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On the representation by sums of algebras of continuous functions

Sur la représentation des algèbres de fonctions continues comme sommes de sous-algèbres

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ABSTRACT

We give a necessary condition for the representation of the space of continuous functions by sums of its k closed subalgebras. A sufficient condition for this representation problem was first obtained by Sternfeld in 1978. In case of two subalgebras ($k = 2$), our necessary condition turns out to be also sufficient. If $k = 1$, our result coincides with a version of the classical Stone–Weierstrass theorem.

RÉSUMÉ

Nous donnons une condition nécessaire pour la représentation d'un espace de fonctions continues comme la somme d'un nombre fini k de ses sous-algèbres fermées. Une condition suffisante pour ce problème a été obtenue par Sternfeld en 1978. Dans le cas de deux sous-algèbres ($k = 2$), notre condition nécessaire se trouve être également suffisante. Dans le cas d'une seule sous-algèbre ($k = 1$), notre résultat coïncide avec une version du théorème de Stone–Weierstrass classique.

1. Introduction

Let X be a compact Hausdorff space and $C(X)$ be the space of continuous real-valued functions on X endowed with the topology of uniform convergence. Assume we are given a finite number of closed subalgebras $A_1, ..., A_k$ of $C(X)$. This paper is devoted to the following problem. What conditions imposed on $A_1, ..., A_k$ are necessary and/or sufficient for the representation $A_1 + \cdots + A_k = C(X)$? The history of this problem goes back to 1937 and 1948 papers by M.H. Stone [30,31]. A version of the corresponding famous result, known as the Stone–Weierstrass theorem, states that a closed subalgebra

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$A \subset C(X)$, which contains a nonzero constant function, coincides with the whole space $C(X)$ if and only if A separates points of X (that is, for any two different points x and y in X, there exists a function $g \in A$ with $g(x) \neq g(y)$). Obviously, in case of k subalgebras $A_1, ..., A_k$ of $C(X)$, the condition of separation of points is necessary also for the representation $A_1 + \cdots + A_k = C(X)$. Indeed, if $A_1 + \cdots + A_k = C(X)$, then for any different $x, y \in X$ there must be a subalgebra A_i, $i \in \{1, ..., k\}$, separating these points; otherwise, we could construct a continuous function f on X with $f(x) \neq f(y)$, which would not belong to $A_1 + \cdots + A_k$. But this condition is far from being sufficient. A trivial example is a compact set $X \subset \mathbb{R}^2$ with interior and the algebras $U = \{u(x)\}$, $V = \{v(y)\}$ of univariate functions defined on the projections of X into the coordinate axes x and y, respectively. Clearly, the tuple (U, V) separates points of X, but $U + V \neq C(X)$. Indeed, there exists a square $[a, b] \times [c, d] \subset X$ and a continuous function $h : X \to \mathbb{R}$ such that $h(a, c) = h(b, d) = 1$, $h(a, d) = h(b, c) = -1$ and $-1 < h(x, y) < 1$ elsewhere on X. Now, since the functional $F(f) = f(a, c) + f(b, d) - f(a, d) - f(b, c)$ annihilates all members of the class $U + V$ and $F(h) \neq 0$, we obtain that $h \notin U + V$. A little more strong necessary condition, in case of k subalgebras, is the separation of disjoint closed sets in X. We say that the k-tuple of algebras $(A_1, ..., A_k)$ separates disjoint closed sets in X if, for any closed $P, Q \subset X$ with $P \cap Q = \emptyset$, there exists an algebra A_i and a function $g \in A_i$ such that the images of g on P and Q are different. This condition is necessary, since a compact Hausdorff space X is a normal topological space and, by Urysohn's lemma, any two disjoint closed sets P and Q in X can be separated by some function $f \in C(X)$. Note that this condition is also not sufficient. Below we give a corresponding example, which we also refer to in the sequel. This highly nontrivial example belongs to S. Ya. Khavinson (see [13]). Let $\Omega \subset \mathbb{R}^2$ consist of a broken line whose sides are parallel to the coordinate axis and whose vertices are

$$(0; 0), (1; 0), (1; 1), (1 + \frac{1}{2^2}; 1), (1 + \frac{1}{2^2}; 1 + \frac{1}{2^2}), (1 + \frac{1}{2^2} + \frac{1}{3^2}; 1 + \frac{1}{2^2}), \ldots$$

