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Remarks on the canonical metrics on the Cartan–Hartogs domains

Remarques sur les métriques canoniques des domaines de Cartan–Hartogs

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A R T I C L E I N F O

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A B S T R A C T

The Cartan–Hartogs domains are defined as a class of Hartogs-type domains over irreducible bounded symmetric domains. For a Cartan–Hartogs domain $\Omega^B(\mu)$ endowed with the natural Kähler metric $g(\mu)$, Zedda conjectured that the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$ if and only if $(\Omega^B(\mu), g(\mu))$ is biholomorphically isometric to the complex hyperbolic space. In this paper, following Zedda's argument, we give a geometric proof of the Zedda's conjecture by computing the curvature tensors of the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$.

R É S U M É

Les domaines de Cartan–Hartogs sont définis comme une classe de domaines de type Hartogs sur les domaines symétriques bornés irréductibles. Pour un domaine de Cartan–Hartogs $\Omega^B(\mu)$ muni de sa métrique de Kähler naturelle $g(\mu)$, Zedda a conjecturé que le coefficient a_2 du développement de la fonction ε de Rawnsley relative au domaine de Cartan–Hartogs $(\Omega^B(\mu), g(\mu))$ est constant sur $\Omega^B(\mu)$ si et seulement si $(\Omega^B(\mu), g(\mu))$ est biholomorphiquement isométrique à l'espace hyperbolique complexe. Dans cet article, en nous appuyant sur ses arguments, nous donnons une preuve géométrique de la conjecture de Zedda en calculant les tenseurs de courbure du domaine de Cartan–Hartogs $(\Omega^B(\mu), g(\mu))$.

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1. Introduction

The expansion of the Bergman kernel has received a lot of attention recently, due to the influential work of Donaldson, see, e.g., [5], about the existence and uniqueness of constant scalar curvature Kähler metrics (cscK metrics). Donaldson used the asymptotics of the Bergman kernel proved by Catlin [4] and Zelditch [23] and the calculation of Lu [12] of the first coefficient in the expansion to give conditions for the existence of cscK metrics.

Assume that D is a bounded domain in $\mathbb{C}^n$ and φ is a strictly plurisubharmonic function on D. Let g be a Kähler metric on D associated with the Kähler form $\omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\varphi$. For $\alpha > 0$, let $\mathcal{H}_\alpha$ be the weighted Hilbert space of square integrable holomorphic functions on (D, g) with the weight $\exp\{-\alpha\varphi\}$, that is

$$\mathcal{H}_\alpha := \left\{ f \in \mathrm{Hol}(D) \;\middle|\; \int_D |f|^2 \exp\{-\alpha\varphi\}\frac{\omega^n}{n!} < +\infty \right\},$$

where $\mathrm{Hol}(D)$ denotes the space of holomorphic functions on D. Let K_α be the Bergman kernel (namely, the reproducing kernel) of $\mathcal{H}_\alpha$ if $\mathcal{H}_\alpha \neq \{0\}$. The Rawnsley's ε-function on D (see Cahen–Gutt–Rawnsley [2] and Rawnsley [17]) associated with that the metric g is defined by

$$\varepsilon_\alpha(z) := \exp\{-\alpha\varphi(z)\}K_\alpha(z, \overline{z}), \;\; z \in D.$$

Note the Rawnsley's ε-function depends only on the metric g and not on the choice of the Kähler potential φ (which is defined up to an addition with the real part of a holomorphic function on D). The asymptotics of the Rawnsley's ε-function ε_α was expressed in terms of the parameter α for compact manifolds by Catlin [4] and Zelditch [23] (for $\alpha \in \mathbb{N}$) and for non-compact manifolds by Ma–Marinescu [13,14]. In some particular cases, it was also proved by Engliš [6,7].

The Cartan–Hartogs domains are defined as a class of Hartogs-type domains over irreducible bounded symmetric domains. Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ of genus γ. The generic norm of Ω is defined by

$$N(z, \xi) := (V(\Omega)K(z, \xi))^{-1/\gamma},$$

where $V(\Omega)$ is the total volume of Ω with respect to the Euclidean measure of $\mathbb{C}^d$ and $K(z, \xi)$ is its Bergman kernel. Thus $0 < N_\Omega(z, \overline{z}) \le 1$ for all $z \in \Omega$ and $N_\Omega(0, 0) = 1$. For an irreducible bounded symmetric domain Ω in $\mathbb{C}^d$ and a positive real number μ, the Cartan–Hartogs domain $\Omega^B(\mu)$ is defined by

$$\Omega^B(\mu) := \left\{(z, w) \in \Omega \times \mathbb{C} : \; |w|^2 < N(z, z)^\mu \right\}.$$

For the Cartan–Hartogs domain $\Omega^B(\mu)$, define

$$\Phi(z, w) := -\log(N(z, z)^\mu - |w|^2).$$

The Kähler form $\omega(\mu)$ on $\Omega^B(\mu)$ is defined by

$$\omega(\mu) := \frac{\sqrt{-1}}{2}\partial\overline{\partial}\Phi.$$

We endow the Cartan–Hartogs domain $\Omega^B(\mu)$ with the Kähler metric $g(\mu)$ associated with the Kähler form $w(\mu)$. For the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$, the Rawnsley's ε-function admits the following finite expansion (e.g., see Th. 3.1 in Feng–Tu [8]):

$$\varepsilon_\alpha(z, w) = \sum_{j=0}^{d+1} a_j(z, w)\alpha^{d+1-j}, \;\; (z, w) \in \Omega^B(\mu). \tag{1.1}$$

