omega \to \Omega$ preserves the measure $\mathbb{P}$ for any $x \in \mathbb{R}^d$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for $x, y \in \mathbb{R}^d$. The maps $A : (\mathbb{R}^d \backslash \{0\}) \times \mathbb{R}^d \times \Omega \to \mathbb{S}^{d,+}$, the set of $d \times d$ real symmetric and nonnegative matrices, and $H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ are supposed to be continuous in all variables and stationary, that is, for all $p \in \mathbb{R}^d \backslash \{0\}, x, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$(A, H)(p, x, \tau_z\omega) = (A, H)(p, x + z, \omega).$$

We also remark that the equations below, unless otherwise specified, are understood in the Crandall–Lions viscosity sense.

To avoid any unnecessary assumptions, in what follows we state a general condition, which we call assumption (**H**), on the support of $\mathbb{P}$.

Assumption (**H**)**:** We assume that, for any $p \in \mathbb{R}^d$, the approximate corrector equation

$$\delta v^{\delta,p} - \mathrm{tr}(A(Dv^{\delta,p}+p,x,\omega)D^2 v^{\delta,p}) + H(Dv^{\delta,p}+p,x,\omega) = 0 \quad \text{in} \quad \mathbb{R}^d, \tag{2.1}$$

has a comparison principle, and that, for any $R > 0$, there exists $C_R > 0$ such that, if $|p| \le R$, then the unique solution $v^{\delta,p}$ to (2.1) satisfies

$$\|\delta v^{\delta,p}\|_\infty + \|Dv^{\delta,p}\|_\infty \le C_R.$$

The conditions ensuring the comparison principle are well documented; see, for instance, the Crandall, Ishii, Lions "User's Guide" [6]. Given the comparison principle, it is well known that

$$\|v^{\delta,p}(\cdot,\omega)\|_\infty \le \sup_{x\in\mathbb{R}^d} |H(0,x,\omega)|/\delta,$$

so that the L^∞-assumption on δv^δ is not very restrictive. The Lipschitz bound, however, is more subtle and relies in general on a coercivity condition on the Hamiltonian. Such a structure condition is discussed, in particular, in [14].

Our main result is stated next.

Theorem 2.1. *Assume* (**H**) *and, in addition, suppose that homogenization holds in probability, that is, for any $p \in \mathbb{R}^d$, the family $(\delta v^{\delta,p}(0,\cdot))_{\delta>0}$ converges, as $\delta \to 0$, in probability to some constant $-\overline{H}(p)$, where $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ is a continuous and coercive map. Let $p \in \mathbb{R}^d$ be an extreme point of the convex hull of the sub level-set $\{q \in \mathbb{R}^d : \overline{H}(q) \le \overline{H}(p)\}$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a corrector $\chi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ associated with p and ω, which is a Lipschitz continuous solution to (1.3) satisfying (1.4).*

Some observations and remarks are in order here.

We begin noting that we do not know if the corrector χ has stationary increment, and we do not expect this to be true in general. Note that the corrector is not necessary unique, even up to an additive constant. However, by a measurable selection argument, we can assume that χ depends on ω in a measurable way.

The existence of a corrector yields that, in fact, the $\delta v^{\delta,p}(0,\omega)$'s converge to $-\overline{H}(p)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$; see Proposition 1.2 in [13]. In the rest of the paper, we will use this fact repeatedly. Note also that convexity plays absolutely no role here.

Our result readily applies to the case where (**H**) holds, the law of the pair (A,H) under $\mathbb{P}$ is radially symmetric, and $A = A(p,x,\omega)$ and $H = H(p,x,\omega)$ are homogeneous in p of degrees 0 and 1, respectively; this is stated in Corollary 3.9. Moreover, since $\overline{H}(p) = \bar{c}|p|$ for some positive $\bar{c}$, Theorem 2.1 implies the existence of a corrector for any direction p. Note that this case covers the homogenization of equations of mean curvature type and the result is new. Other known results for such equations are quantitative.

This result also extends to the case where H satisfies, for all $p, x \in \mathbb{R}^d$, $\omega \in \Omega$ and $\lambda \in [0,1]$,

$$0 \le H(\lambda p, x, \omega) \le \lambda H(p,x,\omega).$$

Then there exists a corrector for any direction p such that $\overline{H}(p)$ is positive. Indeed, following Corollary 3.9, homogenization holds in probability for any direction p and $\overline{H}(p) = \bar{c}(|p|)$ for some map $\bar{c}$ which is increasing when positive.

