



Lie algebras/Topology

Goldman–Turaev formality from the Knizhnik–Zamolodchikov connection



Connexion de Knizhnik–Zamolodchikov et formalité pour la bigèbre de Lie de Goldman–Turaev

Anton Alekseev, Florian Naef

Department of Mathematics, University of Geneva, 2–4, rue du Lièvre, C.P. 64, CH-1211 Geneva, Switzerland

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ABSTRACT

For an oriented 2-dimensional manifold Σ of genus g with n boundary components, the space $\mathbb{C}\pi_1(\Sigma)/[\mathbb{C}\pi_1(\Sigma), \mathbb{C}\pi_1(\Sigma)]$ carries the Goldman–Turaev Lie bialgebra structure defined in terms of intersections and self-intersections of curves. Its associated graded Lie bialgebra (under the natural filtration) is described by cyclic words in $H_1(\Sigma)$ and carries the structure of a necklace Schedler Lie bialgebra. The isomorphism between these two structures in genus zero has been established in [13] using Kontsevich integrals and in [2] using solutions of the Kashiwara–Vergne problem.

In this note, we give an elementary proof of this isomorphism over \mathbb{C} . It uses the Knizhnik–Zamolodchikov connection on $\mathbb{C}\setminus\{z_1, \dots, z_n\}$. We show that the isomorphism naturally depends on the complex structure on the surface. The proof of the isomorphism for Lie brackets is a version of the classical result by Hitchin [9]. Surprisingly, it turns out that a similar proof applies to cobrackets.

Furthermore, we show that the formality isomorphism constructed in this note coincides with the one defined in [2] if one uses the solution of the Kashiwara–Vergne problem corresponding to the Knizhnik–Zamolodchikov associator.

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R É S U M É

Soit Σ une variété orientable de dimension 2, de genre g et avec n composants de bord. L'espace $\mathbb{C}\pi_1(\Sigma)/[\mathbb{C}\pi_1(\Sigma), \mathbb{C}\pi_1(\Sigma)]$ a une structure de bigèbre de Lie de Goldman–Turaev définie par les intersections et les auto-intersections des courbes sur Σ . La bigèbre de Lie graduée associée (par rapport à la filtration naturelle) est décrite par l'espace des mots cycliques en $H_1(\Sigma)$. En genre zéro, l'isomorphisme entre ces deux bigèbres de Lie a été établi dans [13] en utilisant l'intégrale de Kontsevich, et dans [2] en utilisant les solutions du problème de Kashiwara–Vergne.

Dans cette note, nous donnons une démonstration élémentaire de cet isomorphisme sur \mathbb{C} . Notre démonstration utilise la connexion de Knizhnik–Zamolodchikov sur $\mathbb{C}\setminus\{z_1, \dots, z_n\}$. Nous montrons que cet isomorphisme dépend naturellement de la structure complexe sur Σ . La preuve de l'isomorphisme pour le crochet de Lie est une version d'un résultat

E-mail addresses: Anton.Alekseev@unige.ch (A. Alekseev), Florian.Naef@unige.ch (F. Naef).

classique de Hitchin [9]. D'une manière surprenante, un argument similaire s'applique également au cocrochet. De plus, nous montrons que l'isomorphisme de formalité construit dans cette note coïncide avec l'isomorphisme défini dans [2] si on choisit la solution du problème de Kashiwara–Vergne, qui correspond à l'associateur de Knizhnik–Zamolodchikov.

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1. Holonomy maps

In this section we recall the definition of the Knizhnik–Zamolodchikov connection and define the holonomy map using iterated integrals.

1.1. The Knizhnik–Zamolodchikov connection

For $n \in \mathbb{Z}_{\geq 1}$ let \mathfrak{t}_{n+1} be the Drinfeld–Kohno Lie algebra of infinitesimal braids. It has generators $t_{ij} = t_{ji}$ with $i \neq j$ for $i, j = 1, \dots, n + 1$ and relations $[t_{ij}, t_{kl}] = 0$ for all quadruples of distinct labels i, j, k, l and $[t_{ij} + t_{ik}, t_{jk}] = 0$ for all triples i, j, k . The Lie subalgebra of \mathfrak{t}_{n+1} generated by $a_i = t_{i(n+1)}$ for $i = 1, \dots, n$ is a free Lie algebra with n generators.

There is a canonical flat Knizhnik–Zamolodchikov (KZ) connection on the configuration space $\text{Conf}_{n+1}(\mathbb{C})$ of $n + 1$ points $z_1, \dots, z_{n+1} \in \mathbb{C}$ with values in \mathfrak{t}_{n+1} :

$$d + A_{KZ} = d + \frac{1}{2\pi i} \sum_{i < j} t_{ij} \, d \log(z_i - z_j).$$

The fiber of the forgetful map $\text{Conf}_{n+1}(\mathbb{C}) \rightarrow \text{Conf}_n(\mathbb{C})$ (by forgetting the point $z = z_{n+1}$) over $(z_1, \dots, z_n) \in \text{Conf}_n(\mathbb{C})$ is given by $\Sigma = \mathbb{C} \setminus \{z_1, \dots, z_n\}$. The restriction of the KZ connection to the fiber is of the form:

$$d + A = d + \frac{1}{2\pi i} \sum_{i=1}^n a_i \, d \log(z - z_i). \tag{1}$$

Note that the 1-forms $\frac{1}{2\pi i} \, d \log(z - z_i)$ form a basis in the cohomology $H^1(\Sigma)$. Then, we can naturally view a_i 's as a basis of $H_1 = H_1(\Sigma)$. In fact, this is a natural basis of H_1 given by the cycles around the marked points. It is convenient to use the notation

$$A(z) = \frac{1}{2\pi i} \sum_{i=1}^n a_i \, d \log(z - z_i) \in \Omega^1(\Sigma) \otimes H_1.$$

