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Partial differential equations

Non-null-controllability of the Grushin operator in 2D ☆

Non-contrôlabilité à zéro de l'opérateur de Grushin en dimension 2

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A R T I C L E I N F O

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A B S T R A C T

We are interested in the exact null controllability of the equation $\partial_t f - \partial_x^2 f - x^2 \partial_y^2 f = \mathbb{1}_\omega u$, with control u supported on ω. We show that, when ω does not intersect a horizontal band, the considered equation is never null-controllable. The main idea is to interpret the associated observability inequality as an L^2 estimate on polynomials, which Runge's theorem disproves. To that end, we study in particular the first eigenvalue of the operator $-\partial_x^2 + (nx)^2$ with Dirichlet conditions on $(-1, 1)$, and we show a quite precise estimation it satisfies, even when n is in some complex domain.

R É S U M É

Nous nous intéressons à la contrôlabilité exacte à zéro de l'équation $\partial_t f - \partial_x^2 f - x^2 \partial_y^2 f = \mathbb{1}_\omega u$ sur $(-1, 1) \times \mathbf{T}$, avec contrôle u sur ω. Nous démontrons que si ω est le complémentaire d'une bande horizontale, l'équation considérée n'est contrôlable pour aucun temps. L'idée principale est d'interpréter l'inégalité d'observabilité comme une estimation sur les fonctions entières, que nous nions grâce au théorème de Runge. Pour réaliser cette interprétation, nous étudions en particulier la première valeur propre de $-\partial_x^2 + (nx)^2$ avec conditions de Dirichlet sur $]-1, 1[$, et en obtenons une estimation assez précise, y compris pour certains n complexes.

1. Introduction

1.1. The problem of controllability of the Grushin equation

We are interested in the following equation, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\Omega = (-1, 1) \times \mathbb{T}$ and ω is an open subset of Ω:

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$$\begin{aligned} &(\partial_t - \partial_x^2 - x^2\partial_y^2) f(t,x,y) = \mathbb{1}_\omega u(t,x,y) && t \in [0,T], (x,y) \in \Omega \\ &f(t,x,y) = 0 && t \in [0,T], (x,y) \in \partial\Omega. \end{aligned}$$

It is a control problem with state f and control u supported on ω. More precisely, we are interested in the exact null controllability of this equation.

Definition 1. We say that the Grushin equation is null-controllable on ω in time $T > 0$ if for all f_0 in $L^2(\Omega)$, there exists u in $L^2([0,T] \times \omega)$ such that the solution f to:

$$\begin{aligned} &(\partial_t - \partial_x^2 - x^2\partial_y^2) f(t,x,y) = \mathbb{1}_\omega u(t,x,y) && t \in [0,T], (x,y) \in \Omega \\ &f(t,x,y) = 0 && t \in [0,T], (x,y) \in \partial\Omega \\ &f(0,x,y) = f_0(x,y) && (x,y) \in \Omega \end{aligned} \tag{1}$$

satisfies $f(T,x,y) = 0$ for all (x,y) in Ω.

We show in this paper that, if ω does not intersect a horizontal band, then the answer is negative whatever T (Theorem 2).

Theorem 2. *Let $[a,b]$ be a non-trivial segment of $\mathbb{T}$ and $\omega = (-1,1) \times (\mathbb{T} \setminus [a,b])$. Let $T > 0$. The Grushin equation is not null-controllable on ω in time T.*

That is to say, there exists some $f_0 \in L^2(\Omega)$ that no $u \in L^2([0,T] \times \omega)$ can steer to 0 in time T. This can be strengthened by saying that, even if we ask the initial condition to be very regular, it may be impossible to steer it to 0 in finite time. We will state this in a precise way in Proposition 27.

We stated Theorem 2 with $\Omega = (-1,1) \times \mathbb{T}$ as it is (very) slightly easier than $\Omega = (-1,1) \times (0,1)$, but the situation is the same for both cases, and we briefly explain in Appendix C what to do for the latter case.

The proof we provide for this theorem is very specific to the potential x^2: if we replace x^2 in Eq. (1) by, say, $x^2 + \epsilon x^3$, we cannot prove with our method the non-null controllability. However, there is only a single, but crucial argument that prevents us from doing so. We will discuss this a little further after Theorem 22.

1.2. Bibliographical comments

This equation has previously been studied on $(-1,1) \times (0,1)$, and some results already exist for different controllability sets. Controllability holds for large time, but not in small time if $\omega = (a,b) \times (0,1)$ with $0 < a < b$, as shown by Beauchard, Cannarsa, and Guglielmi [5], and holds in any time if $\omega = (0,a) \times (-1,1)$ with $0 < a$, as shown by Beauchard, Miller, and Morancey [7].