If H is convex in p and A is independent of p, our proof implies that, for any $p \in \mathbb{R}^d$, the limit $\lim_{\delta\to 9} \delta v^{\delta,p}(0,\cdot)$ exists in probability; see Proposition 3.10. This result and its the proof are very much in the favor of [15].

Finally, we note that our arguments also yield the existence of a corrector in some directions and, thus, homogenization, for nonconvex Hamiltonians and p-dependent A. More precisely, for any direction p, there exists a constant $\bar{c}$ such that p belongs to the convex hull of directions p' for which a corrector exists with an associated homogenized constant equal to $\bar{c}$; see Corollary 3.8.

3. The Proof of Theorem 2.1

Throughout the section, we assume that condition (**H**) is satisfied, but do not suppose that homogenization holds (even in probability): this condition is added only at the very end of the section.

Fix $R > 0$, let C_R be as in (**H**) and define the metric space

$$\Theta := \left\{\theta \in C^{0,1}(\mathbb{R}^d) : \theta(0) = 0 \text{ and } \|D\theta\|_\infty \le C_R\right\}$$

with distance, for all $\theta_1, \theta_2 \in \Theta$,

$$d(\theta_1,\theta_2) := \sup_{x\in\mathbb{R}^d} \frac{|\theta_1(x) - \theta_2(x)|}{1+|x|^2}.$$

It is immediate that Θ is a compact.

Next we enlarge the probability space to $\widetilde{\Omega} := \Omega \times \Theta \times [-C_R, C_R]$, which is endowed with the one-parameter group of transformations $\widetilde{\tau}_x : \widetilde{\Omega} \to \widetilde{\Omega}$ defined, for $x \in \mathbb{R}^d$, by

$$\widetilde{\tau}_x(\omega, \theta, s) = (\tau_x \omega, \theta(\cdot + x) - \theta(x), s);$$

below, by an abuse of notation, we write $\widetilde{\tau}_x(\theta) = \theta(\cdot + x) - \theta(x)$.

Fix $p \in \mathbb{R}^d$ with $|p| \le R$, let $v^{\delta,p}$ be the solution to (2.1), define the map $\Phi_{\delta,p} : \Omega \to \widetilde{\Omega}$ by

$$\Phi_{\delta,p}(\omega) = (\omega, v^{\delta,p}(\cdot, \omega) - v^{\delta,p}(0, \omega), -\delta v^{\delta,p}(0, \omega)),$$

which is clearly measurable, and consider the push-forward measure

$$\mu_{\delta,p} = \Phi_{\delta,p} \sharp \mathbb{P},$$

which is a Borel probability measure on $\widetilde{\Omega}$.

Note that, since the first marginal of $\mu_{\delta,p}$ is $\mathbb{P}$ and Ω is a Polish space while $\Theta \times [-C_R, C_R]$ is compact, the family of measures $(\mu_{\delta,p})_{\delta>0}$ is tight.

Let μ be a limit, up to a subsequence $\delta_n \to 0$, of the $\mu_{\delta_n,p}$'s.

Lemma 3.1. *For each $x \in \mathbb{R}^d$, the transformation $\widetilde{\tau}_x$ preserves the measure μ.*

Proof. Fix a continuous and bounded map $\xi : \widetilde{\Omega} \to \mathbb{R}$. Since the map $\widetilde{\omega} \to \xi(\tau_x(\widetilde{\omega}))$ is continuous and bounded and $\mu_{\delta_n,p}$ converges weakly to μ, we have

$$\int_{\widetilde{\Omega}} \xi(\widetilde{\omega}) \tau_x \sharp \mu(d\widetilde{\omega}) = \int_{\widetilde{\Omega}} \xi(\tau_x(\widetilde{\omega})) \mu(d\widetilde{\omega}) = \lim_n \int_{\widetilde{\Omega}} \xi(\tau_x(\widetilde{\omega})) \mu_{\delta_n,p}(d\widetilde{\omega}).$$