1.2. Free algebras and iterated integrals

The degree completed Hopf algebra TH_1 over \mathbb{C} is naturally isomorphic to the completed universal enveloping algebra of the free Lie algebra $\mathbb{L}(a_1, \dots, a_n)$ with generators a_1, \dots, a_n . Group-like elements in TH_1 form a group $G_n = \exp(\mathbb{L}(a_1, \dots, a_n))$ isomorphic to $\mathbb{L}(a_1, \dots, a_n)$ equipped with the group law defined by the Baker–Campbell–Hausdorff series.

For a path γ parametrized by $z : [0, 1] \rightarrow \Sigma$, we define the holonomy of the connection $d + A$ using iterated integrals:

$$\begin{aligned} \text{Hol}_\gamma &= \sum_{k=0}^{\infty} \int_{1 \geq t_1 \geq \dots \geq t_k \geq 0} A(z(t_1)) \dots A(z(t_k)) \\ &= \sum_{\mathbf{i}} a_{i_1} \dots a_{i_k} \int_\gamma \frac{dz(t_1)}{z(t_1) - z_{i_1}} \circ \dots \circ \frac{dz(t_k)}{z(t_k) - z_{i_k}} \in G_n, \end{aligned}$$

where $\mathbf{i} = (i_1, \dots, i_k)$. Since the connection $d + A$ is flat, the holonomy Hol_γ is independent of the homotopy deformations of γ with fixed end points. Hence, for small deformations of γ one can view Hol_γ as a functions of its endpoints. The de Rham differential of this function is given by

$$d \text{Hol}_\gamma = A(z(1)) \text{Hol}_\gamma - \text{Hol}_\gamma A(z(0)).$$

Let γ be a closed path and consider the path $\gamma(s)$ that starts at $z(s)$, follows γ and ends at $z(s)$. The corresponding holonomy is denoted by $\text{Hol}_\gamma(s) = \text{Hol}_{\gamma(s)}$, and its de Rham differential has the form

$$d \text{Hol}_\gamma(s) = A(z(s)) \text{Hol}_\gamma - \text{Hol}_\gamma A(z(s)). \tag{2}$$

For a closed path γ and two points $s \neq t \in S^1$, we denote by $\gamma(t \leftarrow s)$ the oriented path starting at $z(s)$, following γ and ending at $z(t)$. The corresponding holonomy is denoted $\text{Hol}_\gamma(t \leftarrow s) = \text{Hol}_{\gamma(t \leftarrow s)}$ and its de Rham differential is given by

$$d \text{Hol}_\gamma(t \leftarrow s) = A(z(t)) \text{Hol}_\gamma(t \leftarrow s) - \text{Hol}_\gamma(t \leftarrow s) A(z(s)).$$

Choose a base point $p \in \Sigma$ and denote by $\pi_1 = \pi_1(\Sigma, p)$ the fundamental group of Σ with base point p . Let γ be a closed path based at p . The map

$$W : \gamma \mapsto \text{Hol}_\gamma$$

descends to a group homomorphism $\pi_1 \rightarrow G_n$ and induces an isomorphism of completed Hopf algebras $\mathbb{C}\pi_1 \rightarrow TH_1$.

Denote by

$$|\mathbb{C}\pi_1| = \mathbb{C}\pi_1 / [\mathbb{C}\pi_1, \mathbb{C}\pi_1]$$

the space spanned by conjugacy classes in π_1 and similarly

$$|TH_1| = TH_1 / [TH_1, TH_1]$$

the space spanned by cyclic words in H_1 . Note that $|\mathbb{C}\pi_1|$ is also isomorphic to the vector space spanned by free homotopy classes of loops in Σ . The map W induces a map $|W| : |\mathbb{C}\pi_1| \rightarrow |TH_1|$, which is independent of the base point p . In what follows, we will study properties of this map.

Remark 1.1. For higher genus surfaces, iterated integrals of the harmonic Magnus connection were studied in [10]. It is not known whether the harmonic Magnus expansion gives rise to a Goldman–Turaev formality map.

2. Goldman–Turaev formality

The space $|\mathbb{C}\pi_1|$ carries the canonical Goldman–Turaev Lie bialgebra structure, which depends on the framing of Σ . Moreover, it is canonically filtered by powers of the augmentation ideal of the group algebra $\mathbb{C}\pi_1$. The space $|TH_1|$ carries the necklace Schedler Lie bialgebra structure (see [14]), which depends on the choice of a basis in H_1 . The main result of this note is an elementary proof of the following theorem:

Theorem 2.1. *The map $|W| : |\mathbb{C}\pi_1| \rightarrow |TH_1|$ is an isomorphism of Lie bialgebras for the Goldman–Turaev Lie bialgebra structure on (completed) $|\mathbb{C}\pi_1|$ defined by the blackboard framing and the necklace Schedler Lie bialgebra structure on $|TH_1|$ defined by the natural basis $\{a_1, \dots, a_n\} \subset H_1$.*

2.1. Kirillov–Kostant–Souriau double bracket

Recall the Kirillov–Kostant–Souriau (KKS) double bracket (in the sense of van den Bergh [16]) on TH_1 , which is completely determined by its values on generators:

$$a_i \otimes a_j \mapsto \{a_i, a_j\} = \delta_{ij}(1 \otimes a_i - a_i \otimes 1) = \delta_{ij}(1 \wedge a_i).$$

One of the key observations is the following lemma (reinterpreting [6]).