By Th. 1.1 of Lu [12], Th. 4.1.2 and Th. 6.1.1 of Ma–Marinescu [13], Th. 3.11 of Ma–Marinescu [14] and Th. 0.1 of Ma–Marinescu [15], see also Th. 3.3 of Xu [20], we have

$$\begin{cases} a_0 = 1, \\ a_1 = \dfrac{1}{2}k_{g(\mu)}, \\ a_2 = \dfrac{1}{3}\Delta k_{g(\mu)} + \dfrac{1}{24}|R_{g(\mu)}|^2 - \dfrac{1}{6}|Ric_{g(\mu)}|^2 + \dfrac{1}{8}k^2_{g(\mu)}, \end{cases} \tag{1.2}$$

where $k_{g(\mu)}$, $\triangle$, $R_{g(\mu)}$ and $Ric_{g(\mu)}$ denote the scalar curvature, the Laplace, the curvature tensor and the Ricci curvature associated with the metric $g(\mu)$ on the Cartan–Hartogs domain $\Omega^B(\mu)$, respectively.

Let B^d be the unit ball in $\mathbb{C}^d$ and let the metric g_{hyp} on B^d be given by

$$ds^2 = -\sum_{i,j=1}^{d} \frac{\partial^2 \ln(1-\|z\|^2)}{\partial z_i \partial \overline{z_j}} dz_i \otimes d\overline{z_j}.$$

We denote by (B^d, g_{hyp}) the complex hyperbolic space. Note that $g_{hyp} = \frac{1}{d+1} g_B$ on B^d for the Bergman metric g_B of B^d. When Ω is the unit ball in $\mathbb{C}^d$ and $\mu = 1$, then the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is the complex hyperbolic space in $\mathbb{C}^{d+1}$. With the exception of the complex hyperbolic space which is obviously homogeneous, each Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is a noncompact, nonhomogeneous, complete Kähler manifold. Further, for some particular value μ_0 of μ, $g(\mu_0)$ is a Kähler–Einstein metric (see Yin–Wang [21]).

Recently, Loi–Zedda [11] and Zedda [22] studied the canonical metrics on the Cartan–Hartogs domains. By calculating the scalar curvature $k_{g(\mu)}$, the Laplace $\Delta k_{g(\mu)}$ of $k_{g(\mu)}$, the norm $|R_{g(\mu)}|^2$ of the curvature tensor $R_{g(\mu)}$ and the norm $|Ric_{g(\mu)}|^2$ of the Ricci curvature $Ric_{g(\mu)}$ of a Cartan–Hartogs domain $(\Omega^{B^{d_0}}(\mu), g(\mu))$, Zedda [22] has proved that if the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$, then $(\Omega^B(\mu), g(\mu))$ is Kähler–Einstein. Moreover, Zedda [22] conjectured that the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$ if and only if $(\Omega^B(\mu), g(\mu))$ is biholomorphically isometric to the complex hyperbolic space.

In 2014, Feng–Tu [8] proved this conjecture by giving the explicit expression of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$. The methods in Feng–Tu [8] are very different from the argument in Zedda [22]. In this paper, following the framework of Zedda [22], we give a geometric proof of Zedda's conjecture by computing the curvature tensors of the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$. We will prove the following result:

Theorem 1.1. *Let $(\Omega^B(\mu), g(\mu))$ be a Cartan–Hartogs domain endowed with the canonical metric $g(\mu)$. Then the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$ if and only if $(\Omega^B(\mu), g(\mu))$ is biholomorphically isometric to the complex hyperbolic space.*

Let Ω be the irreducible bounded symmetric domain endowed with its Bergman metric g_B and let R_{g_B} denote the curvature tensor associated with (Ω, g_B). When the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$, we will use the curvature tensor R_{g_B} on Ω to determine $\Omega^B(\mu)$, which means, in this case, that $(\Omega^B(\mu), g(\mu))$ must be biholomorphically isometric to the complex hyperbolic space.

For general references for this paper, see Feng–Tu [8] and Zedda [22]. For the sake of simplicity, similar to Zedda [22], the Cartan–Hartogs domains in this paper will be restricted to the Hartogs-type domains over the irreducible bounded symmetric domains only with one-dimensional fibers.

2. Preliminaries

Let Ω be an irreducible bounded symmetric domain of $\mathbb{C}^d$ of genus γ and let $N(z,z)$ denote the generic norm of Ω. Define

$$g^{\Omega(\mu)} := \frac{\mu}{\gamma} g_B, \tag{2.1}$$

where g_B is the Bergman metric of Ω. Then, from Zedda [22], we have the following results.