In view of the definition of $\widetilde{\tau}$ and μ_{δ_n}, we get

$$\begin{aligned} \int_{\widetilde{\Omega}} \xi(\tau_x(\widetilde{\omega})) \mu_{\delta_n,p}(d\widetilde{\omega}) &= \int_{\Omega} \xi(\tau_x \omega, v^{\delta_n,p}(x + \cdot, \omega) - v^{\delta_n,p}(x, \omega), -\delta_n v^{\delta_n,p}(0, \omega))\, d\mathbb{P}(\omega) \\ &= \int_{\Omega} \xi(\tau_x \omega, v^{\delta_n,p}(\cdot, \tau_x \omega) - v^{\delta_n,p}(0, \tau_x \omega), -\delta_n v^{\delta_n,p}(-x, \tau_x \omega))\, d\mathbb{P}(\omega) \\ &= \int_{\Omega} \xi(\omega, v^{\delta_n,p}(\cdot, \omega) - v^{\delta_n,p}(0, \omega), -\delta_n v^{\delta_n,p}(-x, \omega))\, d\mathbb{P}(\omega), \end{aligned}$$

the last line being a consequence of the stationarity of $\mathbb{P}$.

Using that $v^{\delta_n,p}$ is Lipschitz continuous uniformly in δ and ξ is continuous on the set $\widetilde{\Omega}$, we find

$$\begin{aligned} \int_{\widetilde{\Omega}} \xi(\tau_x(\widetilde{\omega})) \mu_{\delta_n,p}(d\widetilde{\omega}) &= \int_{\Omega} \xi(\omega, v^{\delta_n,p}(\cdot, \omega) - v^{\delta_n,p}(0, \omega), -\delta_n v^{\delta_n,p}(0, \omega) + O(\delta_n))\, d\mathbb{P}(\omega) \\ &= \int_{\widetilde{\Omega}} \xi(\widetilde{\omega})\, d\mu_{\delta_n,p}(\widetilde{\omega}) + o(1). \end{aligned}$$

Letting $n \to +\infty$, we finally get

$$\int_{\widetilde{\Omega}} \xi(\widetilde{\omega}) \tau_x \sharp \mu(d\widetilde{\omega}) = \int_{\widetilde{\Omega}} \xi(\widetilde{\omega})\, d\mu(\widetilde{\omega}),$$

and, hence, the claim. □

The next lemma asserts that there exists some $\overline{c} = \overline{c}(p)$ such that the restriction of μ to the last component is just a Dirac mass. If we know that homogenization holds, then $\overline{c}(p)$ is of course nothing but $\overline{H}(p)$. Note that in what follows, abusing once again the notation, we denote by μ the restriction of μ to the first two components $\Omega \times \Theta$.

Lemma 3.2. *There exits a constant $\overline{c} = \overline{c}(p, (\delta_n)_{n\in\mathbb{N}})$ such that, for any Borel measurable set $E \subset \Omega \times \Theta$,*

$$\mu(E \times [-M_p, M_p]) = \mu(E \times \{\overline{c}\}).$$

In particular, the sequence $(\delta_n v^{\delta_n,p}(0, \cdot))_{n\in\mathbb{N}}$ converges in probability to $-\overline{c}$.

Proof. Let $n \geq 1$ large and $k \in \{0, \dots, 2n\}$, set $t_k := -M_p + M_p k/n$ and

$$E_k := \{\omega \in \Omega : \exists \theta \in \Theta \text{ and } \exists s \in [t_k, t_{k+1}] \text{ such that } (\omega, \theta, s) \in \text{sppt}(\mu)\}.$$

Since the first marginal of μ is $\mathbb{P}$ and $\bigcup_{l=0}^{2n} E_l \times \Theta \times [-M_p, M_p] \supset \text{sppt}(\mu)$, there exists $k \in \{0, \dots, 2n\}$ such that $\mathbb{P}(E_k) > 0$.

It turns out that E_k is translation invariant, that is, for each $x \in \mathbb{R}^d$, $\tau_x E_k = E_k$. Indeed, if $\omega \in \tau_x E_k$, there exists $\theta \in \Theta$ and $s \in [t_k, t_{k+1}]$ such that $(\tau_{-x}\omega, \theta, s) \in \text{sppt}(\mu)$ and, hence, $\widetilde{\tau}_{-x}(\omega, \theta(\cdot + x) - \theta(x), s)$ belongs to $\text{sppt}(\mu)$. Since μ is invariant under $\widetilde{\tau}_x$, so is its support. Hence $(\omega, \theta(\cdot + x) - \theta(x), s) \in \text{sppt}(\mu)$ and ω belongs to E_k. The opposite implication follows in the same way.

The ergodicity of $\mathbb{P}$ yields that $\mathbb{P}[E_k = 1]$, which means that μ is concentrated in some $E_k \times \Theta \times [t_k, t_{k+1}]$. Thus μ is also concentrated on $\Omega \times \Theta \times [t_k, t_{k+1}]$. Letting $n \to +\infty$ implies that there exists $\bar{c} \in [-M_p, M_p]$ such that μ is concentrated of the set $\Omega \times \Theta \times \{\bar{c}\}$.

