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Powers and logarithms of convex bodies ☆

Puissances et logarithmes de corps convexes

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A R T I C L E I N F O

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A B S T R A C T

Do we have enough examples of convex bodies that we truly understand? Is out standard set of examples diverse enough to understand convexity? In this note, we will dramatically increase our set of examples. More specifically, we will present several new constructions of convex bodies: the geometric mean of two convex bodies, the power function K^α (which in general exists only for $|\alpha| \le 1$), and even the logarithm $\log K$.

R É S U M É

Existe-t-il suffisamment de corps convexes que nous comprenions vraiment ? L'éventail usuel d'exemples est-il assez diversifié pour saisir la notion de convexité ? Dans cette note, nous proposons une augmentation drastique du corpus d'exemples. Plus précisément, nous présentons plusieurs constructions nouvelles de corps convexes : la moyenne géométrique de deux corps convexes, la fonction puissance K^α (qui, en général, n'existe que pour $|\alpha| \le 1$), et même le logarithme $\log K$.

1. Introduction: means of convex bodies

Do we have enough examples of convex bodies that we truly understand? Is our standard set of examples diverse enough to understand convexity? Recently it was realized that the polarity operation in convexity (both in convex analysis and convex geometry) is analogous to the operation $x \mapsto \frac{1}{x}$ defined for $x > 0$ (the inverse operation is a "polarity" on $\mathbb{R}_+$). Let us follow this analogy and think about the polar body K° as the inverse "$\frac{1}{K}$". Surprisingly, this point of view will be used to dramatically increase our set of examples – one may see an example of this idea in the construction of continued fractions by Molchanov ([10]). Our goal for this note is to present some very novel constructions of convex bodies that

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are, on the one hand, "invisible" but, on the other hand, may be studied and should increase the diversity of our examples and help us develop a new intuition. In particular, we will show very unexpected constructions such as the power K^α (for $|\alpha| \le 1$) and the logarithm $\log K$.

We start with a few definitions and some notation. A convex body in $\mathbb{R}^n$ is a set $K \subseteq \mathbb{R}^n$ that is convex, compact, and has a non-empty interior. We will also make the assumption that our convex bodies are origin-symmetric, i.e. $x \in K$ implies $-x \in K$. Let us denote the set of all such bodies in $\mathbb{R}^n$ by $\mathcal{K}_s^n$.

In this introductory section, we recall various ways of averaging convex bodies. For two convex bodies $K, T \in \mathcal{K}_s^n$, let $K + T$ denote their Minkowski sum (for the definition of this notion, and other basic definitions in convexity, the reader may consult [12]). Similarly, for $K \in \mathcal{K}_s^n$, the body λK will denote the dilation of K by a factor $\lambda > 0$. Using this notation, we can of course define the arithmetic mean of K and T as $A(K,T) = \frac{K+T}{2}$.

In order to define the harmonic mean $H(K,T)$, we need an additional ingredient – the existence of an inverse operation $K \mapsto$ "K^{-1}". As was mentioned above, it turns out to be natural to define K^{-1} as the polar body

$$K^\circ = \left\{ y \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } x \in K \right\}$$

(where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$). For more background on the motivation behind this idea, we refer the reader to [9]. We will take it as a fact, and define the harmonic mean as $H(K,T) = \left(\frac{K^\circ + T^\circ}{2}\right)^\circ$. This mean was already considered by Firey in the early 1960s [3].

The construction of the geometric mean is a more delicate matter. In [11] and [8], we prove the following result.