Lemma 2.1 *(Zedda [22], Lemma 4). The scalar curvature $k_{g(\mu)}$ of the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is given by*

$$k_{g(\mu)} = \frac{d(\mu(d+1)-\gamma)}{\mu} \frac{N^\mu - |w|^2}{N^\mu} - (d+2)(d+1). \tag{2.2}$$

Lemma 2.2 *(Zedda [22], Lemma 8). The norm with respect to $g(\mu)$ of the curvature tensor $R_{g(\mu)}$ of the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ when evaluated at any point $(0, w) \in \Omega^B(\mu) \subseteq \Omega \times \mathbb{C}$ is given by*

$$[|R_{g(\mu)}|^2]_{z=0} = (1-|w|^2)^2 |R_{g^{\Omega(\mu)}}|^2 - 4|w|^2(1-|w|^2) k_{g^{\Omega(\mu)}} + 2d(d+1)|w|^4 + 4(d+1), \tag{2.3}$$

where $k_{g^{\Omega(\mu)}}$ is the scalar curvature of $(\Omega, g^{\Omega(\mu)})$ and $|R_{g^{\Omega(\mu)}}|$ is the norm with respect to $g^{\Omega(\mu)}$ of the curvature tensor $R_{g^{\Omega(\mu)}}$ of $(\Omega, g^{\Omega(\mu)})$.

Lemma 2.3 *(Zedda [22], (39) and (40)). For the Cartan–Hartogs domain $\Omega^B(\mu)$ endowed with the Kähler metric $g(\mu)$, we have the following identities*

$$[\Delta k_{g(\mu)}]_{z=0} = -\frac{d(\mu(d+1)-\gamma)}{\mu}(1-|w|^2)((d-1)|w|^2+1), \tag{2.4}$$

$$[|Ric_{g(\mu)}|^2]_{z=0} = d(\frac{d(\mu(d+1)-\gamma)}{\mu})^2(1-|w|^2)^2 + \\ -2d(d+2)\frac{d(\mu(d+1)-\gamma)}{\mu}(1-|w|^2) + (d+1)(d+2)^2. \tag{2.5}$$

3. The proof of the main theorem

In this section, we will give the proof of the main theorem. Firstly, by (1.2), we have

$$a_2(z,w) = \frac{1}{3}\Delta k_{g(\mu)} + \frac{1}{24}|R_{g(\mu)}|^2 - \frac{1}{6}|Ric_{g(\mu)}|^2 + \frac{1}{8}k^2_{g(\mu)}. \tag{3.1}$$

For convenience, denote

$$c := \frac{\mu(d+1)-\gamma}{\mu}. \tag{3.2}$$

If $a_2(z,w)$ is a constant, then $a_2(0,w)$ is also a constant on $|w|<1$. Then by (2.2), (2.3), (2.4) and (2.5), after a straightforward computation, we have the following result.

Proposition 3.1. *Let Ω be an irreducible bounded symmetric domain of $\mathbb{C}^d$ of genus γ. Assume that $(\Omega^B(\mu), g(\mu))$ is a Cartan–Hartogs domain endowed with the canonical metric $g(\mu)$. If the coefficient a_2 of the Rawnsley's ε-function expansion for the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$ is constant on $\Omega^B(\mu)$, then we have*

$$\mu = \frac{\gamma}{d+1}, \quad [|R_{g_B}|^2]_{z=0} = \frac{2d}{d+1}, \tag{3.3}$$

where g_B is the Bergman metric of Ω.

Proof. Firstly, by (2.4), we have

$$\frac{1}{3}[\Delta k_{g(\mu)}]_{z=0} = \frac{1}{3}d\cdot c(d-1)|w|^4 - \frac{1}{3}d\cdot c(d-2)|w|^2 - \frac{1}{3}dc. \tag{3.4}$$

Since $|R_{g^{\Omega(\mu)}}|^2 = |R_{g_B}|^2\frac{\gamma^2}{\mu^2}$ by the definition of $g^{\Omega(\mu)}$ (see (2.1)), from (2.3), we get

$$\frac{1}{24}[|R_{g(\mu)}|^2]_{z=0} = \frac{1}{24}[\frac{\gamma^2}{\mu^2}|R_{g_B}|^2 - 4d\frac{\gamma}{\mu} + 2d(d+1)]|w|^4 + \frac{1}{24}[-2\frac{\gamma^2}{\mu^2}|R_{g_B}|^2 + 4d\frac{\gamma}{\mu}]|w|^2 \\ + \frac{1}{24}[\frac{\gamma^2}{\mu^2}|R_{g_B}|^2 + 4(d+1)]. \tag{3.5}$$

Similarly, from (2.5), one rewrite the $-\frac{1}{6}[|Ric_{g(\mu)}|^2]_{z=0}$ in $|w|^4$ and $|w|^2$ as follows

$$-\frac{1}{6}[|Ric_{g(\mu)}|^2]_{z=0} = -\frac{1}{6}d\cdot c^2|w|^4 - \frac{1}{6}[2d(d+2)c - 2dc^2]|w|^2 + \\ -\frac{1}{6}[dc^2 + (d+1)(d+2)^2 - 2d(d+2)c]. \tag{3.6}$$

Lastly, from (2.2), we have

$$\frac{1}{8}[k^2_{g(\mu)}]_{z=0} = \frac{1}{8}d^2c^2|w|^4 + \frac{1}{8}[2d(d+1)(d+2)c - 2d^2c^2]|w|^2 + \frac{1}{8}[d^2c^2 + (d+2)(d+1)^2]. \tag{3.7}$$