It remains to check that $(\delta_n v^{\delta_n, p}(0, \cdot))_{n \in \mathbb{N}}$ converges in probability to $-\bar{c}$. This is a consequence of the classical Porte-Manteau Theorem, since, for any $\varepsilon > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}[|\delta_n v^{\delta_n,p}(0,\cdot) + \bar{c}| \geq \varepsilon] &= \limsup \mu_{\delta_n,p}[\Omega \times \Theta \times ([-M_p, M_p] \backslash (\bar{c} - \varepsilon, \bar{c} + \varepsilon))] \\ &\leq \mu[\Omega \times \Theta \times ([-M_p, M_p] \backslash (\bar{c} - \varepsilon, \bar{c} + \varepsilon))] = 0. \quad \square\end{aligned}$$

The next lemma is the first step in finding a corrector and possibly identifying $\bar{c}$ and $\overline{H}(p)$, when the latter exists.

Lemma 3.3. *Let $\bar{c}$ be defined by Lemma 3.2. For μ-a.e. $(\omega, \theta) \in \Omega \times \Theta$, θ is a solution to*

$$-\text{tr}(A(D\theta + p, x, \omega)D^2\theta) + H(D\theta + p, x, \omega) = \bar{c} \quad \text{in} \quad \mathbb{R}^d. \tag{3.1}$$

Proof. Fix $R, \varepsilon > 0$ and let $E(R, \varepsilon)$ be the set of $(\omega, \theta) \in \Omega \times \Theta$ such that θ such that, in the open ball $B_R(0)$,

$$-\text{tr}(A(D\theta + p, x, \omega)D^2\theta) + H(D\theta + p, x, \omega) \geq \bar{c} - \varepsilon$$

and

$$-\text{tr}(A(D\theta + p, x, \omega)D^2\theta) + H(D\theta + p, x, \omega) \leq \bar{c} + \varepsilon.$$

Recall that Lemma 3.2 gives that $(\delta_n v^{\delta_n,p}(0, \cdot))_{n\in\mathbb{N}}$ converge in probability to $-\bar{c}$. Since $v^{\delta_n,p}$ solves (2.1) and is uniformly Lipschitz continuous, it follows that, as $n \to \infty$, $\mu_{\delta_n,p}(E(R, \varepsilon)) \to 1$.

Finally observing that $E(R, \varepsilon)$ is closed in $\Omega \times \Theta$, we infer, using again the Porte-Manteau Theorem, that $\mu(E(R, \varepsilon)) = 1$. As R and ε are arbitrary, we conclude that the set (ω, θ), for which the fact that the equation is satisfied in the viscosity sense is of full probability. $\square$

Next we investigate some properties of θ.

Lemma 3.4. *For any $x \in \mathbb{R}^d$, $\mathbb{E}_\mu[\theta(x)] = 0$.*

Proof. Since the map $(\omega, \theta) \to \theta(x)$ is continuous on $\Omega \times \Theta$ and $v^{\delta_n,p}$ is stationary, we have

$$\mathbb{E}_\mu[\theta(x)] = \lim \mathbb{E}_{\mu_{\delta_n,p}}[\theta(x)] = \lim \mathbb{E}_\mathbb{P}\left[v^{\delta_n,p}(x) - v^{\delta_n,p}(0)\right] = 0. \quad \square$$

Lemma 3.5. *For μ-a.e. $\widetilde{\omega} = (\omega, \theta)$ and any direction $q \in \mathbb{Q}^d$, the (random) limit*

$$\rho_{\widetilde{\omega}}(q) := \lim_{t\to\infty} \frac{\theta(tq)}{t}$$

exists. Moreover, $\rho_{\widetilde{\omega}}(q)$ is invariant under $\widetilde{\tau}_x$ for $x \in \mathbb{R}^d$, that is,

$$\rho_{\widetilde{\tau}_x(\widetilde{\omega})}(q) = \rho_{\widetilde{\omega}}(q) \qquad \mu - a.e.$$

Proof. We first show that, for any $r > 0$, the limit

$$\lim_{t\to+\infty} \frac{1}{t}\left(\int\limits_{B_r(0)} \theta(tq + y)dy - \int\limits_{B_r(0)} \theta(y)dy\right)$$

exists $\mathbb{P}$-a.s.