Theorem 1.1. *There exists a map $G : \mathcal{K}_s^n \times \mathcal{K}_s^n \to \mathcal{K}_s^n$ that satisfies the following properties:*

1) *$G(K,K) = K$;*
2) *G is symmetric in its arguments: $G(K,T) = G(T,K)$;*
3) *G is monotone in its arguments: If $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$ then $G(K_1,T_1) \subseteq G(K_2,T_2)$;*
4) *G is continuous in its arguments, with respect to the Hausdorff distance;*
5) *G satisfies the harmonic mean – geometric mean – arithmetic mean inequality*

$$\left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq G(K,T) \subseteq \frac{K+T}{2};$$

6) *$[G(K,T)]^\circ = G(K^\circ, T^\circ)$;*
7) *$G(K, K^\circ) = B_2^n$, where B_2^n denotes the unit Euclidean ball;*
8) *for every linear map $u : \mathbb{R}^n \to \mathbb{R}^n$ we have $G(uK, uT) = u(G(K,T))$;*
9) *for every $\alpha, \beta > 0$, we have $G(\alpha K, \beta T) = \sqrt{\alpha\beta}\, G(K,T)$.*

We call $G(K,T)$ the geometric mean of K and T.

All the properties in the above list are natural properties for the geometric mean to satisfy. Property (7), for example, is the analogue of the fact that the geometric mean of x and $\frac{1}{x}$ is always 1. For property (4), we remind the reader that the Hausdorff distance is defined as

$$d(K,T) = \min\left\{ r > 0 : K \subseteq T + rB_2^n \text{ and } T \subseteq K + rB_2^n \right\}.$$

Whenever we discuss continuity or convergence in $\mathcal{K}_s^n$, we will always have the Hausdorff metric in mind.

Properties (7) and (8) suffice to compute the geometric mean of centered ellipsoids. If $E, F \in \mathcal{K}_s^n$ are ellipsoids, then $G(E,F)$ is also an ellipsoid. Furthermore, if we change the scalar product on $\mathbb{R}^n$ in such a way that $G(E,F)$ is the new unit ball, then $F = E^\circ$. This fact characterizes $G(E,F)$ uniquely.

Let us say a few words about the proof of Theorem 1.1. First, as was done in [9], one defines a simpler construction $g(K,T)$ that satisfies all properties except property (9). To do so, one defines two sequences of convex bodies $\{A_n\}_{n=0}^\infty$ and $\{H_n\}_{n=0}^\infty$ by

$$\begin{aligned} A_0 &= K, & H_0 &= T, \\ A_{n+1} &= A(A_n, H_n), & H_{n+1} &= H(A_n, H_n) \end{aligned}$$

and then sets $g(K,T) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} H_n$. Even though g does not satisfy the important scaling property (9), it is still a useful construction. For example, in [11], it is used to define a new type of a geometric Banach limit on sequences of convex bodies.

There are two possible approaches for passing from g to G. The first one is explained in [11], and uses the aforementioned geometric Banach limit. It has the advantage of also applying to non-symmetric convex bodies. A completely different approach is detailed in [8], and uses ellipsoids as basic ingredients. This second approach is similar in spirit to the ideas discussed in the remainder of this note.

We conclude this section with an open problem: we do not know if the nine properties of Theorem 1.1 suffice to characterize G uniquely.

2. Powers of convex bodies

Our next goal is to construct the power $K^\alpha \in \mathcal{K}_s^n$ for a body $K \in \mathcal{K}_s^n$ and $\alpha \in \mathbb{R}$.

Recall that the support function $h_K : \mathbb{R}^n \to (0,\infty)$ of a body $K \in \mathcal{K}_s^n$ is defined by $h_K(y) = \sup_{x\in K} \langle x, y\rangle$. The support function is related to the Minkowski sum via the relation $h_{\lambda K+\mu T} = \lambda h_K + \mu h_T$. In fact, given $p \geq 1$, $K, T \in \mathcal{K}_s^n$ and $\lambda, \mu > 0$ one may define the p-sum $+_p$ and p-homothety $\cdot_p$ by

$$h^p_{(\lambda\cdot_p K)+_p(\mu\cdot_p T)} = \lambda h_K^p + \mu h_T^p.$$

p-sums were originally defined by Firey ([4]) and studied extensively by Lutwak ([6], [7]).