Combining (3.1), (3.4), (3.5), (3.6) and (3.7), we have

$$[a_2(z,w)]_{z=0} = c_0|w|^4 + c_1|w|^2 + c_2,$$

where

$$c_0 := \frac{1}{3}dc(d-1) + \frac{1}{24}\frac{\gamma^2}{\mu^2}|R_{g_B}|^2 + \frac{1}{12}d(d+1) - \frac{1}{6}d\frac{\gamma}{\mu} - \frac{1}{6}dc^2 + \frac{1}{8}d^2c^2, \tag{3.8}$$

$$c_1 := -\frac{1}{3}dc(d-2) - \frac{1}{12}\frac{\gamma^2}{\mu^2}|R_{g_B}|^2 + \frac{1}{6}d\frac{\gamma}{\mu} - \frac{1}{3}cd(d+2) + \frac{1}{4}d(d+1)(d+2)c - \frac{1}{4}d^2c^2. \tag{3.9}$$

Since $a_2(z,w)$ is a constant, we get that $a_2(0,w)$ is a constant, and thus $c_0 = c_1 = 0$. Hence, from $2c_0 + c_1 = 0$, we have

$$\frac{2}{3}dc(d-1) + \frac{1}{6}dc = \frac{2}{3}d^2c - \frac{1}{4}dc(d+1)(d+2),$$

in which we use the fact $\frac{\gamma}{\mu} = (d+1) - c$ (see (3.2)). If $c \neq 0$, then we have

$$\frac{2}{3}(d-1) + \frac{1}{6} = \frac{2}{3}d - \frac{1}{4}(d+1)(d+2).$$

Thus $d = 0$ or $d = -3$, which is impossible. Therefore, we have $c = 0$, and furthermore, from (3.2), we have

$$\mu = \frac{\gamma}{d+1}. \tag{3.10}$$

By putting $c = 0$ and $\frac{\gamma}{\mu} = d+1$ into $c_0 = 0$, we get

$$\frac{1}{24}(d+1)^2|R_{g_B}|^2 - \frac{1}{12}d(d+1) = 0.$$

That is

$$[|R_{g_B}|^2]_{z=0} = \frac{2d}{d+1}. \tag{3.11}$$

This proves the proposition. □

Now we will use (3.3) to determine the Cartan–Hartogs domain $(\Omega^B(\mu), g(\mu))$.

Case 1. For $\Omega = D^I_{m,n} := \{z \in M_{m\times n} : I - z\bar{z}^t > 0\}$ $(1 \le m \le n)$, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{2mn(mn+1)}{(m+n)^2}.$$

By (3.11) (note $d = mn$ and $\gamma = n + m$ in this case), we get $(mn+1)\frac{2mn(mn+1)}{(m+n)^2} = 2mn$, which implies $m = 1$ or $n = 1$. So we get $m = 1$. Then $\gamma = n+1$, and by (3.10), $\mu = 1$. Hence the Cartan–Hartogs domain is the complex hyperbolic space.

Case 2. For $\Omega = D^{II}_n := \{z \in M_{n,n} :\ z^t = -z, I - z\bar{z}^t > 0\}$ $(n \ge 4)$, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{n(n+1)(n^2-5n+12) - 16n}{4(n-1)^2}.$$

By (3.11) (note $d = n(n-1)/2$ in this case), we have $n^5 - 5n^4 + 5n^3 + 5n^2 - 6n = 0$, which has no positive integer solution for $n \ge 4$ (note that $n^5 - 5n^4 + 5n^3 + 5n^2 - 6n = n(n-1)(n-2)(n+1)(n-3)$ has no positive integer zero for $n \ge 4$).

Case 3. For $\Omega = D^{III}_n := \{z \in M_{n,n} :\ z^t = z, I - z\bar{z}^t > 0\}$ $(n \ge 2)$, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{n(n+1)(n^2+19n-60) + 96n}{4(n+1)^2}.$$

By (3.11) (note $d = n(n+1)/2$ in this case), we have $n^5 + 21n^4 - 27n^3 + 11n^2 - 70n + 64 = 0$, which has no positive integer solution for $n \ge 2$ (note that $n^5 + 21n^4 - 27n^3 + 11n^2 - 70n + 64 = (n-1)(n^4 + 22n^3 - 5n^2 + 6n - 64)$ has no positive integer zero for $n \ge 2$).

Case 4. For $\Omega = D^{IV}_n := \{z \in \mathbb{C}^n : 1 - 2z\bar{z}^t + |zz^t|^2 > 0, z\bar{z}^t < 1\}$ $(n \ge 5)$, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{3n-2}{n}.$$

By (3.11), we get $n^2 + n - 2 = 0$, which has no positive integer solution for $n \ge 5$.

