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Harmonic analysis

Sharp weighted estimates involving one supremum

Estimations pondérées précisées associées à un seul supremum

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we study the sharp weighted estimate involving one supremum. In particular, we give a positive answer to an open question raised by Lerner and Moen. We also extend the result to rough homogeneous singular integral operators.

R É S U M É

Nous étudions dans cette note les estimations pondérées précisées associées à un seul supremum. En particulier, nous résolvons par l'affirmative un probléme ouvert posé par Lerner et Moen. Nous étendons également le résultat aux opérateurs intégraux singuliers homogènes rugueux.

1. Introduction and main result

Our main object is the following so-called sparse operator:

$$A_{\mathcal{S}}(f)(x) = \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \chi_Q(x),$$

where $\mathcal{S} \subset \mathcal{D}$ is a sparse family, i.e. for all (dyadic) cubes $Q \in \mathcal{S}$, there exist $E_Q \subset Q$ which are pairwise disjoint and $|E_Q| \ge \gamma|Q|$ with $0 < \gamma < 1$, and $\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f$. We only consider the sparse operator, because it dominates the Calderón–Zygmund operator pointwisely, see [2,14,9,8,11].

We are going to study the sharp weighted bounds of $A_{\mathcal{S}}$. Before that, let us recall

$$[w]_{A_p} = \sup_Q A_p(w, Q) := \sup_Q \langle w\rangle_Q \langle w^{1-p'}\rangle_Q^{p-1},$$

$$[w]_{A_\infty} = \sup_Q A_\infty(w, Q) := \sup_Q \frac{\langle M(w\chi_Q)\rangle_Q}{\langle w\rangle_Q}.$$

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In [6], Hytönen and Lacey proved the following estimate:

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_n [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [w^{1-p'}]_{A_\infty}^{\frac{1}{p}}), \tag{1}$$

which generalizes the famous A_2 theorem, obtained by Hytönen in [5]. (We also remark that when $p=2$, (1) was obtained by Hytönen and Pérez in [7].) It was also conjectured in [6] that

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_n ([w]_{A_p^{\frac{1}{p}} A_\infty^{\frac{1}{p'}}} + [w^{1-p'}]_{A_{p'}^{\frac{1}{p'}} A_\infty^{\frac{1}{p}}}),$$

where

$$[w]_{A_p^\alpha A_r^\beta} := \sup_Q A_p(w,Q)^\alpha A_r(w,Q)^\beta .$$

This conjecture, which is usually referred to as the one supremum conjecture, is still open. Before this conjecture was formulated, Lerner [10] obtained the following mixed A_p–A_r estimate:

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_{n,p,r} ([w]_{A_p^{\frac{1}{p-1}} A_r^{1-\frac{1}{p-1}}} + [w^{1-p'}]_{A_{p'}^{\frac{1}{p'-1}} A_r^{1-\frac{1}{p'-1}}}),$$

which was further extended by Lerner and Moen [13] to the $r=\infty$ case with Hruščëv A_∞ constant:

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_{n,p} ([w]_{A_p^{\frac{1}{p-1}} (A_\infty^{\exp})^{1-\frac{1}{p-1}}} + [w^{1-p'}]_{A_{p'}^{\frac{1}{p'-1}} (A_\infty^{\exp})^{1-\frac{1}{p'-1}}}),$$

where $A_\infty^{\exp}(w,Q) = \langle w\rangle_Q \exp(\langle \log w^{-1}\rangle_Q)$. Some further extension can also be found in [15]. Comparing this result with the one supremum conjecture, besides replacing the Fujii–Wilson A_∞ constant by Hruščëv A_∞ constant, the main difference is that the power of A_p constant is larger, leading to a result which is weaker than the one-supremum conjecture. However, there is also another idea, which is replacing A_p by A_q, where $q<p$. Our main result follows from this idea and it is formulated as follows.