For us, an ellipsoid will always mean a centered ellipsoid, i.e. a linear image of the unit ball B_2^n. If E is an ellipsoid, then $h_E(y) = \sqrt{\langle u\cdot y, y\rangle}$ for a uniquely defined positive-definite matrix $u : \mathbb{R}^n \to \mathbb{R}^n$. We will also write u_E instead of u to emphasize the dependence on E. If E and F are ellipsoids and $\lambda, \mu > 0$ then $(\lambda \cdot_2 E) +_2 (\mu \cdot_2 F)$ is also an ellipsoid and $u_{(\lambda\cdot_2 E)+_2(\mu\cdot_2 F)} = \lambda u_E + \mu u_F$. On the other hand, for 1-sum, $\lambda E + \mu F$ is usually not an ellipsoid.

If $f : (0,\infty) \to \mathbb{R}$ is any function, one can always apply it to the positive-definite matrix u. If $f > 0$, then the matrix $f(u)$ will again be positive-definite. This allows us to apply f to ellipsoids by setting $u_{f(E)} = f(u_E)$. In geometric terms, $f(E)$ is obtained from E by fixing the directions of the principal axes and applying f to the *square* of their lengths.

By choosing $f(x) = x^\alpha$, we can define E^α for every ellipsoid E and every $\alpha \in \mathbb{R}$. Concentrating for the moment on the case $\alpha > 0$, there is an important difference between the cases $0 < \alpha < 1$ and $\alpha > 1$. To explain this difference, we need the following definition.

Definition 2.1 *(Löwner [5]).* A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone if, for every positive definite matrices u and v (of any size), $u \succeq v$ implies $f(u) \succeq f(v)$.

Recall that $u \succeq v$ in the matrix order means that $u - v$ is positive semi-definite. Since $u_E \succeq u_F$ if and only if $E \supseteq F$, we see that an operator monotone function is also monotone with respect to inclusion on the class of ellipsoids.

It is well known that the function $f(x) = x^\alpha$ is operator monotone if $0 \leq \alpha \leq 1$, but not if $\alpha > 1$ (Löwner, see also e.g. [1], Section V.1). This monotonicity allows us to extend the power operation to general convex bodies. To do so, we first define the upper pre-power as

$$\overline{P}_\alpha(K) = \bigcap \left\{ E^\alpha : \ E \text{ is an ellipsoid and } K \subseteq E \right\}.$$

Of course, $\overline{P}_\alpha(K)$ is always a convex body. The map $K \mapsto \overline{P}_\alpha(K)$ satisfies some natural properties, such as monotonicity, but it does not satisfy the power law $P_{\alpha\beta}(K) = P_\beta(P_\alpha(K))$, which is the analogue of the identity $x^{\alpha\beta} = \left(x^\alpha\right)^\beta$. Instead, the pre-powers only satisfy the inclusion $P_{\alpha\beta}(K) \supseteq P_\beta(P_\alpha(K))$.

In order to create a better construction, we continue as follows: for a fixed finite partition of $[\alpha, 1]$,

$$\Pi : \ \alpha = t_0 < t_1 < \cdots < t_m = 1,$$

we set $s_i = t_{i-1}/t_i$ for $i = 1, 2, ..m$, and define

$$\overline{P}_\Pi(K) = \left(\overline{P}_{s_1} \circ \overline{P}_{s_2} \circ \cdots \circ \overline{P}_{s_m}\right)(K).$$

We then have the following theorem, which is a slight extension of the results of [8].

Theorem 2.2. *Fix $\mathcal{K}_s^n$ and $0 \leq \alpha \leq 1$. Then the limit*

$$K^{\overline{\alpha}} := \lim_{\lambda(\Pi)\to 0} P_\Pi(K) \in \mathcal{K}_s^n$$

exists in the Hausdorff sense. Here $\lambda(\Pi) = \max |t_{i+1} - t_i|$ denotes the length of the longest interval in Π. Furthermore, the maps $K \mapsto K^{\overline{\alpha}}$ have the following properties:

1) *for every $0 < \alpha < 1$, if $K \subseteq T$ then $K^{\overline{\alpha}} \subseteq T^{\overline{\alpha}}$;*
2) *for every $0 < \alpha < 1$, every $K \in \mathcal{K}_s^n$ and every $\lambda > 0$ we have $(\lambda K)^{\overline{\alpha}} = \lambda^\alpha K^{\overline{\alpha}}$;*
3) *for every $0 < \alpha, \beta < 1$ and every $K \in \mathcal{K}_s^n$ we have $\left(K^{\overline{\alpha}}\right)^{\overline{\beta}} = K^{\overline{\alpha\beta}}$;*
4) *if $E \in \mathcal{K}_s^n$ is an ellipsoid, then $E^{\overline{\alpha}}$ agrees with its linear algebra definition for every $0 < \alpha < 1$;*
5) *$K \supseteq B_2^n$ implies $K^{\overline{\alpha}} \subseteq K$, and $K \subseteq B_2^n$ implies $K^{\overline{\alpha}} \supseteq K$ for all $0 \leq \alpha \leq 1$.*

We call the map $K \mapsto K^{\overline{\alpha}}$ the upper power. As the name suggests, one can also define lower powers, by taking

$$\underline{P}_\alpha(K) = \mathrm{conv} \bigcup \left\{ E^\alpha : \ E \text{ is an ellipsoid and } K \supseteq E \right\}.$$

and carrying out the rest of the construction in the same way. The lower powers $K^{\underline{\alpha}}$ will have the same good properties as the upper powers, and of course $K^{\underline{\alpha}} \subseteq K^{\overline{\alpha}}$ for all $K \in \mathcal{K}_s^n$ and $0 \le \alpha \le 1$.

However, the upper and lower powers still do not satisfy all properties one may expect. To see this, recall that we interpret K^{-1} to be K°. Therefore, the power law $\left(K^\alpha\right)^\beta = K^{\alpha\beta} = \left(K^\beta\right)^\alpha$ should imply that $\left(K^\alpha\right)^\circ = (K^\circ)^\alpha$, and there is no reason for the upper and lower powers to satisfy this property.

We will now present an alternative construction that does commute with polarity. We begin by fixing a free ultrafilter $\mathcal{U}$ on the natural numbers $\mathbb{N}$. The Blaschke selection theorem states that, for every $R > 0$, the set

$$\left\{K \in \mathcal{K}_s^n : \ K \subseteq R \cdot B_2^n\right\}$$

is compact (see, e.g., [12]). Therefore, if $\{K_m\}_{m=1}^\infty$ is a uniformly bounded sequence of convex bodies, then the ultralimit $\lim_{\mathcal{U}} K_m$ always exists and is convex.

Fix a number $0 < \gamma < 1$ (say $\gamma = \frac{1}{2}$). For every $K \in \mathcal{K}_s^n$ and every $0 < \alpha < 1$, we set

$$Q_m(K, \alpha) = \left(\underline{P}_{\gamma^{1/m}} \circ \overline{P}_{\gamma^{1/m}}\right)^\ell (K),$$

where $\ell = \ell(m)$ is the biggest integer such that $\gamma^{2\ell/m} \ge \alpha$. Finally, we define $K^\alpha = \lim_{\mathcal{U}} Q_m(K, \alpha)$.

Theorem 2.3. *The maps $K \mapsto K^\alpha$ that send convex bodies to convex bodies satisfy properties (1)–(5) from Theorem 2.2, together with the additional property:*

6) *for every $0 < \alpha < 1$ and every $K \in \mathcal{K}_s^n$, we have $\left(K^\alpha\right)^\circ = (K^\circ)^\alpha$.*

Let us mention that the identity $\left(K^\alpha\right)^\circ = (K^\circ)^\alpha$ follows from the following elementary estimate: if $\frac{1}{R} \cdot B_2^n \subseteq K \subseteq R \cdot B_2^n$ for some $R > 0$, then

$$\left(\frac{1}{R}\right)^{2-2\gamma^{1/m}} \cdot Q_m\left(K, \alpha\right)^\circ \subseteq Q_m\left(K^\circ, \alpha\right) \subseteq R^{2-2\gamma^{1/m}} \cdot Q_m\left(K, \alpha\right)^\circ .$$

As is the case for the geometric mean, we do not know if the six properties of Theorem 2.3 characterize the powers $K \mapsto K^\alpha$ uniquely. Note that if this is indeed the case, then the definition of K^α does not depend on the choice of the ultrafilter $\mathcal{U}$, and the limit $\lim_{m\to\infty} Q_m(K, \alpha)$ actually exists in the Hausdorff sense.