Cases 5 and 6. For an irreducible bounded symmetric domain Ω, we have that $|[R_{g_B}]_{\alpha\bar{\beta}\upsilon\bar{\delta}}(0)|^2$ is an integer with respect to (Ω, g_B) and

$$[|R_{g_B}|^2]_{z=0} = \frac{1}{\gamma^4}\sum_{\alpha,\beta,\upsilon,\delta} |[R_{g_B}]_{\alpha\bar{\beta}\upsilon\bar{\delta}}(0)|^2.$$

For $\Omega = D^V(16) = E_6/Spin(10) \times T^1$ (in this case, $d = 16$ and $\gamma = 12$), by (3.11), we have

$$[|R_{g_B}|^2]_{z=0} = \frac{32}{17}.$$

So $\frac{32}{17}\gamma^4 = \sum_{\alpha,\beta,\upsilon,\delta} |[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer, which is impossible for $\gamma = 12$.

For $\Omega = D^{VI}(27) = E_7/E_6 \times T^1$ (in this case, $d = 27$ and $\gamma = 18$), by (3.11), we have

$$[|R_{g_B}|^2]_{z=0} = \frac{27}{14}.$$

So $\frac{27}{14}\gamma^4 = \sum_{\alpha,\beta,\upsilon,\delta} |[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer, which is impossible for $\gamma = 18$.

Combining the above results, we get that if a_2 is a constant, then the Cartan–Hartogs domain is the complex hyperbolic space.

Since the complex hyperbolic space is the unit ball equipped with the hyperbolic metric, we have that $a_2(z, w)$ is a constant for the complex hyperbolic space. So we have proved the main theorem.

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Appendix A

For completeness, we will give $[|R_{g_B}|^2]_{z=0}$ for a classical symmetric domain Ω with the Bergman metric g_B and prove $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer with respect to (Ω, g_B) for an irreducible bounded symmetric domain Ω. In fact, they can be found in some standard literatures (e.g., Helgason [9] and Mok [16]).

I. Here, we will give $[|R_{g_B}|^2]_{z=0}$ for a classical symmetric domain Ω with the Bergman metric g_B. By definition (see (13) in [22] for reference), we have

$$|R_{g_B}|^2 = \sum_{\alpha,\beta,\eta,\theta,\zeta,\nu,\xi,\tau} \overline{g_B^{\alpha\bar{\zeta}}}\, g_B^{\beta\bar{\nu}}\, \overline{g_B^{\eta\bar{\xi}}}\, g_B^{\theta\bar{\tau}} R_{\alpha\bar{\beta}\eta\bar{\theta}} \overline{R_{\zeta\bar{\nu}\xi\bar{\tau}}}.$$

The curvature tensor R_{g_B} of (Ω, g_B) at 0 can be found in section 2 in Calabi [3].

Case 1. For $\Omega = D^I_{m,n} := \{z \in M_{m\times n} : I - z\bar{z}^t > 0\}$ (here $\gamma = m + n$, $d = mn$). Furthermore, we can give the following identity

$$\begin{aligned}\log K(z,z) &= \log \frac{1}{V(\Omega)} \det(I - z\bar{z}^t)^{-(n+m)} \\ &= \log \frac{1}{V(\Omega)} + (m+n)\sum_{\alpha,\beta} |z_{\alpha\beta}|^2 + \frac{m+n}{2} \sum_{\alpha,\beta,\upsilon,\lambda} \bar{z}_{\alpha\upsilon} z_{\alpha\lambda} \bar{z}_{\beta\lambda} z_{\beta\upsilon} \\ &\quad + \text{higher-order terms.}\end{aligned}$$

Therefore we get $[g_B]_{\alpha\beta,\overline{\lambda\sigma}}(0) = (m+n)\delta^\alpha_\lambda \delta^\beta_\sigma$. Moreover, we have $[R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}}(0) = -(m+n)(\delta^\alpha_\beta \delta^\lambda_\mu \delta^\upsilon_\tau \delta^\rho_\sigma + \delta^\alpha_\mu \delta^\beta_\lambda \delta^\upsilon_\rho \delta^\sigma_\tau)$. Hence, the following identity is established

$$\begin{aligned}[|R_{g_B}|^2]_{z=0} &= \sum_{\alpha,\beta,\lambda,\mu=1}^{m} \sum_{\upsilon,\rho,\sigma,\tau=1}^{n} \frac{1}{(m+n)^2} (\delta^\alpha_\beta \delta^\lambda_\mu \delta^\upsilon_\tau \delta^\rho_\sigma + \delta^\alpha_\mu \delta^\beta_\lambda \delta^\upsilon_\rho \delta^\sigma_\tau)^2 \\ &= \frac{2mn(mn+1)}{(m+n)^2}.\end{aligned}$$

Case 2. For $\Omega = D^{II}_n := \{z \in M_{n,n}, z^t = -z, I - z\bar{z}^t > 0\}$ $(n \geq 4)$ (here $\gamma = 2(n-1), d = \frac{1}{2}n(n-1)$). Similar to Case 1, we have

$$\begin{aligned}\log K(z,z) &= \log \frac{1}{V(\Omega)} \det(I - z\bar{z}^t)^{-(n-1)} \\ &= -\log V(\Omega) + (n-1)\sum_{\alpha<\beta} 2|z_{\alpha\beta}|^2 + \frac{n-1}{2} \sum_{\alpha,\beta,\upsilon,\lambda} \bar{z}_{\alpha\upsilon} z_{\alpha\lambda} \bar{z}_{\beta\lambda} z_{\beta\upsilon} \\ &\quad + \text{higher-order terms.}\end{aligned}$$