We add to this line the limit point of the vertices $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$. Clearly, Ω is a compact set. Let U and V be the algebras considered above. Then it is not difficult to see that the tuple (U, V) separates disjoint closed sets of Ω. Besides, every function f on Ω is of the form $s(x) + t(y)$. Indeed, we can put $s(0) = a$, where a is any real number, and define s and t uniquely from the equation $f(x, y) = s(x) + t(y)$. Now construct a function f_0 on Ω as follows. On the link joining $(0; 0)$ to $(1; 0)$, f_0 continuously increases from 0 to 1; on the link from $(1; 0)$ to $(1; 1)$ it continuously decreases from 1 to 0; on the link from $(1; 1)$ to $(1 + \frac{1}{2^2}; 1)$ it increases from 0 to $\frac{1}{2}$; on the link from $(1 + \frac{1}{2^2}; 1)$ to $(1 + \frac{1}{2^2}; 1 + \frac{1}{2^2})$ it decreases from $\frac{1}{2}$ to 0; on the next link it increases from 0 to $\frac{1}{3}$, etc. At the point $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$ set the value of f_0 equal to 0. Obviously, f_0 is a continuous function on Ω. In addition, by the above argument, $f_0(x, y) = s(x) + t(y)$. But s and t cannot be chosen as continuous functions, since they get unbounded as x and y tend to $\frac{\pi^2}{6}$.

The above simple separation conditions were pointed out and generalized by Y. Sternfeld in a number of papers. He obtained necessary and sufficient separation conditions for the representation of the classes of bounded and continuous functions. For the problem of representation $A_1 + \cdots + A_k = C(X)$, he proved that the representation holds if and only if $(A_1, ..., A_k)$ separates regular Borel measures on X. In order to formulate his condition, we continue with the definition of some notions associated with the algebras A_i, $i = 1, ..., k$. First define the equivalence relation R_i, $i = 1, ..., k$, for elements in X by setting

$$a \overset{R_i}{\sim} b \text{ if } f(a) = f(b) \text{ for all } f \in A_i. \tag{1.1}$$

Obviously, for each $i = 1, ..., k$, the quotient space $X_i = X/R_i$ with respect to the relation R_i, equipped with the quotient space topology, is compact. In addition, the natural projections $s_i : X \to X_i$ are continuous. Note that the quotient spaces X_i are not only compact but also Hausdorff (see, e.g., [14, p. 54]).

In view of the Stone–Weierstrass theorem, we can write that

$$A_i = \{f(s_i(x)) : \ f \in C(X_i)\}, \ i = 1, ..., k. \tag{1.2}$$

Let $C^*(X)$ denote the class of regular Borel measures on X (that is, measures defined on the smallest σ-algebra that contains the open sets of X) and $\mathcal{S} = \{s\}$ be a family of mappings defined on X. We say that $\mathcal{S}$ uniformly separates measures of $C^*(X)$ if there exists a number $0 < \lambda \leq 1$ such that, for each $\mu \in C^*(X)$, the equality $\|\mu \circ s^{-1}\| \geq \lambda \|\mu\|$ holds for some $s \in \mathcal{S}$. Sternfeld proved that $A_1 + \cdots + A_k = C(X)$ if and only if the family $\{s_1, \ldots, s_k\}$ uniformly separates measures of the class $C(X)^*$ (see [29]).

Although the above separation condition of Sternfeld is both necessary and sufficient for the representation, it is hardly practical. Sproston and Straus [27] gave a practically convenient sufficient condition for the sum $A_1 + \cdots + A_k$ to be the whole of $C(X)$. To describe the condition, define the set functions

$$\tau_i(Z) = \{x \in Z : \ |s_i^{-1}(s_i(x)) \bigcap Z| \geq 2\}, \quad Z \subset X, \ i = 1, \ldots, k,$$

where $|Y|$ denotes the cardinality of a considered set Y. Define $\tau(Z)$ to be $\bigcap_{i=1}^{k} \tau_i(Z)$ and define $\tau^2(Z) = \tau(\tau(Z))$, $\tau^3(Z) = \tau(\tau^2(Z))$ and so on inductively. The result of [27] says that $A_1 + \cdots + A_k = C(X)$ provided that $\tau^n(X) = \emptyset$ for some positive integer n. In fact, this condition first appeared in the work of Sternfeld [28], where the author proved that $\tau^n(X) = \emptyset$ (for some n) guarantees that the family $\{s_1, \ldots, s_k\}$ uniformly separates regular Borel measures if X is a compact metric space.