We conclude this section with two remarks.

Remark 1. In general, it is impossible to define K^α for $\alpha > 1$, at least if one wants to keep the power law $\left(K^\alpha\right)^\beta = K^{\alpha\beta}$. To see this, let $Q = [-1, 1]^n$ be the cube, and assume that Q^2 is well defined. Then by John's theorem, there exists an ellipsoid E such that $E \subseteq Q^2 \subseteq \sqrt{n}E$ (see, e.g., [12]). It follows that

$$E^{1/2} \subseteq \left(Q^2\right)^{1/2} = Q \subseteq \left(\sqrt{n}E\right)^{1/2} = n^{1/4}E^{1/2},$$

and since $E^{1/2}$ is also an ellipsoid, these inclusions are impossible: it is well known that $d_{\mathrm{BM}}\left(Q, B_2^n\right) = n^{1/2}$, where d_{BM} denotes the Banach–Mazur distance.

Remark 2. Our construction of the upper power is somewhat related to the logarithmic mean of Böröczky, Lutwak, Yang, and Zhang ([2]). Recall that for $K, T \in \mathcal{K}_s^n$ and $0 < \alpha < 1$, the logarithmic mean is defined by

$$L_\alpha(K, T) = \left\{x \in \mathbb{R}^n : \ \langle x, \theta\rangle \le h_K(\theta)^\alpha h_T(\theta)^{1-\alpha} \text{ for all } \theta \in S^{n-1}\right\}$$

(where S^{n-1} denotes the unit sphere). In other words, $L = L_\alpha(K, T)$ is the largest convex function such that $h_L(\theta) \le h_K(\theta)^\alpha h_T(\theta)^{1-\alpha}$ for all $\theta \in S^{n-1}$.

To see the relation with our construction, notice that a slab $S = \{x : \ |\langle x, \theta\rangle| \le c\}$ can be viewed as a "degenerate ellipsoid". By approximating S with proper ellipsoids, we see that

$$S^\alpha = \left\{x : \ |\langle x, \theta\rangle| \le c^\alpha\right\}.$$

It follows that

$$L_\alpha(K, B_2^n) = \bigcap \left\{S^\alpha : \ S \text{ is a slab and } K \subseteq S\right\},$$

which is similar to the upper pre-power $\overline{P}_\alpha(K)$. It follows that we have

$$K^{\overline{\alpha}} \subseteq \overline{P}_\alpha(K) \subseteq L_\alpha(K, B_2^n)$$

for all $K \in \mathcal{K}_s^n$ and $0 < \alpha < 1$.

In the same way, one can construct variants of the other definitions in this note using only slabs instead of ellipsoids. We will not pursue this point further.

3. Logarithms of convex bodies

The final goal of this note is to discuss the construction of the logarithms of convex bodies. We would like these logarithms to be convex bodies themselves. For an ellipsoid E, one may define $\log E$ in the same way as in the previous section, by setting $u_{\log E} = \log(u_E)$. The problem, of course, is that $\log(u_E)$ is positive-definite only if $u_E \succeq Id$. Therefore $\log E$ can only be defined for ellipsoids E such that $E \supseteq B_2^n$.

The reader may check that $\log\left(rB_2^n\right) = \sqrt{2\log r}B_2^n$ for $r \geq 1$. Moreover, for every $E \supseteq B_2^n$ we have $\log\left(E^\alpha\right) = \alpha \cdot_2 \log E = \sqrt{\alpha}\log E$. The appearance of the 2-homothety $\cdot_2$ is not surprising, given the relation between the 2-sum and ellipsoids. It is known that the function $f(x) = \log x$ is operator monotone (again, see e.g. [1]), so $E \supseteq F$ implies $\log E \supseteq \log F$.