Theorem 1.1. *Let* $1 \le q < p$ *and* $w \in A_q$. *Then*

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_{n,p,q} [w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} .$$

This result was conjectured by Lerner and Moen, see [13, p.251]. It improves the previous result of Duoandikoetxea [3], i.e.

$$\|A_{\mathcal{S}}\|_{L^p(w)} \le c_{n,p,q} [w]_{A_q},$$

proved by means of extrapolation. In the next section, we will give a proof for this theorem. Extensions to rough homogeneous singular integrals will be provided in Section 3.

2. Proof of Theorem 1.1

Before we state our proof, we would like to demonstrate our understanding of this A_q condition, which allows us to avoid using extrapolation or interpolation completely. We can rewrite the A_q condition in the following form:

$$\begin{aligned}\langle w\rangle_Q \langle w^{1-q'}\rangle_Q^{q-1} &= \langle w\rangle_Q \langle w^{(1-p')\frac{p-1}{q-1}}\rangle_Q^{q-1} \\ &:= \langle w\rangle_Q \langle \sigma^{\frac{1}{p'}}\rangle_{\bar{A},Q}^{p},\end{aligned}$$

where $\bar{A}(t) = t^{p'(p-1)/(q-1)} = t^{\frac{p}{q-1}}$ and as usual, $\sigma = w^{1-p'}$. So we have seen that the A_q condition is actually the power bumped A_p condition! Now we are ready to present our proof. Without loss of generality, we can assume $f \ge 0$. By duality, we have

$$\begin{aligned}\|A_{\mathcal{S}}(f)\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w)}=1} \int A_{\mathcal{S}}(f) g w \\ &= \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle g\rangle_Q^w w(Q) \\ &\le \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q\in\mathcal{S}} \langle f w^{\frac{1}{p}}\rangle_{A,Q} \langle w^{-\frac{1}{p}}\rangle_{\bar{A},Q} \langle g\rangle_Q^w \langle w\rangle_Q |Q|\end{aligned}$$

$$
\begin{aligned}
&\times \exp(\langle \log w^{-1}\rangle_Q)^{\frac{1}{p'}} \exp(\langle \log w\rangle_Q)^{\frac{1}{p'}}\\
&\le [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}} \sup_{\|g\|_{L^{p'}(w)}=1} \Big(\sum_{Q\in\mathcal{S}} \langle f w^{\frac{1}{p}}\rangle_{A,Q}^p |Q|\Big)^{\frac{1}{p}}\\
&\times \Big(\sum_{Q\in\mathcal{S}} (\langle g\rangle_Q^w)^{p'} \exp(\langle \log w\rangle_Q)|Q|\Big)^{\frac{1}{p'}}\\
&\le c_n \gamma^{-1} p \|M_A\|_{L^p} [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}} \|f\|_{L^p(w)},
\end{aligned}
$$

where in the last step, we have used the sparsity and the Carleson embedding theorem.

3. Rough homogeneous singular integral operators

Recall that the rough homogeneous singular integral operator T_Ω is defined by

$$
T_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\,\mathrm{d}y,
$$

where $\int_{S^{n-1}} \Omega = 0$. The quantitative weighted bound of T_Ω with $\Omega \in L^\infty$ has been studied in [8], based on refinement of the ideas in [4]; see also a recent paper by the author, Pérez, Rivera-Ríos and Roncal [16], relying upon the sparse domination formula established in [1].

Our main result in this section is stated as follows.

Theorem 3.1. *Let $1 \le q < p$, $w \in A_q$ and $\Omega \in L^\infty(S^{n-1})$. Then*

$$
\|T_\Omega\|_{L^p(w)} \le c_{n,p,q} [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}.
$$

Proof. The proof is again based on the sparse domination formula [1] (see also a very recent paper by Lerner [12]). It suffices to prove

$$
\|A_{r,\mathcal{S}}\|_{L^p(w)} \le c_{n,p,r,q} [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}},
$$

where $1 < r < \frac{p}{q}$ and

$$
A_{r,\mathcal{S}}(f) = \sum_{Q\in\mathcal{S}} \langle |f|^r\rangle_Q^{\frac{1}{r}} \chi_Q.
$$