Since the uniform converge of uniformly Lipschitz continuous maps implies the L^∞-weak $\star$ convergence of their gradients, the map $\xi : \Omega \times \Theta \to \mathbb{R}$ defined by

$$\xi((\omega,\theta)) := \int_{B_r(0)} D\theta(y)\cdot q \,\mathrm{d}y$$

is continuous and bounded on $\Omega \times \Theta$.

Moreover,

$$\frac{1}{t}\left(\int_{B_r(0)} \theta(tq+y)\mathrm{d}y - \int_{B_r(0)} \theta(y)\mathrm{d}y\right) = \frac{1}{t}\int_0^t \int_{B_r(0)} D\theta(sq+y)\cdot q\,\mathrm{d}y\,\mathrm{d}s = \frac{1}{t}\int_0^t \xi(\widetilde{\tau}_{sq}(\widetilde{\omega}))\mathrm{d}s.$$

It follows from the ergodic theorem that the above expression has, as $t\to\infty$ and μ-a.s. a limit $\rho_{\widetilde{\omega}}(q,r)$.

Choosing $r=1/n$ and letting $n\to+\infty$, we also find that, as $t\to+\infty$, $\theta(tq)/t$ has μ-a.s., a limit $\rho_{\widetilde{\omega}}(q) = \lim_{n\to infty}\rho_{\widetilde{\omega}}(q,1/n)$ because θ is C_R-Lipschitz continuous.

Fix $x\in\mathbb{R}^d$ and $\widetilde{\omega}\in\widetilde{\Omega}$ for which $\rho_{\widetilde{\omega}}(q)$ and $\rho_{\widetilde{\tau}_x(\widetilde{\omega})}(q)$ are well defined; recall that this holds for μ-a.e. $\widetilde{\omega}$.

Then, in view of the Lipschitz continuity of θ, we have

$$\rho_{\widetilde{\tau}_x(\widetilde{\omega})}(q) = \lim_{t\to+\infty}\frac{\widetilde{\tau}_x(\theta)(tq)}{t} = \lim_{t\to+\infty}\frac{1}{t}(\theta(x+tq)-\theta(x)) = \rho_{\widetilde{\omega}}(q). \quad \square$$

Lemma 3.6. *There exists a random vector $\mathbf{r}\in L^\infty_\mu(\widetilde{\Omega};\mathbb{R}^d)$ such that, μ-a.s. and for any direction $v\in\mathbb{R}^d$,*

$$\lim_{t\to+\infty}\frac{\theta(tv)}{t} = \mathbf{r}_{\widetilde{\omega}}\cdot v.$$

Proof. Since θ is C_R-Lipschitz continuous, it is enough to check that the map $q\to\rho_{\widetilde{\omega}}(q)$ is linear on $\mathbb{Q}^d$ for μ-a.e. $\widetilde{\omega}$.

Let $\widetilde{\Omega}_0$ be a set of μ-full probability in Ω such that the limit $\rho_{\widetilde{\omega}}(q)$ in Lemma 3.5 exists for any $q\in\mathbb{Q}^d$.

Restricting further the set Ω_0 if necessary; we may also assume (see, for instance, the proof of Lemma 4.1 in [1]) that, for any $\eta, M>0$ and $\widetilde{\omega}=(\omega,\theta)\in\Omega_0$, there exists $T>0$ such that, for all $q\in\mathbb{Q}^d$ with $|q|\le M$, all $x\in\mathbb{R}^d$ and $t\ge T$,

$$\left|\frac{\theta(x+tq)-\theta(x)}{t} - \rho_{\widetilde{\omega}}(q)\right| \le \eta(|x|+1).$$

Fix $\eta, M>0$, $q_1,q_2\in\mathbb{Q}^d$ with $|q_1|,|q_2|\le M$, $\widetilde{\omega}\in\Omega_0$ and $\eta>0$, and let T be associated with η, M as above. Then, for any $t\ge T$, we have

$$\theta(t(q_1+q_2)) = \theta(t(q_1+q_2)) - \theta(tq_2) + \theta(tq_2).$$

Thus

$$\begin{aligned}
&\left|\frac{\theta(t(q_1+q_2))}{t} - \rho_{\widetilde{\omega}}(q_1) - \rho_{\widetilde{\omega}}(q_2)\right| \\
&\quad \le \left|\frac{\theta(t(q_1+q_2))-\theta(tq_2)}{t} - \rho_{\widetilde{\omega}}(q_1)\right| + \left|\frac{\theta(tq_2)}{t} - \rho_{\widetilde{\omega}}(q_2)\right| \\
&\quad \le \eta(|q_2|+t^{-1}) + \eta
\end{aligned}$$