Lemma 2.2.

$$\{A(z), A(w)\} = \frac{1}{2\pi i} (1 \wedge (A(z) - A(w))) \text{dlog}(z - w).$$

Proof. The proof is by the direct computation:

$$\begin{aligned} \{A(z), A(w)\} &= \frac{1}{(2\pi i)^2} \left\{ \sum_i a_i \text{dlog}(z - z_i), \sum_j a_j \text{dlog}(w - z_j) \right\} \\ &= \frac{1}{(2\pi i)^2} \sum_i (1 \wedge a_i) \text{dlog}(z - z_i) \text{dlog}(w - z_i) \\ &= \frac{1}{(2\pi i)^2} \sum_i (1 \wedge a_i) (\text{dlog}(z - z_i) - \text{dlog}(w - z_i)) \text{dlog}(z - w) \\ &= \frac{1}{2\pi i} (1 \wedge (A(z) - A(w))) \text{dlog}(z - w). \quad \square \end{aligned}$$

2.2. Variations of the holonomy map

In order to proceed we will need some notation from non-commutative differential geometry. Let $\partial_i : TH_1 \rightarrow TH_1 \otimes TH_1$ for $i = 1, \dots, n$ denote double derivations with the property $\partial_i a_j = \delta_{ij}(1 \otimes 1)$. They induce maps (denoted by the same letter) $\partial_i : |TH_1| \rightarrow TH_1$, and by composition $\partial_i \partial_j : |TH_1| \rightarrow TH_1 \otimes TH_1$. In what follows we will use the formulas for the first and second derivatives of the elements $|W_\gamma|$ for γ a closed path parametrized by a map $z : S^1 \rightarrow \Sigma$:

$$\begin{aligned} \partial_j |W_\gamma| &= \int_{S^1} \text{Hol}_\gamma(s) \, d\log(z(s) - z_j), \\ \partial_i \partial_j |W_\gamma| &= \int_{S^1 \times S^1} (\text{Hol}_\gamma(t \leftarrow s) \otimes \text{Hol}_\gamma(s \leftarrow t)) \, d\log(z(s) - z_i) \, d\log(z(t) - z_j), \end{aligned} \tag{3}$$

where in the expression for the second derivative the torus $S^1 \times S^1$ is oriented by the volume form $ds dt$.

2.3. Necklace Schedler Lie bialgebra

We will also make use of the maps $\text{Tr} : TH_1 \otimes TH_1 \rightarrow |TH_1|$ given by $\text{Tr}(a \otimes b) = |ab|$ and $\text{Tr}^{12} : TH_1 \otimes TH_1 \rightarrow |TH_1| \otimes |TH_1|$ which is the componentwise projection $\text{Tr}^{12}(a \otimes b) = |a| \otimes |b|$. In terms of the KKS double bracket, the necklace Schedler Lie bialgebra structure looks as follows: for $\psi, \psi' \in |TH_1|$, we have

$$\begin{aligned} [\psi, \psi'] &= \sum_{ij} \text{Tr}((\partial_i \psi \otimes \partial_j \psi')\{a_i, a_j\}), \\ \delta \psi &= \sum_{ij} \text{Tr}^{12}((\partial_i \partial_j \psi)\{a_i, a_j\}). \end{aligned}$$

Remark 2.3. The necklace Lie algebra structure corresponding to the KKS double bracket first appeared in [5]. It was then generalized to other double brackets and other quivers (a quiver is a part of a general definition of a necklace Lie algebra) in [12,7,4]. The first description of the cobracket is in [14]. The formulas above represent the KKS necklace Schedler Lie bialgebra structure in a form convenient for our purposes.

2.4. Proof of Theorem 2.1

We are now ready to give a proof of Theorem 2.1:

Proof of the Lie homomorphism. Let γ and γ' be two closed loops in Σ with a finite number of transverse intersections parametrized by $z, w : [0, 1] \rightarrow \Sigma$. Denote by (s_i, t_i) pairs of parameters corresponding to intersection points, $p_i = z(s_i) = w(t_i)$ and by $D_i(\varepsilon) \subset S^1 \times S^1$ small discs of radius ε around (s_i, t_i) with boundaries small circles $S_i(\varepsilon)$ positively oriented under the orientation defined by the volume form $ds dt$ on the torus $S^1 \times S^1$. Let $\epsilon_i = +1$ if the orientation induced by the form $ds dt$ on $T_{p_i} \Sigma$ coincides with the blackboard orientation and $\epsilon_i = -1$ otherwise.