Denote $\bar{B}(t) = t^{\frac{p'(p-1)}{r(q-1)}} = t^{\frac{p}{r(q-1)}}$. Again, we assume $f \ge 0$. By duality, we have

$$
\begin{aligned}
\|A_{r,\mathcal{S}}(f)\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w)}=1} \int A_{r,\mathcal{S}}(f) g w\\
&= \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q\in\mathcal{S}} \langle f^r\rangle_Q^{\frac{1}{r}} \langle g\rangle_Q^w w(Q)\\
&\le \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q\in\mathcal{S}} \langle f^r w^{\frac{r}{p}}\rangle_{B,Q}^{\frac{1}{r}} \langle w^{-\frac{r}{p}}\rangle_{\bar{B},Q}^{\frac{1}{r}} \langle g\rangle_Q^w \langle w\rangle_Q |Q|\\
&\times \exp(\langle \log w^{-1}\rangle_Q)^{\frac{1}{p'}} \exp(\langle \log w\rangle_Q)^{\frac{1}{p'}}\\
&\le [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}} \sup_{\|g\|_{L^{p'}(w)}=1} \Big(\sum_{Q\in\mathcal{S}} \langle f^r w^{\frac{r}{p}}\rangle_{B,Q}^{\frac{p}{r}} |Q|\Big)^{\frac{1}{p}}\\
&\times \Big(\sum_{Q\in\mathcal{S}} (\langle g\rangle_Q^w)^{p'} \exp(\langle \log w\rangle_Q)|Q|\Big)^{\frac{1}{p'}}\\
&\le c_n \gamma^{-1} p \|M_B\|_{L^{p/r}}^{\frac{1}{r}} [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}} \|f\|_{L^p(w)},
\end{aligned}
$$

where again, in the last step we have used the sparsity and the Carleson embedding theorem. □

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References

[1] J.M. Conde-Alonso, A. Culiuc, F. Di Plinio, Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE 10 (5) (2017) 1255–1284.
[2] J.M. Conde-Alonso, G. Rey, A pointwise estimate for positive dyadic shifts and some applications, Math. Ann. 365 (2016) 1111–1135.
[3] J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, J. Funct. Anal. 260 (2011) 1886–1901.
[4] J. Duoandikoetxea, J.L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986) 541–561.
[5] T.P. Hytönen, The sharp weighted bound for general Calderón–Zygmund operators, Ann. of Math. (2) 175 (3) (2012) 1473–1506.
[6] T.P. Hytönen, M.T. Lacey, The A_p–A_∞ inequality for general Calderón–Zygmund operators, Indiana Univ. Math. J. 61 (6) (2012) 2041–2092.
[7] T.P. Hytönen, C. Pérez, Sharp weighted bounds involving A_∞, Anal. PDE 6 (4) (2013) 777–818.
[8] T.P. Hytönen, L. Roncal, O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, Isr. J. Math. 218 (2017) 133–164.
[9] M.T. Lacey, An elementary proof of the A_2 bound, Isr. J. Math. 217 (2017) 181–195.
[10] A.K. Lerner, Mixed A_p–A_r inequalities for classical singular integrals and Littlewood–Paley operators, J. Geom. Anal. 23 (3) (2013) 1343–1354.
[11] A.K. Lerner, On pointwise estimates involving sparse operators, N.Y. J. Math. 22 (2016) 341–349.
[12] A.K. Lerner, A weak type estimate for rough singular integrals, preprint, available at arXiv:1705.07397.
[13] A.K. Lerner, K. Moen, Mixed A_p–A_∞ estimates with one supremum, Stud. Math. 219 (3) (2013) 247–267.
[14] A.K. Lerner, F. Nazarov, Intuitive dyadic calculus: the basics, preprint, available at arXiv:1508.05639, 2015.
[15] K. Li, Two weight inequalities for bilinear forms, Collect. Math. 68 (2017) 129–144.
[16] K. Li, C. Pérez, I.P. Rivera-Ríos, L. Roncal, Weighted norm inequalities for rough singular integral operators, preprint, available at arXiv:1701.05170.