Letting $t\to+\infty$ and $\eta\to 0$ yields the claim, since η and M are arbitrary. $\square$

Lemma 3.7. *Let $\mathbf{r}$ be defined as in Lemma 3.6. Then $\mathbb{E}_\mu[\mathbf{r}]=0$.*

Proof. Lemma 3.4 yields that, for any $v\in\mathbb{R}^d$,

$$0 = \lim_{t\to+\infty}\mathbb{E}_\mu\left[\frac{\theta(tv)}{t}\right] = \mathbb{E}_\mu\left[\lim_{t\to+\infty}\frac{\theta(tv)}{t}\right] = \mathbb{E}_\mu[\mathbf{r}\cdot v] = \mathbb{E}_\mu[\mathbf{r}]\cdot v. \quad \square$$

As a straightforward consequence of the previous results, we have the existence of a corrector and, hence, homogenization for at least one vector p'.

Corollary 3.8. *For μ-a.e. $\widetilde{\omega}=(\omega,\theta,\overline{c})$, $\lim_{\delta\to 0}\delta v^{\delta,p'}(0,\omega)$ exists for $p':=p+\mathbf{r}_{\widetilde{\omega}}$ and is given by $\overline{c}$. Moreover, $\theta'(x):=\theta(x)-\mathbf{r}_{\widetilde{\omega}}\cdot x$ is a corrector for p', in the sense that*

$$-\mathrm{tr}(A(D\theta'+p',x,\omega)D^2\theta')+H(D\theta'+p',x,\omega)=\overline{c} \quad \text{in} \quad \mathbb{R}^d \ \text{with} \ \lim_{|x|\to+\infty}\theta'(x)/|x|=0.$$

Another consequence of the above results is that homogenization holds if the law of (A,H) under $\mathbb{P}$ is a radially symmetric. By this we mean that, for any rotation matrix R, the law of (A,H) is the same as the law of the pair $(\widetilde{A},\widetilde{H})$ given by

$$(\widetilde{A},\widetilde{H})(p,x,\omega):=(R^T AR,H)(Rp,Rx,\omega).$$

Note that this implies that $v^{\delta,Rp}(0,\cdot)$ has the same law as $v^{\delta,p}(0,\cdot)$.

Corollary 3.9. *Assume that, $\mathbb{P}$-a.s., $A=A(p,x,\omega)$ is 0-homogeneous in p, H satisfies, for all $\lambda\in[0,1]$,*

$$0\le H(\lambda p,x,\omega)\le \lambda H(p,x,\omega) \tag{3.2}$$

and suppose that the law of (A,H) under $\mathbb{P}$ is radially symmetric. Then homogenization holds in probability, that is, for any $p\in\mathbb{R}^d$, $\lim_{\delta\to 0}-\delta v^{\delta,p}(0,\cdot)=\overline{c}(|p|)$ in probability. Moreover, the map $s\to\overline{c}(s)$ satisfies, for any $0<s_1<s_2$,

$$0\le \overline{c}(s_1)/s_1\le \overline{c}(s_2)/s_2.$$

Note that the map $\overline{c}$ is increasing as soon as it is positive. Moreover, one easily checks that, if, in addition, H is 1-homogeneous in p and coercive, then $\overline{c}(s)=\overline{c}s$ for some positive constant $\overline{c}$.

Proof. It follows from the assumed bounds and the stationarity, that there exists a set Ω_0 with $\mathbb{P}[\Omega_0]=1$ such that, for any $p\in\mathbb{R}^d$ and $\omega\in\Omega_0$, $c^+(p):=\limsup_{\delta\to 0}-\delta v^{\delta,p}(0,\omega)$ and $c^-(p):=\liminf_{\delta\to 0}-\delta v^{\delta,p}(0,\omega)$ exist and are deterministic. The radial symmetry assumption and as well as (3.9) imply that $c^\pm(p)=c^\pm(|p|)$ and, in addition, for all $\lambda\in[0,1]$,

$$0\le c^\pm(\lambda s)\le\lambda c^\pm(s).$$

Also note that the maps $s\to c^\pm(s)$ are nondecreasing. Indeed given $0<s_1<s_2$, choosing $s=s_2$ and $\lambda=s_1/s_2$, we find

$$c^\pm(s_1)/s_1\le c^\pm(s_2)/s_2\le c^\pm(s_2)/s_1.$$

It follows that $c^\pm$ is increasing as soon as it is positive.