We compute the necklace Lie bracket of the elements $|W_\gamma|$ and $|W_{\gamma'}|$:

$$\begin{aligned} [|W_\gamma|, |W_{\gamma'}|] &= \text{Tr}((\partial_i |W_\gamma| \otimes \partial_j |W_{\gamma'}|)\{a_i, a_j\}) \\ &= \int_{S^1 \times S^1} \text{Tr}((\text{Hol}_\gamma(s) \otimes \text{Hol}_{\gamma'}(t))\{A(z(s)), A(w(t))\}) \\ &= \frac{1}{2\pi i} \int_{S^1 \times S^1} \text{Tr}((\text{Hol}_\gamma(s) \otimes \text{Hol}_{\gamma'}(t))(1 \wedge (A(z(s)) - A(w(t)))) \, d\log(z - w) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{S^1 \times S^1 \setminus \cup_i D_i(\varepsilon)} d|\text{Hol}_\gamma(s) \text{Hol}_{\gamma'}(t)| \, d\log(z - w) \\ &= - \sum_i |\text{Hol}_\gamma(s_i) \text{Hol}_{\gamma'}(t_i)| \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{S_i(\varepsilon)} d\log(z - w) \\ &= \sum_i |\text{Hol}_\gamma(s_i) \text{Hol}_{\gamma'}(t_i)| \epsilon_i. \end{aligned}$$

Here, in the first line, we use the definition of the necklace Schedler Lie bracket, in the second line, the expression (3) for the first derivative of $|W_\gamma|$ and the definition of $A(z)$, in the third line, Lemma 2.2, in the fourth line, formula (2) for the differential of the holonomy map, in the fifth line, the Stokes formula, and in the sixth line the residue formula. Note that the result of the calculation is exactly the expression for the image under the map W of the Goldman bracket [8] of the loops γ and γ' . \square

Proof of the cobracket homomorphism. Next, consider a closed path γ parametrized by $z : S^1 \rightarrow \Sigma$. Assume that γ is an immersion with finitely many transverse self-intersections $p_i = z(s_i) = z(t_i)$ (it is convenient to have each point appear twice in the count with $s_j = t_i$ and $t_j = s_i$). Denote $\alpha^\pm(\varepsilon)$ the circles on $S^1 \times S^1$ corresponding to the parameters $t = s \pm \varepsilon$ and by β the strip between them. Similar to the proof above, let $D_i(\varepsilon)$ be small discs of radius ε around the self-intersection points, $S_i(\varepsilon)$ their positively oriented boundaries and ϵ_i signs of self-intersections defined as above. Note that $\epsilon_j = -\epsilon_i$ for i and j representing the same self-intersection point.

We compute the necklace cobracket of the element $|W_\gamma|$:

$$\begin{aligned} \delta|W_\gamma| &= \text{Tr}^{12}(\partial_i \partial_j |W_\gamma| \{a_i, a_j\}) \\ &= \int_{S^1 \times S^1} \text{Tr}^{12}(\text{Hol}_\gamma(t \leftarrow s) \otimes \text{Hol}_\gamma(s \leftarrow t) \{A(z(s)), A(z(t))\}) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{S^1 \times S^1 \setminus \beta(\varepsilon) \cup (\cup_i D_i(\varepsilon))} d(\text{Tr}^{12}(\text{Hol}_\gamma(t \leftarrow s) \otimes \text{Hol}_\gamma(s \leftarrow t)) \text{dlog}(z(s) - z(t))) \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\cup_i S_i(\varepsilon) \cup \alpha^+(\varepsilon) \cup \alpha^-(\varepsilon)} (|\text{Hol}(t \leftarrow s)| \otimes |\text{Hol}(s \leftarrow t)|) \text{dlog}(z(s) - z(t)) \\ &= \sum_i |\text{Hol}(t_i \leftarrow s_i)| \otimes |\text{Hol}(s_i \leftarrow t_i)| \epsilon_i + (1 \wedge |W_\gamma|) \frac{1}{2\pi i} \int_\gamma \text{dlog}(\dot{z}(t)) \\ &= \sum_i |\text{Hol}(t_i \leftarrow s_i)| \otimes |\text{Hol}(s_i \leftarrow t_i)| \epsilon_i + (|W_\gamma| \wedge |1|) \text{rot}(\gamma). \end{aligned}$$

Here, in the first line, we use the definition of the necklace Schedler cobracket, in the second line, the expression (3) for the second derivative of $|W_\gamma|$, in the third line, Lemma 2.2 and the expression (2) for the de Rham differential of the holonomy map, in the fourth line, the Stokes formula, and, in the fifth line, the definition of the rotation number $\text{rot}(\gamma)$. The resulting expression is exactly the Turaev cobracket of the path γ with respect to the blackboard framing, which defines the rotation number. \square

Remark 2.4. In the proof above, the calculation of the Goldman bracket of the holonomies follows the standard scheme, see [9], [3]. The calculation of the Turaev cobracket uses the same techniques, but it seems to be new.

Remark 2.5. In [15], the cobracket is defined on the space $|\mathbb{C}\pi_1|/|\mathbb{C}1|$, where $|1|$ is the homotopy class of the trivial loop on Σ . If one fixes a framing on Σ , this definition can be lifted to $|\mathbb{C}\pi_1|$ by the formula above (for details, see [11]).

3. The z-dependence of the holonomy map

In this section, we study the dependence of the holonomy map $|W| : |\mathbb{C}\pi_1| \rightarrow |TH_1|$ on the positions of the poles z_1, \dots, z_n in the connection (1). We will use the notation $|W^{(z)}|$ to make this z -dependence more explicit.

3.1. The bundle \mathcal{G}_n

Consider the natural fibration $\text{Conf}_{n+1} \rightarrow \text{Conf}_n$ and assign to a point of the base (z_1, \dots, z_n) the Goldman–Turaev Lie algebra of the fiber $|\mathbb{C}\pi_1(\Sigma)|$ for $\Sigma = \mathbb{C} \setminus \{z_1, \dots, z_n\}$. This defines a bundle of Lie bialgebras \mathcal{G}_n over Conf_n .