Sproston and Straus proved the last statement for X being a compact Hausdorff space. For $k = 2$, the condition is also necessary for the representation, but not in general if $k > 2$ (see the counterexample in [27]).

Note that the above condition $\tau^n(X) = \emptyset$ is more geometric than measure theoretic. It holds if points of X are of a certain geometrical structure. This is easily seen in the case of two subalgebras. For $k = 2$, the condition $\tau^n(X) = \emptyset$ can be expressed in terms of sets of points in X that are geometrically explicit. In the special case of the algebras U and V considered above, these points were introduced in the literature under different names such as "permissible lines" [4] "bolts of lightning" [1,6,7,13,14,21,22], "trips" [20], "paths" [5,8,10,18,19], "links" [3,15], etc. The term bolt of lightning is the most common one and is due to Arnold [1]. It first appeared in his solution to Hilbert's thirteenth problem. Note that a *bolt of lightning* is a finite ordered subset $L = \{p_1, p_2, \cdots p_n\}$ in $\mathbb{R}^2$ such that $p_i \neq p_{i+1}$, each line segment $p_i p_{i+1}$ (unit of the bolt) is parallel to the coordinate axis x or y, and two adjacent units $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$ are perpendicular. A bolt of lightning L is said to be closed if $p_n p_1 \perp p_1 p_2$ (in this case, n is an even number). For a compact set $X \subset \mathbb{R}^2$ and the algebras $U = \{u(x)\}$, $V = \{v(y)\}$, it is not difficult to prove that $\tau^n(X) = \emptyset$ if and only if there are no closed bolts in X and the lengths (number of points) of all bolts are uniformly bounded (see [14]).

The purpose of this paper is to obtain a necessary condition of the type "$\tau^n(X) = \emptyset$" for the representation $A_1 + \cdots + A_k = C(X)$. For this purpose, we introduce in the next section new objects called "cycles" and "semicycles" with respect to finitely many subalgebras of $C(X)$.

2. Cycles and semicycles with respect to a family of algebras

We begin this section with the definition of two objects, which are essential for the further analysis of the considered representation problem. Assume, as above, X is a compact Hausdorff space, $C(X)$ is the space of continuous real-valued functions on X and A_i, $i = 1, ..., k$, are closed subalgebras of $C(X)$ that contain the constants. As it is shown above these algebras can be written in the form (1.2).

Cycles with respect to the algebras A_i, $i = 1, ..., k$, are defined as follows.

Definition 2.1. A set of points $l = (x_1, \ldots, x_n) \subset X$ is called a cycle with respect to the algebras A_i, $i = 1, ..., k$, if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with the nonzero integer coordinates λ_j such that

$$\sum_{j=1}^{n} \lambda_j \delta_{s_i(x_j)} = 0, \quad \text{for all } i = 1, \ldots, k.$$

Here, δ_a is a characteristic function of the unit set $\{a\}$.

For example, the set $l = \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (1,1,1)\}$ is a cycle in $\mathbb{I}^3$, $\mathbb{I} = [0,1]$, with respect to the algebras $A_i = \{p(z_i)\colon\ p \in C[0,1]\}$, $i = 1, 2, 3$. The vector λ in Definition 2.1 can be taken as $(-2, 1, 1, 1, -1)$.

The idea of cycles with respect to k directions in $\mathbb{R}^d$ was first implemented by Braess and Pinkus [2] in a work devoted to ridge function interpolation. Klopotowski, Nadkarni, Rao [16] defined cycles of minimal lengths with respect to canonical projections and called them *loops*. In Ismailov [11], these objects (under the name of *closed paths*) have been generalized to those having association with k arbitrary functions. It was proven in [2] that the nonexistence of cycles with respect to k directions is necessary and sufficient for interpolation by ridge functions. It was proven in [16] that the nonexistence of cycles with respect to canonical projections in $\mathbb{R}^k$ is necessary and sufficient for representation of multivariate functions by sums of univariate functions. It was proven in [11] that the nonexistence of cycles with respect to k arbitrary functions is necessary and sufficient for representation by linear superpositions.

Note that results of the above-mentioned works [2,11,16] are topology-free. The above example of Khavinson shows that consideration of only cycles is not enough for investigating the problems of representation when the topology of continuity is involved (see also [12]). The set Ω does not contain closed bolts (that is, cycles with respect to the algebras U and V), but at the same time $U + V \neq C(\Omega)$. This tells us that to approach the problem of the representation $A_1 + \cdots + A_k = C(X)$, we need more general objects than cycles.