The controllability of the Grushin equation is part of the larger field of the controllability of degenerate parabolic partial differential equations of hypoelliptic type. For the non-degenerate case, controllability is known since 1995 to hold when Ω is any bounded C^2 domain, in any open control domain and in arbitrarily small time [17,14]. For parabolic equations degenerating on the boundary, the situation is well understood in dimension one [10] and in dimension two [11]. For parabolic equations degenerating inside the domain, in addition to the already mentioned two articles on the Grushin equation, we mention articles on Kolmogorov-type equations [3,6], the heat equation on the Heisenberg group [4], and quadratic hypoelliptic equations on the whole space [8,9].

1.3. Outline of the proof, structure of the article

As usual in controllability problems, we focus on the following observability inequality on the adjoint system, which is equivalent by a duality argument to the null-controllability (Definition 1, see [12, Theorem 2.44] for a proof of this equivalence): there exists $C > 0$ such that for all f_0 in $L^2(\Omega)$, the solution f to:

$$\begin{aligned} &(\partial_t - \partial_x^2 - x^2\partial_y^2) f(t,x,y) = 0 && t \in [0,T], (x,y) \in \Omega \\ &f(t,x,y) = 0 && t \in [0,T], (x,y) \in \partial\Omega \\ &f(0,x,y) = f_0(x,y) && (x,y) \in \Omega \end{aligned} \tag{2}$$

satisfies:

$$\int_\Omega |f(T,x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \le C \int_{[0,T]\times\omega} |f(t,x,y)|^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$

Therefore, Theorem 2 can be stated the following way.

Theorem 3. *There exists a sequence $(f_{k,0})$ in $(L^2(\Omega))^{\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, the solution f_k to the Grushin equation (2) with initial condition $f_{k,0}$ satisfies $\sup_k |f_k|_{L^2([0,T]\times\omega)} < +\infty$ and $|f_k(T,\cdot,\cdot)|_{L^2(\Omega)} \to +\infty$ as $k \to +\infty$.*

To find such a sequence, we look for solutions to the Grushin equation (2) that concentrate near $x = 0$. To that end, we remark that, denoting $v_{n,k}$ an eigenfunction of the operator $-\partial_x^2 + (nx)^2$ with Dirichlet boundary conditions on $(-1,1)$ associated with eigenvalue $\lambda_{n,k}$, $\Phi_{n,k}(x,y) = v_{n,k}(x)\mathrm{e}^{iny}$ is an eigenfunction of the Grushin operator $-\partial_x^2 - x^2\partial_y^2$ with eigenvalue $\lambda_{n,k}$. In addition, we expect that the first eigenfunction $v_n = v_{n,0}$ of $-\partial_x^2 + (nx)^2$ on $(-1,1)$ is close to the first eigenfunction of the same operator on $\mathbb{R}$, that is, $v_n \sim (\frac{n}{4\pi})^{1/4}\mathrm{e}^{-nx^2/2}$, and that the associated eigenvalue $\lambda_n = \lambda_{n,0}$ is close to n. So it is natural to look for a counterexample of the observability inequality (3) among the linear combinations of $\Phi_n(x,y) = v_n(x)\mathrm{e}^{iny}$ for $n \geq 0$.

In Section 2.1, we will see by heuristic arguments and with the help of these approximations that the problem of the controllability of the Grushin equation is close to the controllability of the square root of minus the Laplacian, and show that this model is not null controllable. As another warm-up, we will show in Section 2.2 that the method used for treating the square root of minus the Laplacian allows us to treat with little changes the case of the Grushin equation for $(x,y) \in \mathbb{R}\times\mathbb{T}$.

The case of the Grushin equation for $(x,y) \in (-1,1)\times\mathbb{T}$ (Theorem 2) gave us much more trouble, but in Section 2.3 we are able to adapt the method used in the previous two cases. To achieve that, we use some technical tools that are proved in later sections. First, in Section 3, estimates on polynomials of the form $|\sum \gamma_n a_n z^n|_{L^\infty(U)} \leq C|\sum a_n z^n|_{L^\infty(U^\delta)}$, under a simple geometric hypothesis on U, and some general—although somewhat hard to prove—hypotheses on the sequence (γ_n) (Theorem 18). Second, in Section 4, a spectral analysis of the operator $-\partial_x^2 + (nx)^2$ on $(-1,1)$; most importantly, an asymptotic expansion of the first eigenvalue λ_n of the form $\lambda_n = n + \gamma(n)\mathrm{e}^{-n}$ with $\gamma(n) \sim 4\pi^{-1/2}n^{3/2}$, and γ having a particular holomorphic structure (Theorem 22).