In a neighborhood of a point $z = (z_1, \dots, z_n) \in \text{Conf}_n$ the bundle \mathcal{G}_n can be trivialized as follows. Choose small open disks D_i centered at z_i such that $D_i \cap D_j = \emptyset$ for all $i \neq j$ and define a local chart $U^{(z)}$ around z that consists of points $w = (w_1, \dots, w_n) \in \text{Conf}_n$ such that $w_i \in D_i$. Note that the surface with boundary $\tilde{\Sigma} = \mathbb{C} \setminus \cup_i D_i$ is homotopically equivalent to $\Sigma^{(w)} = \mathbb{C} \setminus \{w_1, \dots, w_n\}$ for all $w \in U^{(z)}$ via the inclusion $\tilde{\Sigma} \subset \Sigma^{(w)}$. The canonical isomorphism $|\mathbb{C}\pi_1(\tilde{\Sigma})| \cong |\mathbb{C}\pi_1(\Sigma^{(w)})|$ defines a trivialization of \mathcal{G}_n in this chart.

The bundle \mathcal{G}_n carries a canonical flat connection defined by the following family of local flat sections. Let $\gamma : S^1 \rightarrow \mathbb{C}$ be a closed curve. Then, the homotopy class $[\gamma]$ defines a flat section over the complement of the subset of Conf_n where $z_i \in \gamma(S^1)$ for some i . Moreover, it is easy to see that there is a unique local flat section passing through each point of \mathcal{G}_n .

The canonical flat connection is compatible with the Goldman–Turaev Lie bialgebra structure on the fibers. Indeed, on flat sections over $U^{(z)}$ defined by closed curves in $\tilde{\Sigma}$, the Goldman bracket and Turaev cobracket take the same value independent of w (sufficiently close to z).

The flat bundle \mathcal{G}_n admits the following description: the fundamental group of Conf_n is the pure braid group PB_n . The bundle \mathcal{G}_n is induced by the natural representation of PB_n on the free group $\pi_1(\Sigma)$.

3.2. The bundle \mathcal{H}_n

The natural action of PB_n on the homology $H_1(\Sigma)$ is trivial. This trivial action lifts to $|TH_1|$ and gives rise to the trivial vector bundle $\mathcal{H}_n \rightarrow \text{Conf}_n$ with fiber $|TH_1|$.

The bundle \mathcal{H}_n carries a canonical flat connection. Recall that the Drinfeld–Kohno Lie algebra \mathfrak{t}_n acts on the free Lie algebra $\mathbb{L}(a_1, \dots, a_n)$ via the identification $a_i = t_{i(n+1)} \in \mathfrak{t}_{n+1}$. Since this action is induced by the adjoint action of \mathfrak{t}_{n+1} on itself, we will use the notation ad . For instance, $\text{ad}(t_{12})a_1 = [t_{12}, t_{1(n+1)}] = -[t_{2(n+1)}, t_{1(n+1)}] = -[a_2, a_1]$. In particular, the central element of \mathfrak{t}_n

$$c = \sum_{i < j} t_{ij}$$

acts by the inner derivation $\text{ad}(c)a = [a, a_1 + \dots + a_n]$.

The canonical connection on \mathcal{H}_n is (a version of) the KZ connection:

$$\nabla_n = d + \frac{1}{2\pi i} \sum_{i < j} \text{ad}(t_{ij}) \text{dlog}(z_i - z_j). \tag{4}$$

It is flat because the original KZ connection is flat and because the map ad is a Lie algebra homomorphism.

Remark 3.1. The image of the Lie algebra \mathfrak{t}_n under the map ad is contained in the Kashiwara–Vergne Lie algebra \mathfrak{kv}_n acting on $|TH_1|$ by derivations of the necklace Schedler Lie bialgebra structure [2]. Hence, the flat connection (4) is compatible with the Lie bialgebra structure on the fibers of \mathcal{H}_n .

3.3. The holonomy is an isomorphism of flat bundles with connection

For each $z = (z_1, \dots, z_n) \in \text{Conf}_n$ the holonomy map $|W^{(z)}|$ maps the fiber $|\mathbb{C}\pi_1(\Sigma^{(z)})|$ of \mathcal{G}_n to the fiber $|TH_1|$ of \mathcal{H}_n . This collection of maps for $z \in \text{Conf}_n$ defines a smooth bundle map $|W| : \mathcal{G}_n \rightarrow \mathcal{H}_n$.

Theorem 3.2. *The bundle map $|W|$ intertwines the canonical flat connection on \mathcal{G}_n and the connection ∇_n on \mathcal{H}_n .*

Proof. It is sufficient to compare the canonical connections and ∇_n on local charts. In more detail, let $\gamma : S^1 \rightarrow \tilde{\Sigma} = \sigma \setminus \cup_i D_i$ be a closed curve. Then, $[\gamma]$ defines a flat section of \mathcal{G}_n over $U^{(z)}$. It is then sufficient to show that $|W_\gamma^{(w)}|$ is a ∇_n -flat section of \mathcal{H}_n for $w \in U^{(z)}$.