Definition 2.2. A set of points $l = (x_1, \ldots, x_n) \subset X$ is called a semicycle with respect to the algebras A_i, $i = 1, ..., k$, if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with the nonzero integer coordinates λ_j such that for any $i = 1, ..., k$,

$$\sum_{j=1}^{n} \lambda_j \delta_{s_i(x_j)} = \sum_{t=1}^{r_i} \lambda_{i_t} \delta_{s_i(x_{i_t})}, \quad \text{where } r_i \leq k. \tag{2.1}$$

Note that for $i = 1, \ldots, k$, the set $\{\lambda_{i_t},\ t = 1, ..., r_i\}$ is a subset of the set $\{\lambda_j,\ j = 1, ..., n\}$. This means that, for each i, we have at most k terms in the sum $\sum_{j=1}^{n} \lambda_j \delta_{s_i(x_j)}$. Further note that in (2.1) the sum $\sum_{t=1}^{r_i} \lambda_{i_t} \delta_{s_i(x_{i_t})}$ is allowed over an empty subset of the set $\lambda = (\lambda_1, \ldots, \lambda_n)$ with value zero. Thus we see that every cycle is also a semicycle.

Assume, for example, that we are given two algebras A_1 and A_2 with quotient mappings s_1 and s_2, respectively. Assume $l = \{x_1, x_2, ..., x_n\}$ is an ordered set with the property

$$s_1(x_1) = s_1(x_2), s_2(x_2) = s_2(x_3), s_1(x_3) = s_1(x_4), ..., s_2(x_{n-1}) = s_2(x_n).$$

It is not difficult to see that, for a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ with the components $\lambda_i = (-1)^i$,

$$\sum_{j=1}^{n} \lambda_j \delta_{s_1(x_j)} = \lambda_n \delta_{s_1(x_n)},$$

$$\sum_{j=1}^{n} \lambda_j \delta_{s_2(x_j)} = \lambda_1 \delta_{s_2(x_1)}.$$

Thus, by Definition 2.2, the set $l = \{x_1, \dots, x_n\}$ forms a semicycle with respect to the algebras A_1 and A_2.

Note that in Marshall and O'Farrell [21], a finite sequence $(x_1, ..., x_n)$ with $x_i \neq x_{i+1}$ satisfying either $s_1(x_1) = s_1(x_2)$, $s_2(x_2) = s_2(x_3)$, $s_1(x_3) = s_1(x_4)$, ..., or $s_2(x_1) = s_2(x_2)$, $s_1(x_2) = s_1(x_3)$, $s_2(x_3) = s_2(x_4)$, ..., is called a *bolt* with respect to (A_1, A_2). If $(x_1, ..., x_n, x_1)$ is a bolt and n is an even number, then the bolt $(x_1, ..., x_n)$ is called closed. These objects are straightforward generalization of classical bolts (see Introduction) and appeared in several results concerning the density of $A_1 + A_2$ in $C(X)$. Bolts with respect to (A_1, A_2) are essential for the description of regular Borel measures orthogonal to $A_1 + A_2$ (see [21]).

A cycle (or semicycle) l is called a *q-cycle* (*q-semicycle*) if the vector λ associated with l can be chosen so that $|\lambda_i| \le q$, $i = 1, ..., n$, and q is the minimal number with this property.

The semicycle considered above is a 1-semicycle. If, in that example, $s_2(x_{n-1}) = s_2(x_1)$, then the set $\{x_1, x_2, ..., x_{n-1}\}$ is a 1-cycle. Let us give a simple example of a 2-cycle with respect to the algebras $U = \{u(x)\}$, $V = \{v(y)\}$ considered above. Consider the union

$$\{0, 1\}^2 \cup \{0, 2\}^2 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0), (0, 2), (2, 0)\}.$$

Clearly, this set is a 2-cycle with the associated vector $(2, 1, 1, -1, -1, -1, -1)$. Similarly, one can construct a q-cycle or q-semicycle for any positive integer q.

Each semicycle $l = (x_1, \dots, x_n)$ and an associated vector $\lambda = (\lambda_1, \dots, \lambda_n)$ generate the following functional

$$F_{l,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad f \in C(X). \tag{2.2}$$

Obviously, $F_{l,\lambda}$ is a bounded linear functional with norm $\sum_{j=1}^{n} |\lambda_j|$.