2. Proof of the non-null controllability of the Grushin equation

2.1. The toy model

Let us write the observability inequality on functions of the form $\sum a_n v_n(x)\mathrm{e}^{iny}$ (keeping in mind that $v_n(x)$ is real, and noting $\omega_y = \mathbb{T}\setminus[a,b]$ so that $\omega = (-1,1)\times\omega_y$):

$$\sum_n |a_n|^2 \mathrm{e}^{-2\lambda_n T} \leq C \sum_{n,m} a_n \overline{a_m} \int_{-1}^{1} v_n(x)v_m(x)\,\mathrm{d}x \int_0^T \mathrm{e}^{-(\lambda_n+\lambda_m)t}\,\mathrm{d}t \int_{y\in\omega_y} \mathrm{e}^{\mathrm{i}(n-m)y}\,\mathrm{d}y. \tag{4}$$

Now let us proceed by heuristic arguments to see what we can expect from the estimates on the eigenvalues λ_n and the eigenfunctions v_n that we mentioned in Section 1.3. We imagine that in the previous inequality, $\lambda_n = n$ and $\int_{-1}^1 v_n v_m = \frac{1}{\sqrt{4\pi}}(nm)^{1/4}\int_{\mathbb{R}} \mathrm{e}^{-(n+m)x^2/2}\,\mathrm{d}x = \sqrt{2}\frac{(nm)^{1/4}}{\sqrt{n+m}}$, which does not decay very fast off-diagonal, so we further imagine that $\int_{-1}^1 v_n v_m = 1$. Then, with these approximations, the previous observability inequality reads:

$$\sum_n |a_n|^2 \mathrm{e}^{-2nT} \leq C \sum_{n,m} a_n\overline{a_m} \int_{[0,T]\times\omega_y} \mathrm{e}^{-(n+m)t+\mathrm{i}(n-m)y}\,\mathrm{d}t\,\mathrm{d}y = C \int_{[0,T]\times\omega_y} \left|\sum a_n \mathrm{e}^{-nt+\mathrm{i}ny}\right|^2 \mathrm{d}t\,\mathrm{d}y. \tag{5}$$

This suggests that the controllability problem of the Grushin equation (1) is similar to the following model control problem: let us consider the Hilbert space $\{\sum_{n\geq 0} a_n \mathrm{e}^{iny}, \sum |a_n|^2 < +\infty\}$, D the unbounded operator on this space with domain $\{\sum a_n \mathrm{e}^{iny}, \sum n^2|a_n|^2 < +\infty\}$ defined by $D(\sum a_n \mathrm{e}^{iny}) = \sum n a_n \mathrm{e}^{iny}$. Then the null controllability of the equation $(\partial_t + D) = \mathbb{1}_\omega u$ on an open set $\omega = \mathbb{T}\setminus[a,b]$ in time T is equivalent to the previous "simplified" observability inequality (5), which does not hold (Theorem 4).

Theorem 4. *Let $[a,b]$ be a nontrivial segment of $\mathbb{T}$, $\omega_y = \mathbb{T}\setminus[a,b]$ and $T > 0$. The equation $(\partial_t + D)f = \mathbb{1}_{\omega_y} u$ is not null controllable on ω_y in time T.*

Incidentally, this is an answer to a specific case of an open problem mentioned by Miller [20, section 3.3] and again by Duyckaerts and Miller [13, remark 6.4].

Proof. The right-hand side of the observability inequality (5) suggests to make the change of variables $z = \mathrm{e}^{-t+\mathrm{i}y}$, for which[1] $\mathrm{d}t\,\mathrm{d}y = |z|^{-2}\,\mathrm{d}\lambda(z)$, and that maps $[0,T]\times\omega_y$ to $\mathcal{D} = \{\mathrm{e}^{-T} < |z| < 1, \arg(z) \in \omega_y\}$ (see Fig. 1). So, the right-hand side of the

[1] We denote λ the Lebesgue measure on $\mathbb{C}$; that is if $(\mu,\nu) \mapsto f(\mu+\mathrm{i}\nu)$ is integrable on $\mathbb{R}^2$, $\int_{\mathbb{C}} f(z)\,\mathrm{d}\lambda(z) = \int_{\mathbb{R}^2} f(\mu+\mathrm{i}\nu)\,\mathrm{d}\mu\,\mathrm{d}\nu$.

observability inequality (5) is equal to:

$$\int_{[0,T]\times\omega_y} \left|\sum a_n \mathrm{e}^{-nt+\mathrm{i}ny}\right|^2 \mathrm{d}t\,\mathrm{d}y = \int_{\mathcal{D}} \left|\sum a_n z^n\right|^2 |z|^{-2}\,\mathrm{d}\lambda(z). \tag{6}$$