Let $w : (-1, 1) \rightarrow \text{Conf}_n(\mathbb{C})$ be a smooth path in $U^{(z)}$ with $w(0) = z$ and the first derivative $w'(0) = u = (u_1, \dots, u_n)$. Choose a starting point on the curve γ and denote the corresponding map by $\hat{\gamma} : [0, 1] \rightarrow \tilde{\Sigma}$.

Consider the KZ connection on $\text{Conf}_{n+1}(\mathbb{C})$ and define the map $\mu = w \times \hat{\gamma} : (-1, 1) \times S^1 \rightarrow \text{Conf}_{n+1}(\mathbb{C})$. The pull-back of the KZ connection under μ is flat. For a given $s \in (-1, 1)$ the holonomy of the induced connection along $\{s\} \times \hat{\gamma}$, denoted by $\text{Hol}_\gamma(s)$, takes values in TH_1 and its projection to $|TH_1|$ coincides with $|W^{(w(s))}|$ (under the identification $t_{i(n+1)} = a_i$).

Since the pull-back connection is flat, the holonomies for different values of s can be compared using the following formula:

$$\text{Hol}_\gamma(s) = \text{Hol}_w(s \leftarrow 0) \text{Hol}_\gamma(0) \text{Hol}_w(0 \leftarrow s),$$

where $\text{Hol}_w(0 \leftarrow s) = \text{Hol}_w(s \leftarrow 0)^{-1}$ is the holonomy of the KZ connection for $n + 1$ points along the part of the path w between 0 and s . Differentiating this formula in s at $s = 0$, we obtain

$$\text{Hol}'_\gamma(0) = \frac{1}{2\pi i} \sum_{1 \leq i < j \leq n+1} \frac{u_i - u_j}{z_i - z_j} \text{ad}(t_{ij}) \text{Hol}_\gamma(0),$$

where $u_{n+1} = 0$ and $z_{n+1} = \hat{\gamma}(0)$. Projecting this equation to $|TH_1|$ yields

$$(\partial_s |W^{(w(s))}|)_{s=0} = \frac{1}{2\pi i} \sum_{1 \leq i < j \leq n} \frac{u_i - u_j}{z_i - z_j} \text{ad}(t_{ij}) |W^{(w(0))}|.$$

Here we have used the fact that operators $\text{ad}(t_{i(n+1)})$ are inner derivations of TH_1 and they act by zero on $|TH_1|$. The formula above shows that $|W_\gamma^{(w(s))}|$ is a flat section over $(-1, 1)$. Since $w(s)$ was an arbitrary path, this shows that $|W_\gamma^{(w)}|$ is a flat section for ∇_n . \square

Remark 3.3. The KZ connection on \mathcal{H}_n is induced by the flat connection on the trivial bundle $\mathcal{L}_n = \text{Conf}_n \times \mathbb{L}(a_1, \dots, a_n)$ defined by the same formula (4).

Let \mathfrak{g} be a Lie algebra (possibly $\mathbb{L}(a_1, \dots, a_n)$ itself) and consider the bundle of fiberwise Lie algebra homomorphisms of \mathcal{L}_n into \mathfrak{g} . This is the trivial bundle $\text{Conf}_n \times \mathfrak{g}^n$, where we identify a Lie algebra homomorphism $x : \mathbb{L}(a_1, \dots, a_n) \rightarrow \mathfrak{g}$ with the n -tuple $(x_1, \dots, x_n) = (x(a_1), \dots, x(a_n))$. The KZ connection induces a connection on this bundle, which is computed as follows:

$$\begin{aligned} ((d + A_{\text{KZ}})x)(a_k) &= dx_k + \frac{1}{2\pi i} \sum_{i < j} (t_{ij} \cdot x)(a_k) \, d\log(z_i - z_j) \\ &= dx_k + \frac{1}{2\pi i} \sum_{i < j} x(-[t_{ij}, a_k]) \, d\log(z_i - z_j) \\ &= dx_k + \frac{1}{2\pi i} \sum_i x([a_i, a_k]) \, d\log(z_i - z_k) \\ &= dx_k + \frac{1}{2\pi i} \sum_i [x_i, x_k] \, d\log(z_i - z_k). \end{aligned}$$

Flat sections of this connection are solutions of the Schlesinger equations for isomonodromic deformations:

$$\frac{\partial x_k}{\partial z_i} = \frac{1}{2\pi i} \frac{[x_k, x_i]}{z_k - z_i}, \quad \frac{\partial x_k}{\partial z_k} = -\frac{1}{2\pi i} \sum_{i \neq k} \frac{[x_k, x_i]}{z_k - z_i}. \tag{5}$$

For $\mathfrak{g} = \mathbb{L}(a_1, \dots, a_n)$, let $x_k = x_k(a_1, \dots, a_n, z_1, \dots, z_n) = x_k(z)$ be local solutions to (5), such that they generate $\mathbb{L}(a_1, \dots, a_n)$ in each fiber. We call (x_1, \dots, x_n) isomonodromic coordinates.

Let $\gamma \subset \Sigma^{(z)}$ be a closed curve. It defines a local flat section $[\gamma]$ of \mathcal{G}_n and by Theorem 3.2 it gives rise to a local flat section $|W_\gamma|$ of \mathcal{H}_n . By the considerations above, $|W_\gamma|$ is represented by a constant (independent of z_1, \dots, z_n) function of the isomonodromic coordinates (x_1, \dots, x_n) .