From Definition 2.2, it follows that, for each function $g_i \in A_i$, $i = 1, \dots, k$,

$$F_{l,\lambda}(g_i) = \sum_{j=1}^{n} \lambda_j g_i(x_j) = \sum_{t=1}^{r_i} \lambda_{i_t} g_i(x_{i_t}), \tag{2.3}$$

where $r_i \le k$. That is, for each algebra A_i, $F_{l,\lambda}$ is a linear combination of point evaluation functionals, where not more than k points of the semicycle l are used. Note that if l is a cycle, then automatically $F_{l,\lambda}(g_i) = 0$ for all $g_i \in A_i$, $i = 1, \dots, k$. Hence, $F_{l,\lambda}(g) = 0$, for any $g \in A_1 + \dots + A_k$.

Remark 1. Assume $f \in C(X)$ and for $i = 1, ..., k$, A_i is a subalgebra of $C(X)$ generated by one element $w_i \in A_i$. Following Khavinson, we say that an algebra $A \subset C(X)$ is generated by an element $w \in A$ if $A = \{h(w(x) : h \in C(\mathbb{R})\}$ (see [14, p. 33]). Note that $a \overset{R_i}{\sim} b$ if and only if $w_i(a) = w_i(b)$; thus any cycle with respect to the algebras A_i is a cycle with respect to the real-valued functions w_i and vice versa. The latter is defined similarly provided that in Definition 2.1 we replace s_i with w_i. If $F_{l,\lambda}(f) = 0$, for any cycle $l \subset X$, then $f = \sum_{i=1}^{k} h_i \circ w_i$, where $h_i : \mathbb{R} \to \mathbb{R}$ are some functions (not necessarily continuous) depending on f (see [11]). It follows that f is decomposed into the sum $\sum_{i=1}^{k} f_i \circ s_i$, where $s_i : X \to X_i$, $i = 1, ..., k$, are the natural projections defined above and $f_i : X_i \to \mathbb{R}$. But this does not mean that we can always choose f_i continuous on X_i (see Khavinson's example in Introduction). We conclude that, in general, f may not belong to $A_1 + \dots + A_k$ even if $F_{l,\lambda}(f) = 0$ for all cycles l in X.

The following theorem is valid.

Theorem 2.1. *Let $A_1 + \dots + A_k = C(X)$. Then*

(Z_1) *there are no cycles in X;*
(Z_2) *for each $q \in \mathbb{N}$, the lengths (number of points) of all q-semicycles in X are uniformly bounded.*

Proof. The part (Z_1) is obvious. Indeed, let $l = (x_1, \dots, x_n)$ be a cycle in X and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a vector associated with it. As it is shown above, $F_{l,\lambda}(g) = 0$ for all functions $g \in A_1 + \dots + A_k$. Let g_0 be a continuous function such that $g_0(x_j) = 1$

if $\lambda_j > 0$ and $g_0(x_j) = -1$ if $\lambda_j < 0$, $j = 1, \dots, n$. Since $F_{l,\lambda}(g_0) \neq 0$, the function g_0 cannot be in $A_1 + \dots + A_k$. Therefore, $A_1 + \dots + A_k \neq C(X)$. But this contradicts the hypothesis of the theorem.

Let us prove (Z_2)-part of the theorem. Consider the linear space

$$A = \prod_{i=1}^{k} A_i = \{(g_1, \dots, g_k) : \; g_i \in A_i, \; i = 1, \dots, k\}$$

endowed with the norm

$$\|(g_1, \dots, g_k)\| = \|g_1\| + \dots + \|g_k\|.$$

By A^* we denote the dual of the space A. Obviously, each functional $G \in A^*$ can be written as the sum

$$G = G_1 + \dots + G_k,$$

where the functionals $G_i \in A_i^*$ and

$$G_i(g_i) = G[(0, \dots, g_i, \dots, 0)], \quad i = 1, \dots, k.$$

Thus, the functional G determines the collection $(G_1, \dots, G_k)$, and, vice versa, every collection $(G_1, \dots, G_k)$ of continuous linear functionals $G_i \in A_i^*$, $i = 1, \dots, k$, determines the functional $G_1 + \dots + G_k$ on A. Considering this, in what follows, the elements of A^* will be denoted by $(G_1, \dots, G_k)$.