Hence we get $[g_B]_{\alpha\beta,\overline{\lambda\sigma}}(0) = 2(n-1)\delta^\alpha_\lambda\delta^\beta_\sigma(\alpha<\beta,\ \lambda<\sigma)$ and similar to (3.9) in Calabi [3], we have

$$[R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}}(0) = 2(n-1)(-\delta^{\beta\rho}_{\alpha\sigma}\delta^{\mu\tau}_{\lambda\upsilon} - \delta^{\mu\tau}_{\beta\sigma}\delta^{\beta\rho}_{\lambda\upsilon} + \delta^{\beta\rho}_{\alpha\lambda}\delta^{\mu\tau}_{\sigma\upsilon} + \delta^{\mu\tau}_{\alpha\lambda}\delta^{\beta\rho}_{\sigma\upsilon}),$$

where the precise definition of $\delta^{\alpha\beta}_{\rho\sigma}(=\frac{\partial z_{\alpha\beta}}{\partial z_{\rho\sigma}} = \delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)$ can be found in Calabi [3]. Here we must note that $z_{\alpha\beta} = -z_{\beta\alpha}$ and $\rho<\sigma$. Hence, a long computation yields the following result

$$\begin{aligned}[][|R_{g_B}|^2]_{z=0} &= \sum_{\alpha<\upsilon\ \beta<\rho\ \lambda<\sigma\ \mu<\tau} g_B^{\overline{\alpha\upsilon,\overline{\alpha\upsilon}}} g_B^{\beta\rho,\overline{\beta\rho}} g_B^{\overline{\lambda\sigma,\overline{\lambda\sigma}}} g_B^{\mu\tau,\overline{\mu\tau}} [R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}} \overline{[R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}}} \\ &= \frac{n(n+1)(n^2-5n+12)-16n}{4(n-1)^2}.\end{aligned}$$

Therefore, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{n(n+1)(n^2-5n+12)-16n}{4(n-1)^2}.$$

Case 3. For $\Omega = D^{III}_n := \{z\in M_{n,n}, z^t = z, I - z\bar{z}^t > 0\}$ $(n\ge 2)$ (here $\gamma = n+1, d = \frac{1}{2}n(n+1)$). Similarly, $\log K(z,z)$ is given by

$$\begin{aligned}\log K(z,z) &= \log\frac{1}{V(\Omega)}\det(I-z\bar{z}^t)^{-(n+1)} \\ &= -\log V(\Omega) + (n+1)\sum_{\alpha<\beta}2|z_{\alpha\beta}|^2 + (n+1)\sum_{\alpha=\beta}|z_{\alpha\beta}|^2 \\ &\quad + \frac{n+1}{2}\sum_{\alpha,\beta,\lambda,\upsilon}\bar{z}_{\alpha\upsilon}z_{\alpha\lambda}\bar{z}_{\beta\lambda}z_{\beta\upsilon} + \text{higher-order terms.}\end{aligned}$$

Hence we have

$$[g_B]_{\alpha\beta,\overline{\lambda\sigma}}(0) = \begin{cases} 2(n+1)\delta^\alpha_\lambda\delta^\beta_\sigma, & \alpha<\beta,\ \lambda<\sigma, \\ (n+1)\delta^\alpha_\lambda, & \alpha=\beta,\ \lambda=\sigma. \end{cases}$$

Similar to (3.12) in Calabi [3], we have $[R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}}(0)$ equals

$$\begin{cases} -2(n+1)(e^{\beta\rho}_{\alpha\sigma}e^{\mu\tau}_{\lambda\upsilon} + e^{\beta\rho}_{\lambda\upsilon}e^{\mu\tau}_{\alpha\sigma} + e^{\beta\rho}_{\alpha\lambda}e^{\mu\tau}_{\sigma\upsilon} + e^{\beta\rho}_{\sigma\upsilon}e^{\mu\tau}_{\alpha\lambda}), & \alpha<\upsilon, \beta<\rho, \lambda<\sigma, \mu<\tau, \\ -2(n+1)(e^{\alpha\upsilon}_{\beta\tau}e^{\lambda\sigma}_{\mu\beta} + e^{\alpha\upsilon}_{\mu\beta}e^{\lambda\sigma}_{\beta\tau}), & \alpha<\upsilon, \beta=\rho, \lambda<\sigma, \mu<\tau, \\ -2(n+1)\delta^\beta_\alpha(e^{\mu\tau}_{\alpha\lambda} + e^{\mu\tau}_{\sigma\alpha}), & \alpha=\upsilon, \beta=\rho, \lambda<\sigma, \mu<\tau, \\ -(n+1)(e^{\alpha\upsilon}_{\beta\mu}e^{\lambda\sigma}_{\mu\beta} + e^{\alpha\upsilon}_{\mu\beta}e^{\lambda\sigma}_{\beta\mu}), & \alpha<\upsilon, \beta=\rho, \lambda<\sigma, \mu=\tau, \\ 0, & \alpha<\upsilon, \beta=\rho, \mu=\tau, \lambda=\sigma, \\ -2(n+1), & \alpha=\upsilon, \beta=\rho, \mu=\tau, \lambda=\sigma, \end{cases}$$