On the other hand, for any $p\in\mathbb{R}^d$, we can find a subsequence of $\mu_{\delta,p}$ converging to some measure μ_p as δ tends to 0. By a diagonal argument, we can assume that this is the same subsequence for any $p\in\mathbb{Q}^d$.

Let $\overline{c}(p)$ be associated with the limit measure μ_p as above. It follows from (**H**) that the map $p\to\overline{c}(p)$ is uniformly continuous and thus can be continuously extended to $\mathbb{R}^d$. Moreover, the assumed radial symmetry yields that $\overline{c}(p)=\overline{c}(|p|)$. Finally, note that, for all $s\ge 0$,

$$0\le c^-(s)\le\overline{c}(s)\le c^+(s). \tag{3.3}$$

Let $\sigma:=\inf\{s\ge 0,\ c^+(s)>0\}$. Then (3.3) implies $c^-=c^+=\overline{c}=0$ on $[0,\sigma]$.

Fix $p\in\mathbb{Q}^d$ with $|p|>\sigma$, and let μ_p, $\overline{c}(|p|)$ and $\mathbf{r}$ be associated with p. For μ_p-a.e. $\widetilde{\omega}=(\omega,\theta)$ with $\omega\in\Omega_0$, $\theta'(x):=\theta(x)-\mathbf{r}_{\widetilde{\omega}}\cdot x$ is a corrector for $p':=p+\mathbf{r}_{\widetilde{\omega}}$ with associated ergodic constant $\overline{c}(|p|)$. It follows that $\lim_{\delta\to 0}(-\delta v^{\delta,p'}(0,\omega))=\overline{c}(|p|)$, and, hence,

$$c^+(|p|)\ge\overline{c}(|p|)=c^+(|p+\mathbf{r}|) \qquad \mu_p-\text{a.s.}$$

Since c^+ is increasing on $(\sigma,+\infty)$, this inequality implies that $|p+\mathbf{r}|=|p|$ a.s. Then $\mathbb{E}[p+\mathbf{r}]=p$ gives $\mathbf{r}=0$ μ_p-a.s., so that, for μ_p-a.e. (θ,ω), θ is a corrector for p with associated ergodic constant $\overline{c}(|p|)$. It follows that $\lim_{\delta\to 0}(-\delta v^{\delta,p}(0,\omega))=\overline{c}(|p|)$. In conclusion, $c^+(|p|)=c^-(|p|)$ for any $p\in\mathbb{Q}^d$, and thus, by continuity, for any $p\in\mathbb{R}^d$, which, in turn, proves that homogenization holds. □

Another application of the previous results is the convergence in law of the random variable $\delta v^{\delta,p}(0,\cdot)$ when H is convex in the gradient variable. The argument is a variant of [15]. Of course, the result is much weaker than the a.s. convergence is established in [14]; see also [1,2]. The proof is, however, rather simple.

Proposition 3.10. *Assume that, $\mathbb{P}$-a.e., $H=H(p,x,\omega)$ is convex in the p variable and that $A=A(x,\omega)$ does not depend on p. Then, for any $p\in\mathbb{R}^d$, homogenization holds in probability, that is there exists $\overline{H}(p)$ such that $\lim_{\delta\to 0}\delta v^{\delta}(0,\cdot)=-\overline{H}(p)$ in probability.*

Proof. Let μ be a measure built as in the beginning of the section. It follows that there exists a random family of measures μ_ω on Θ such that, for any continuous map $\phi : \Omega \times \Theta \to \mathbb{R}$, one has

$$\int_{\Omega\times\Theta} \phi(\omega,\theta)\,\mathrm{d}\mu(\omega,\theta) = \int_{\Omega} \left[\int_{\Theta} \phi(\omega,\theta)\,\mathrm{d}\mu_\omega(\theta) \right] \mathrm{d}\mathbb{P}(\omega).$$