It is not difficult to verify that

$$\|(G_1, \dots, G_k)\| = \max\{\|G_1\|, \dots, \|G_k\|\}. \tag{2.4}$$

Consider the operator

$$T : A \to C(X), \quad T[(g_1, \dots, g_k)] = g_1 + \dots + g_k.$$

Clearly, T is a linear continuous operator with norm $\|T\| = 1$. In addition, since $A_1 + \dots + A_k = C(X)$, T is a surjection. Let us consider also the conjugate operator

$$T^* : C(X)^* \to A^*, \; T^*[H] = (G_1, \dots, G_k),$$

where $G_i(g_i) = H(g_i)$, for any $g_i \in A_i$, $i = 1, \dots, k$. Let H be an arbitrary functional $F_{l,\lambda}$ of the form (2.2), where $l = (x_1, \dots, x_n)$ is a q-semicycle. Set $T^*[F_{l,\lambda}] = (F_1, \dots, F_k)$. From (2.3) we obtain that

$$|F_i(g_i)| = |F_{l,\lambda}(g_i)| \leq \|g_i\| \sum_{t=1}^{r_i} |\lambda_{i_t}| \leq r_i q \|g_i\| \leq kq\|g_i\|, \quad i = 1, \dots, k.$$

Hence,

$$\|F_i\| \leq kq, \quad i = 1, \dots, k.$$

From (2.4), it follows that

$$\|T^*[F_{l,\lambda}]\| = \|(F_1, \dots, F_k)\| \leq kq. \tag{2.5}$$

Since T is a surjection, there exists a number $\epsilon > 0$ such that

$$\|T^*[H]\| \geq \epsilon \|H\|$$

for any functional $H \in C(X)^*$ (see Rudin [26]). Considering the equality $\|F_{l,\lambda}\| = \sum_{j=1}^{n} |\lambda_j|$, for the functional $H = F_{l,\lambda}$ we can write that

$$\|T^*[F_{l,\lambda}]\| \geq \epsilon \sum_{j=1}^{n} |\lambda_j|. \tag{2.6}$$

We obtain from (2.5) and (2.6) that

$$\epsilon \leq \frac{kq}{\sum_{j=1}^{n} |\lambda_j|}.$$

Since $\epsilon > 0$, it follows from the last inequality that n cannot be arbitrarily large. Thus we conclude that the lengths of all q-semicycles in X must be uniformly bounded. □

Corollary 2.1. *If $k = 2$, then the conditions (Z_1) and (Z_2) together are both necessary and sufficient for the representation $A_1 + \dots + A_k = C(X)$. Moreover, in (Z_2), the consideration of only 1-semicycles suffices.*

Proof. Necessity is obvious (it follows directly from Theorem 2.1). To prove the sufficiency, note that a bolt with different points is a 1-semicycle and if X does not contain closed bolts, then it does not contain bolts with overlapping points. This is because a bolt with overlapping points always contains a closed bolt. Thus, it immediately follows that X does not contain closed bolts and that the lengths of all bolts with different points are uniformly bounded by some positive integer N.

For $i = 1, 2$, let X_i be the quotient space of X induced by the equivalence relation (1.1) and s_i be the corresponding quotient mappings. Note that the relation $x \sim y$ when x and y belong to some bolt in X defines an equivalence relation. Following Marshall and O'Farrell [20], let us call equivalence classes *orbits*. For a point $x \in X$ set $Y_1 = s_1^{-1}(s_1[x])$, $Y_2 = s_2^{-1}(s_2[Y_1])$, $Y_3 = s_1^{-1}(s_1[Y_2])$, ... By $O(x)$ denote the orbit of X containing x. Since the lengths of all bolts in X are not greater than N, we conclude that $O(x) = Y_N$. Since X is compact, the sets $Y_1, Y_2, ..., Y_N$, hence $O(x)$, are topologically closed sets. In [20], Marshall and O'Farrell proved the following result (see Proposition 2 in [20]): let X be a compact Hausdorff space. Let A_1 and A_2 be closed subalgebras of $C(X)$ that contain the constants. Suppose all orbits are closed. Then $A_1 + A_2$ is uniformly dense in $C(X)$ if and only if X contains no closed bolt with respect to (A_1, A_2).