where the exact description of $e^{\alpha\beta}_{\rho\sigma}(=\frac{\partial z_{\alpha\beta}}{\partial z_{\rho\sigma}})$ can also be consulted in Calabi [3]. Here $z_{\alpha\beta} = z_{\beta\alpha}$ and $\rho\le\sigma$. Hence, after a complicated computation, we have

$$\begin{aligned}[][|R_{g_B}|^2]_{z=0} &= \sum_{\alpha\le\upsilon\ \beta\le\rho\ \lambda\le\sigma\ \mu\le\tau} g_B^{\overline{\alpha\upsilon,\overline{\alpha\upsilon}}} g_B^{\beta\rho,\overline{\beta\rho}} g_B^{\overline{\lambda\sigma,\overline{\lambda\sigma}}} g_B^{\mu\tau,\overline{\mu\tau}} [R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}} \overline{[R_{g_B}]_{\alpha\upsilon,\overline{\beta\rho},\lambda\sigma,\overline{\mu\tau}}} \\ &= \frac{n(n+1)(n^2+19n-60)+96n}{4(n+1)^2}.\end{aligned}$$

Case 4. For $\Omega = D^{IV}_n := \{z\in\mathbb{C}^n : 1 - 2z\bar{z}^t + |zz^t|^2 > 0, z\bar{z}^t < 1\}$ $(n\ge 5)$ (here $\gamma = n, d = n$). Moreover, $\log K(z,z)$ can be expressed by

$$\begin{aligned}\log K(z,z) &= \log\frac{1}{V(\Omega)}(1-2z\bar{z}^t + |zz^t|^2)^{-n} \\ &= -\log V(\Omega) + 2n\sum_i|z_i|^2 - n|\sum_i z_i^2|^2 + 2n(\sum_i|z_i|^2)^2 + \text{higher-order terms.}\end{aligned}$$

Hence, we have $[g_B]_{\alpha\beta}(0) = 2n\delta^\alpha_\beta$ and $[R_{g_B}]_{\alpha\bar\rho\beta\bar\sigma}(0) = -4n(\delta^\alpha_\rho\delta^\beta_\sigma + \delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\beta\delta^\rho_\sigma)$. Hence

$$[|R_{g_B}|^2]_{z=0} = \frac{1}{16n^4}\sum_{\alpha,\beta,\rho,\sigma} 16n^2(\delta^\alpha_\rho\delta^\beta_\sigma + \delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\beta\delta^\rho_\sigma)^2 = \frac{3n-2}{n}.$$

II. Here, for an irreducible bounded symmetric domain Ω, we will prove that $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer with respect to (Ω, g_B) and

$$[|R_{g_B}|^2]_{z=0} = \frac{1}{\gamma^4}\sum_{\alpha,\beta,\upsilon,\delta} |[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2.$$

Firstly, we will express the curvature tensor in terms of Lie brackets of root vectors. All the following conventions can be found in Siu [18], Borel [1] and the book by Helgason [9]. So we will not explain it explicitly, and we will just compute the curvature tensor of the irreducible compact Hermitian symmetric manifold G/K, which will not affect our conclusions.

Let Ψ denote the set of nonzero roots of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$. Write Δ to denote that of nonzero noncompact roots and Δ^+ that of all positive noncompact roots. Moreover, there exists a set Λ of strongly orthogonal noncompact positive roots. For every $\alpha \in \Delta^+$, let e_α denote the root vector for the root α, $e_{-a} = \overline{e_\alpha}$ denotes the root vectors for the root $-\alpha$. Then we have the (direct sum) root space decomposition

$$\mathfrak{g}_c = \mathfrak{t}_c + \sum_{\alpha\in\Psi}\mathbb{C}e_\alpha.$$

This decomposition is orthogonal with respect to the Killing form $B(\cdot,\bar\cdot)$. Since the Killing form on $\mathfrak{g}$ is negative definite, then we can modify [9] (p. 176), Thm. 5.5 by the following results.

Theorem 4.1 *(see also [19], Lemma 4.3.22 and Thm. 4.3.26). For each $\alpha \in \Delta^+$, let X_α be any root vector, then we have*

$$[X_\alpha, X_{-\beta}] = \begin{cases} N_{\alpha,-\beta}X_{\alpha-\beta}, & \alpha-\beta\in\Psi,\ \alpha\neq\beta,\\ 0, & \alpha-\beta\notin\Psi,\ \alpha\neq\beta,\\ B(X_\alpha, X_{-\alpha})H_\alpha\in\mathfrak{t}_c, & \alpha=\beta,\end{cases}$$

$$N^2_{\alpha,-\beta} = -\frac{q(1-p)}{2}(\alpha,\alpha)B(X_\alpha, X_{-\alpha}),$$

where $n\alpha-\beta (p\le n\le q)$ is the α-series containing β and $(\alpha,\alpha) = B(H_\alpha, H_\alpha)$.