Set $\widehat{\theta}(x,\omega) := \int_\Theta \theta(x)\,\mathrm{d}\mu_\omega(\theta)$. Since $\mathbb{P}$ and μ are invariant with respect to $(\tau_z)_{z\in\mathbb{R}^d}$ and $(\widetilde{\tau}_z)_{z\in\mathbb{R}^d}$ respectively, for any bounded measurable map $\phi = \phi(\omega)$ and any $z \in \mathbb{R}^d$, we have

$$\begin{aligned}\int_{\Omega} \phi(\omega)(\widehat{\theta}(x+z,\omega) - \widehat{\theta}(z))\,\mathrm{d}\mathbb{P}(\omega) &= \int_{\Omega\times\Theta} \phi(\omega)(\theta(x+z) - \theta(z))\,\mathrm{d}\mu(\omega,\theta) \\ &= \int_{\Omega\times\Theta} \phi(\tau_{-z}\omega)\theta(x)\widetilde{\tau}_z\sharp\mathrm{d}\mu(\omega,\theta) = \int_{\Omega} \phi(\tau_{-z}\omega)\widehat{\theta}(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\Omega} \phi(\omega)\widehat{\theta}(x,\tau_z\omega)\,\mathrm{d}\mathbb{P}(\omega).\end{aligned}$$

This shows that $\widehat{\theta}$ has stationary increments. Moreover, in view of Lemma 3.4, $\widehat{\theta}$ has mean zero, and, hence, $D\widehat{\theta}$ is stationary with average 0. In particular, $\widehat{\theta}$ is $\mathbb{P}$-a.s. strictly sublinear at infinity. Since, for μ-a.e. (ω,θ), θ is a solution to (3.3) and H is convex in the gradient variable, $\widehat{\theta}$ is a subsolution to (3.3) and, thus a subcorrector. Following [15], this implies that

$$\liminf_{\delta\to 0} \delta v^{\delta,p}(0,\omega) \geq -\overline{c}.$$

In particular, for any sequence $(\delta'_n)_{n'\in\mathbb{N}}$ that tends to 0 such that $(\mu_{\delta'_n,p})_{n'\in\mathbb{N}}$ and $(\delta'_n v^{\delta'_n,p}(0))_{n'\in\mathbb{N}}$ converge respectively to a measure μ' and a constant $-\overline{c}'$, we have $\overline{c}' \leq \overline{c}$. Exchanging the roles of $(\delta_n)_{n\in\mathbb{N}}$ and $(\delta'_n)_{n'\in\mathbb{N}}$ leads to the equality $\overline{c} = \overline{c}'$. The conclusion now follows. □

We are now ready to prove the main result.

Proof of Theorem 2.1. We assume that homogenization holds in probability and $p \in \mathbb{R}^d$ is an extreme point of the convex hull of the set $S := \{q \in \mathbb{R}^d : \overline{H}(q) \leq \overline{H}(p)\}$.

Let μ be a measure built as in the beginning of the section and $\mathbf{r}$ be defined by Lemma 3.6.

Then $\overline{H}(p+\mathbf{r}) = \overline{H}(p)$ μ-a.s., that is $p+\mathbf{r}$ belongs to S μ-a.s. Indeed Lemma 3.3 gives $\overline{c} = \overline{H}(p)$ and

$$-\mathrm{tr}(A(D\theta + p, x, \omega)D^2\theta) + H(D\theta + p, x, \omega) = \overline{c} \;\text{ in }\; \mathbb{R}^d,$$

while, in view of Lemma 3.6, for all $x \in \mathbb{R}^d$,

$$\lim_{t\to+\infty} \frac{\theta(tx)}{t} = \mathbf{r}\cdot x.$$

Thus $\widetilde{\theta}(x) := \theta(x) - \mathbf{r}\cdot x$ is a corrector for $p+\mathbf{r}$, that is it satisfies

$$-\mathrm{tr}(A(D\widetilde{\theta} + p + \mathbf{r}, x, \omega)D^2\widetilde{\theta}) + H(D\widetilde{\theta} + p + \mathbf{r}, x, \omega) = \overline{c} \quad\text{in}\;\; \mathbb{R}^d \;\text{ and }\; \lim_{|x|\to+\infty} \widetilde{\theta}(x)/|x| = 0.$$

It follows that $\overline{H}(p+\mathbf{r}) = \overline{H}(p)$ μ-a.s.

Next we recall (Lemma 3.7) that $\mathbb{E}_\mu[p+\mathbf{r}] = p$. Since $p+\mathbf{r} \in S$ μ-a.s. and p is an extreme point of the convex hull of S, the equality $\mathbb{E}_\mu[p+\mathbf{r}] = p$ implies that $\mathbf{r} = 0$ μ-a.s. Therefore $\lim_{|x|\to+\infty} \theta(x)/|x| = 0$ μ-a.s., which, together with the fact that θ solves the corrector equation for p, implies that θ is a corrector for p itself. □

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