It follows from this proposition that $\overline{A_1 + A_2} = C(X)$. Note that under the hypothesis of the corollary, $A_1 + A_2$ is closed in $C(X)$. The closedness follows from the result of Medvedev (see Theorem 1 in [22]): the sum $A_1 + A_2$ is closed in $C(X)$ if and only if there exists a positive integer N such that the lengths of bolts in X are bounded by N. Thus we obtain that $A_1 + A_2$ is both dense and closed in $C(X)$. Hence $A_1 + A_2 = C(X)$. The sufficiency is proved. □

Remark 2. Assume A is a closed subalgebra of $C(X)$ that contains the constants. A version of the Stone–Weierstrass theorem states that A coincides with the whole space $C(X)$ if and only if A separates points of X (that is, for any two different points x and y in X, there exists a function $g \in A$ such that $g(x) \neq g(y)$). Note that any bolt with respect to (A, A) consisting of two points x_1 and x_2 is automatically closed. Indeed, in this case, if (x_1, x_2) is a bolt, then (x_1, x_2, x_1) is also a bolt. On the other hand, (x_1, x_2) is a bolt with respect to (A, A) if and only if $f(x_1) = f(x_2)$ for all $f \in A$. Thus, we conclude that the above version of the Stone–Weierstrass theorem is equivalent to Corollary 2.1, provided that $A_1 = A_2$.

Remark 3. For the case of a compact set $X \subset \mathbb{R}^2$ and the algebras $U = \{u(x)\}$, $V = \{v(y)\}$ of univariate functions defined on the projections of X into the coordinate axes x and y, respectively, Corollary 2.1 was first obtained by Khavinson (see [14]). Implementing the separation theory of Sternfeld [28], Khavinson [14] extended his result also to the case of linear superpositions. Using ideas of Khavinson and Marshall O'Farrell's lightning bolt principle (see [20,21]), one of the authors [9] proved Corollary 2.1 for ridge functions and linear superpositions.

Remark 4. We see that the conditions (Z_1) and (Z_2) of Theorem 2.1 are sufficient for the equality $A_1 + A_2 = C(X)$. This means that in the case $k = 2$, these conditions are equivalent to the condition "$\tau^n(X) = \emptyset$" of Sternfeld. Note that for $k > 2$, they are not equivalent, since the condition of Sternfeld is not necessary for the representation $A_1 + \cdots + A_k = C(X)$ (see [27]). One may ask if, for $k > 2$, the conditions (Z_1) and (Z_2) are sufficient for the representation $A_1 + \cdots + A_k = C(X)$. This question, unfortunately, has a negative answer. To see this, let $M(X)$ denote the space of bounded functions on X. Consider the spaces

$$B_i = \{f(s_i(x)) : f \in M(X_i)\}, \ i = 1, ..., k,$$

and also the space $B_1 + \cdots + B_k$. Clearly, $B_1 + \cdots + B_k \subset M(X)$. It can be proven by the same way that the conditions (Z_1) and (Z_2) are necessary for the equality $B_1 + \cdots + B_k = M(X)$. If the conditions ($Z_1$) and ($Z_2$) had been sufficient for $A_1 + \cdots + A_k = C(X)$, they would have been also sufficient for $B_1 + \cdots + B_k = M(X)$, since the representation $A_1 + \cdots + A_k = C(X)$ implies the representation $B_1 + \cdots + B_k = M(X)$ (see [28]). Then we would obtain that the conditions (Z_1) and (Z_2) are necessary and sufficient for both the equalities $A_1 + \cdots + A_k = C(X)$ and $B_1 + \cdots + B_k = M(X)$. But it was shown in Sternfeld [29] that for $k > 2$, these two equalities are not equivalent.

Remark 5. If $C(X) = A_1 + \cdots + A_k$, then the map $\varphi : x \to (s_1(x), ..., s_k(x))$ is a continuous one-to-one map from the compact space X into the compact space $X_1 \times \cdots \times X_k$, hence a homeomorphism of X onto $\phi(X) \subset X_1 \times \cdots \times X_k$. Thus one can identify X with $\phi(X)$ and s_i with the projection of $\phi(X)$ onto X_i. In the light of this, some relevant results were obtained in [17,23–25]. In particular, the survey paper of Nadkarni [23] formulates definitions of "path" and "geodesic" (a path of shortest length) for $k \geq 2$, which agrees with the known definitions for $k = 2$. It further discusses sufficient conditions for $C(X) = A_1 + \cdots + A_k$, in terms of uniform boundedness of lengths of geodesics.

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