Let $\mathfrak{p}_+ = \bigoplus_{\alpha\in\Delta^+}\mathbb{C}e_\alpha$ and $\mathfrak{p}_- = \bigoplus_{-\alpha\in\Delta^+}\mathbb{C}e_\alpha$. Then from [18], we have

$$T^{1,0}_p\Omega = \mathfrak{p}_+,\ T^{0,1}_p\Omega = \mathfrak{p}_-.$$

Moreover, $-B(\cdot,\bar\cdot)$ induces an invariant metric on Ω. Thus the Hermitian metric $\langle\cdot,\cdot\rangle$ on $T^{1,0}_p\Omega$ is defined by

$$\langle e_\alpha, e_\beta\rangle = \langle e_\alpha, \overline{e_\beta}\rangle_R = -B(e_\alpha, e_{-\alpha})\ \ (\alpha,\beta\in\Delta^+).$$

The curvature tensor R is given by $R(X,Y)Z = -[[X,Y],Z]$. The paper [18] also tells us that $R_{\alpha\overline{\beta}\upsilon\overline{\delta}}$ with respect to the $\langle\cdot,\cdot\rangle$ can be expressed by

$$R_{\alpha\overline{\beta}\upsilon\overline{\delta}} = -\langle[e_\alpha, e_{-\beta}], [e_\delta, e_{-\upsilon}]\rangle. \tag{4.1}$$

It is well known that the Bergman metric is an invariant metric. Hence, by [16] (Chapter 3, 2.1), we have $g_B = a\langle\cdot,\cdot\rangle$, where a is a positive constant. By the Proposition 2 in [10], we know that $[R_{g_B}]_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(0) = -2\gamma(\alpha\in\Lambda)$ and $[g_B]_{\alpha\overline{\beta}}(0) = \gamma\delta_{\alpha\beta}$. Thus, by the definition, we have

$$[|R_{g_B}|^2]_{z=0} = \frac{1}{\gamma^4}\sum_{\alpha,\beta,\upsilon,\delta} |[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2.$$

Without loss of generality, we can assume that $\{e_\alpha\}$ constitutes the corresponding basis. Hence, we have

$$[g_B]_{\alpha\overline{\beta}}(0) = a\langle e_\alpha, e_\beta\rangle = \gamma\delta_{\alpha\beta}. \tag{4.2}$$

Thus we get $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}|^2(0) = a^2|R_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$. Now combined with [1], Lemma 2.1, we have the following result

Theorem 4.2. *For an irreducible bounded symmetric domain Ω, we have $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer with respect to (Ω, g_B).*

Proof. Firstly, by the [1], Lemma 2.1, Theorem 4.1 and (4.1), it is not hard to get that, for any $\alpha, \beta, \upsilon, \delta \in \Delta^+$,

$$|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}|^2(0) = \begin{cases} \gamma^2 N^2_{\alpha,-\beta} N^2_{\delta,-\upsilon}, & \alpha - \beta = \delta - \upsilon,\ \delta \neq \upsilon, \\ 0, & \alpha - \beta \neq \delta - \upsilon, \\ \gamma^2 N^4_{\alpha,-\upsilon}, & \alpha = \beta, \upsilon = \delta, \alpha \neq \upsilon, \\ a^2 B(e_\alpha, e_{-\alpha})^4 \langle H_\alpha, H_\alpha \rangle^2, & \alpha = \beta = \upsilon = \delta. \end{cases} \tag{4.3}$$

For the classical irreducible bounded symmetric domains, we have that $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}(0)|^2$ is an integer with respect to (Ω, g_B) by the above arguments. For the two exceptional bounded symmetric domains, by Helgason [9] (p. 523.7), we know that $(\alpha, \alpha) = B(H_\alpha, H_\alpha) = \frac{1}{\gamma}$ for all $\alpha \in \Delta^+$. What's more, by (4.2), we know that $B(e_\alpha, e_{-\alpha}) = -\frac{\gamma}{a}$. Hence, combined with [10], Proposition 2 and (4.3), for $\alpha \in \Lambda$, we have

$$|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}|^2(0) = a^2 \frac{\gamma^4}{a^4} B(H_\alpha, \overline{H_\alpha})^2 = 4\gamma^2. \tag{4.4}$$

Since $B(H_\alpha, \overline{H_\alpha}) = -B(H_\alpha, H_\alpha) = -\frac{1}{\gamma}$. Hence, we have $a = \frac{1}{2}$ and $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}|^2(0) = 4\gamma^2$ for $\alpha \in \Delta^+$.

Furthermore, by Theorem 4.1, we know that

$$N^2_{\alpha,-\beta} = -\frac{q_1(1-p_1)}{2} \alpha(H_\alpha) B(e_\alpha, e_{-\alpha}) = \frac{q_1(1-p_1)}{2\gamma} 2\gamma = q_1(1-p_1).$$

Hence we have $N^2_{\alpha,-\beta}$ is an integer. Similarly, we have $N^2_{\delta,-\upsilon}$ and $N^2_{\alpha,-\upsilon}$ are integers. Then, combined with (4.3) and (4.4), it is easy to show that $|[R_{g_B}]_{\alpha\overline{\beta}\upsilon\overline{\delta}}|^2(0)$ is an integer for the two exceptional bounded symmetric domains. So far we complete the proof. $\square$

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