We now want to extend the definition from ellipsoids to general convex bodies K such that $K \supseteq B_2^n$. Assume that E is any ellipsoid such that $E^\alpha \supseteq K^{\overline{\alpha}}$ for some $0 < \alpha < 1$. If the logarithm for convex bodies behaves like it does for ellipsoids we expect to have

$$\alpha \cdot_2 \log K = \log\left(K^{\overline{\alpha}}\right) \subseteq \log\left(E^\alpha\right) = \alpha \cdot_2 \log E,$$

so $\log K \subseteq \log E$. This suggests the following definitions:

Definition 3.1.

1. For $K \in \mathcal{K}_s^n$, the core family $\mathcal{E}(K)$ is the family of all ellipsoids E such that $E^\alpha \supseteq K^{\overline{\alpha}}$ for some $0 < \alpha < 1$.
2. If $K \supseteq B_2^n$, the (upper) logarithm of K is defined by

$$\log K = \bigcap \{\log E : \; E \in \mathcal{E}(K)\}.$$

We then have the following result:

Theorem 3.2. *The map $K \mapsto \log K$ satisfies the following properties:*

1) *if $B_2^n \subseteq K \subseteq T$, then $\log K \subseteq \log T$;*
2) *if $E \in \mathcal{K}_s^n$ is an ellipsoid and $E \supseteq B_2^n$, then $\log E$ agrees with its linear algebra definition;*
3) *for every $K \supseteq B_2^n$ and every $0 < \alpha < 1$, one has $\log\left(K^{\overline{\alpha}}\right) = \alpha \cdot_2 \log K$;*
4) *for every $K \supseteq B_2^n$ and every $t \geq 1$, we have*

$$\log(t \cdot_2 K) \supseteq \left(\log t \cdot_2 B_2^n\right) +_2 \log K$$

Property (4) is of course related to the standard identity $\log(tx) = \log t + \log x$ for $t, x > 0$. There exists a variant of our definition of the logarithm that transforms the inclusion in (4) into an equality.

The core family $\mathcal{E}(K)$ is not easy to compute in practice. To illustrate that point, and to present another interesting construction, let us define the core of a convex body K as

$$\mathrm{core}(K) = \bigcap \{E : \; E \in \mathcal{E}(K)\}.$$

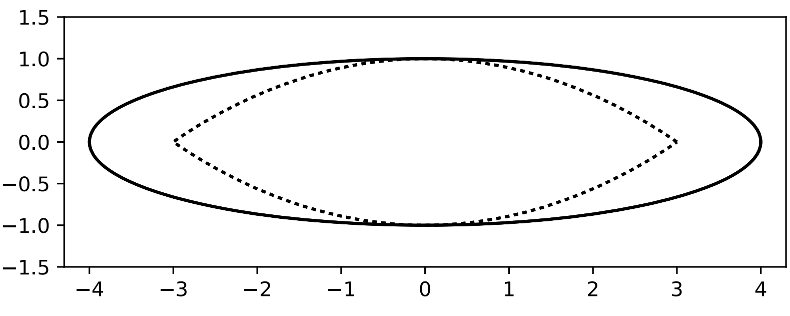

Fig. 1. An ellipse and its core.

It is easy to check that $\mathrm{core}(rB_2^n) = rB_2^n$. However, even for other ellipsoids E, the body $\mathrm{core}(E)$ does not seem to have a simple description. Fig. 1 shows the 2-dimensional ellipse E with axes of lengths 1 and 4 together with its core $\mathrm{core}(E)$, as was computed numerically.

The fact that $\mathrm{core}(E) \neq E$ is a geometric manifestation of the non-commutativity of matrix multiplication. Indeed, if $F \in \mathcal{E}(E)$ and u_E and u_F commute, then we must have $F \supseteq E$. However this is no longer true if u_E and u_F do not commute, which explains why we may have $\mathrm{core}(E) \neq E$.

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