3.4. The moduli space of curves

Recall that the moduli space of genus zero curves with $n + 1$ marked points $\mathcal{M}_{0,n+1}$ can be defined as the quotient of the configuration space of $n + 1$ points on $\mathbb{C}P^1$ modulo the natural PSL_2 -action by Möbius transformations:

$$\mathcal{M}_{0,n+1} = \text{Conf}_{n+1}(\mathbb{C}P^1) / \text{PSL}_2.$$

Equivalently, one can fix one of the points to be at infinity of $\mathbb{C}P^1$ and consider:

$$\mathcal{M}_{0,n+1} \cong \text{Conf}_n(\mathbb{C}) / \Gamma,$$

where Γ is the group of translations and dilations

$$\Gamma = \{z \mapsto az + b; a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

The action of the group Γ lifts to the bundles \mathcal{G}_n and \mathcal{H}_n as follows. The bundle $\mathcal{H}_n = \text{Conf}_n \times |TH_1|$ is trivial and one extends the action of Γ on Conf_n by the trivial action on the fibers. For the bundle \mathcal{G}_n , the group Γ acts by diffeomorphisms $\Sigma^{(z)} \rightarrow \Sigma^{(az+b)}$, where $(az + b) = (az_1 + b, \dots, az_n + b)$. The action of Γ on \mathcal{G}_n is the induced action on the homotopy classes of curves

$$[\gamma(s)] \mapsto [\gamma_{a,b}(s) = a\gamma(s) + b].$$

In conclusion, \mathcal{G}_n and \mathcal{H}_n become equivariant vector bundles under the action of Γ and they give rise to vector bundles over $\mathcal{M}_{0,n+1}$ (that we again denote by \mathcal{G}_n and \mathcal{H}_n).

The canonical flat connection on \mathcal{G}_n descends from Conf_n to $\mathcal{M}_{0,n+1}$ because the action of Γ maps flat sections to flat sections. It turns out that the KZ connection on \mathcal{H}_n also descends to the moduli space.

Proposition 3.4. *The flat connection*

$$\nabla_n = d + \frac{1}{2\pi i} \sum_{i < j} \text{ad}(t_{ij}) \, d\log(z_i - z_j)$$

descends to a flat connection on $\mathcal{M}_{0,n+1}$.

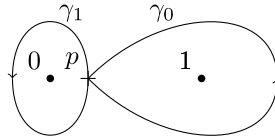


Fig. 1. Definition of γ_0 and γ_1 .

Proof. It is obvious that the connection 1-form is basic under the action of the group of translations $z \mapsto z + b$ and invariant under the action of dilations $z \mapsto \lambda z$. In order to see that it is horizontal under the action of dilations, define the Euler vector field $v = \sum_{i=1}^n z_i \partial_i$. Contracting it with the connection 1-form yields

$$\iota(v) \sum_{i < j} \text{ad}(t_{ij}) \text{dlog}(z_i - z_j) = \sum_{i < j} \text{ad}(t_{ij}) = \text{ad}(c) = 0,$$

where we have used the fact that c acts by zero on $|TH_1|$. \square

Finally, the holonomy map descends to a bundle map over the moduli space as well.

Proposition 3.5. *The holonomy map $|W|$ descends to $\mathcal{M}_{0,n+1}$.*

Proof. The group Γ acts on the complex plane by diffeomorphisms and the connection

$$A^{(z_1, \dots, z_n)} = \frac{1}{2\pi i} \sum_{i=1}^n a_i \text{dlog}(z - z_i)$$

is mapped to $A^{(az+b)}$. Hence, the holonomy map $W^{(z)}$ evaluated on a curve $\gamma : [0, 1] \rightarrow \Sigma^{(z)}$ coincides with the holonomy map $W^{(az+b)}$ evaluated on the curve $\gamma_{a,b} : [0, 1] \rightarrow \Sigma^{(az+b)}$. Applying projection $TH_1 \rightarrow |TH_1|$, we conclude that $|W|$ intertwines the actions of Γ on \mathcal{G}_n and \mathcal{H}_n . \square

We can summarize the statements above as follows:

Theorem 3.6. *The flat bundles with connections \mathcal{G}_n and \mathcal{H}_n and the bundle map $|W|$ over the configuration space Conf_n descend to the moduli space of genus zero curves $\mathcal{M}_{0,n+1}$.*

Remark 3.7. By Remark 3.1, the pure braid group PB_n (the fundamental group of the configuration space Conf_n) acts on $|TH_1|$ by automorphisms of the necklace Schedler Lie bialgebra structure. Theorem 3.6 shows that this action only depends on the complex structure on the surface (that is, on the point of $\mathcal{M}_{0,n+1}$).

4. Relation to the KZ associator and the Kashiwara–Vergne problem

In this section we establish the relation between the map $|W| : |\mathbb{C}\pi_1| \rightarrow TH_1$ and the solution of the Kashiwara–Vergne problem defined by the KZ associator.

Recall the following statements.

Theorem 4.1 (Theorem 5 in [1]). *For every Drinfeld associator $\Phi(x, y)$, the automorphism F_Φ of the free Lie algebra $\mathcal{L}(x, y)$ defined by formulas*

$$\begin{aligned} x &\mapsto \Phi(x, -x - y) x \Phi(x, -x - y)^{-1}, \\ y &\mapsto e^{-(x+y)/2} \Phi(y, -x - y) y \Phi(y, -x - y)^{-1} e^{(x+y)/2}, \end{aligned}$$

is a solution of the Kashiwara–Vergne problem.

For the second statement, we need the following notation: let $\Sigma = \mathbb{C} \setminus \{0, 1\}$, choose a base point $p \in [0, 1]$ and two generators $\gamma_0, \gamma_1 \in \pi_1$, as shown in Fig. 1.

Theorem 4.2 (Theorem 7.6 in [2]). *Let F be an automorphism of the free Lie algebra $\mathcal{L}(x, y)$ that solves the Kashiwara–Vergne problem. Then, the map $\rho_F : \mathbb{C}\pi_1 \rightarrow TH_1$ defined on generators by*

$$\rho_F(\gamma_0) = \exp(F(x)), \quad \rho_F(\gamma_1) = \exp(F(y))$$

induces an isomorphism of Lie bialgebras $|\mathbb{C}\pi_1| \rightarrow |TH_1|$.

We will now prove the following proposition.

Proposition 4.3. For $\Sigma = \mathbb{C} \setminus \{0, 1\}$ the Lie bialgebra isomorphism $|W|$ coincides with the one induced by the solution of the Kashiwara–Vergne problem corresponding to the KZ associator $\Phi_{\text{KZ}}(x, y)$.

Let $n = 2$ and $z_1 = 0, z_2 = 1$. Denote $x = a_1, y = a_2$. The connection $d + A$ acquires the form:

$$d + A = d + \frac{1}{2\pi i} (x \operatorname{dlog}(z) + y \operatorname{dlog}(z - 1)).$$

Following Drinfeld [5], define fundamental solutions of the equation $(d + A)\Psi = 0$ with asymptotics

$$\Psi_0(z) = (1 + O(z))z^{\frac{x}{2\pi i}}, \quad \Psi_1(z) = (1 + O(z))(z - 1)^{\frac{y}{2\pi i}}.$$

Choose the base point $p \in [0, 1]$ and a basis in π_1 represented by the curves γ_0 which surrounds 0 and γ_1 which surrounds 1. Define the representation $\rho_{\text{KZ}} : \pi_1 \rightarrow G_2$ as follows:

$$\rho_{\text{KZ}}(\gamma_0) = e^x, \quad \rho_{\text{KZ}}(\gamma_1) = \Phi_{\text{KZ}}(x, y)^{-1} e^y \Phi_{\text{KZ}}(x, y),$$

where Φ_{KZ} is the KZ associator.

Lemma 4.4. The representations ρ_{KZ} and W are equivalent under conjugation by $\Psi_0(p)$. The corresponding maps $|\mathbb{C}\pi_1| \rightarrow TH_1$ coincide.

Proof. By definition of the holonomy,

$$\Psi_0(\gamma_0 \cdot p) = \operatorname{Hol}_{\gamma_0} \Psi_0(p), \quad \Psi_1(\gamma_1 \cdot p) = \operatorname{Hol}_{\gamma_1} \Psi_1(p).$$

In combination with

$$\Psi_0(\gamma_0 \cdot p) = \Psi_0(p) e^x, \quad \Psi_1(\gamma_1 \cdot p) = \Psi_1(p) e^y, \quad \Psi_1(p) = \Psi_0(p) \Phi_{\text{KZ}}(x, y)$$

we obtain

$$\rho_{\text{KZ}}(\gamma_0) = e^x = \Psi_0(p)^{-1} \Psi_0(\gamma_0 \cdot p) = \Psi_0(p)^{-1} \operatorname{Hol}_{\gamma_0} \Psi_0(p)$$

and

$$\begin{aligned} \rho_{\text{KZ}}(\gamma_1) &= \Phi_{\text{KZ}}^{-1} e^y \Phi_{\text{KZ}} = \Phi_{\text{KZ}}^{-1} \Psi_1(p)^{-1} \Psi_1(\gamma_1 \cdot p) \Phi_{\text{KZ}} \\ &= (\Psi_1(p) \Phi_{\text{KZ}})^{-1} \operatorname{Hol}_{\gamma_1} (\Psi_1(p) \Phi_{\text{KZ}}) = \Psi_0(p)^{-1} \operatorname{Hol}_{\gamma_1} \Psi_0(p), \end{aligned}$$

as required.

Since the two representations of π_1 are equivalent, they descend to the same map on $|\mathbb{C}\pi_1|$. \square

Proof of Proposition. To complete the proof of Proposition, we remark that the representation ρ_{KZ} is equivalent to the representation

$$\begin{aligned} \rho(\gamma_0) &= \Phi(x, -x - y) e^x \Phi(x, -x - y)^{-1}, \\ \rho(\gamma_1) &= e^{-(x+y)/2} \Phi(y, -x - y) e^y \Phi(y, -x - y)^{-1} e^{(x+y)/2}. \end{aligned}$$

The equivalence is given by conjugation with $e^{x/2} \Phi(-x - y, x)$, which corresponds to moving the base point from the neighborhood of 0 to the neighborhood of ∞ .

The map $|\mathbb{C}\pi_1| \rightarrow |TH_1|$ induced by ρ coincides with the one induced by the solution of the Kashiwara–Vergne problem corresponding to the KZ associator $\Phi_{\text{KZ}}(x, y)$. \square

Remark 4.5. G. Massuyeau explained to us that, for $n = 2$, the Lie bialgebra isomorphism $|\mathbb{C}\pi_1| \rightarrow |TH_1|$ constructed in [13] using the Kontsevich integral coincides with the map described above if one chooses the associator $\Phi = \Phi_{\text{KZ}}$.

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