About the left-hand side, we first note that, by writing the integral on a disk $D = D(0,r)$ of $z^n\bar{z}^m$ in polar coordinates, we find that the functions $z \mapsto z^n$ are orthogonal on $D(0,r)$. So, we have for all polynomials $\sum_{n\geq 1} a_n z^n$ with a zero at 0:

$$\int_{D(0,\mathrm{e}^{-T})} \left|\sum a_n z^n\right|^2 |z|^{-2}\,\mathrm{d}\lambda(z) = \sum |a_n|^2 \int_{D(0,\mathrm{e}^{-T})} |z|^{2n-2}\,\mathrm{d}\lambda(z)$$

and, combined with the fact that by another computation in polar coordinates, for $n \geq 1$, $\int_{D(0,\mathrm{e}^{-T})} |z|^{2n-2}\,\mathrm{d}\lambda(z) = \frac{\pi}{n}\mathrm{e}^{-2nT}$:

$$\int_{D(0,\mathrm{e}^{-T})} \left|\sum a_n z^n\right|^2 |z|^{-2}\,\mathrm{d}\lambda(z) \leq \pi \sum |a_n|^2 \mathrm{e}^{-2nT}. \tag{7}$$

So, thanks to Eqs. (6) and (7), the observability inequality (5) implies that, for some $C' > 0$ and for all polynomials f with $f(0) = 0$,

$$\int_{D(0,\mathrm{e}^{-T})} |f(z)|^2 |z|^{-2}\,\mathrm{d}\lambda(z) \leq C' \int_{\mathcal{D}} |f(z)|^2 |z|^{-2}\,\mathrm{d}\lambda(z).$$

By the change of indices $n' = n - 1$ in the sum $f(z) = \sum_{n\geq 1} a_n z^n$, we rewrite this “holomorphic observability inequality” in the following, slightly simpler way: for every polynomials f,

$$\int_{D(0,\mathrm{e}^{-T})} |f(z)|^2\,\mathrm{d}\lambda(z) \leq C' \int_{\mathcal{D}} |f(z)|^2\,\mathrm{d}\lambda(z). \tag{8}$$

This is the main idea of the proof: the observability inequality of the control problem is almost the same as an L^2 estimate on polynomials. We will disprove it thanks to Runge's theorem, whose proof can be found in Rudin's famous textbook [21, theorem 13.9]. More specifically, we will need the following special case.

Proposition 5 *(Runge's theorem). Let U be a connected and simply connected open subset of $\mathbb{C}$, and let f be a holomorphic function on U. There exists a sequence (f_n) of polynomials that converges uniformly on every compact subsets of U to f.*

Let $\theta \in \mathbb{T}$ non-adherent to ω_y (for instance $\theta = (a+b)/2$). We choose in the previous theorem $U = \mathbb{C} \setminus \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ (see Fig. 1) and $f(z) = \frac{1}{z}$. Since $z \mapsto \frac{1}{z}$ is bounded on $\mathcal{D}$, f_n is uniformly bounded on $\mathcal{D}$ and the right-hand side of the holomorphic observability inequality (8) $\int_{\mathcal{D}} |f_n|^2\,\mathrm{d}\lambda(z)$ stays bounded. But since $z \mapsto \frac{1}{z}$ has infinite L^2 norm on $D(0,\mathrm{e}^{-T})$, and thanks to Fatou's lemma, the left-hand side $\int_{D(0,\mathrm{e}^{-T})} |f_n|^2\,\mathrm{d}\lambda(z)$ tends to infinity as n tends to infinity. □

Remark 6. This proof is specific to the one dimensional case, as it relies on the observation that the solutions to the equation $(\partial_t + D)f = 0$ are holomorphic in $z = \mathrm{e}^{-t+\mathrm{i}y}$. As far as the author knows, this argument does not generalize to higher dimension.

2.2. From the toy model to the Grushin equation: the case of the Grushin equation on $\mathbb{R}\times\mathbb{T}$

We show here that the method we used for the toy model is also effective to prove that the Grushin equation for $(x, y) \in \mathbb{R}\times\mathbb{T}$, i.e. the equation

$$\begin{aligned} &(\partial_t - \partial_x^2 - x^2\partial_y^2) f(t,x,y) = 0 \qquad && t \in [0,T], (x,y) \in \mathbb{R}\times\mathbb{T} \\ &f(t,\cdot,\cdot) \in L^2(\mathbb{R}\times\mathbb{T}) && \\ &f(0,x,y) = f_0(x,y) && (x,y) \in \Omega, \end{aligned} \tag{9}$$

where we choose $\omega = \mathbb{R}\times\omega_y = \mathbb{R}\times(\mathbb{T}\setminus[a,b])$, is not null-controllable in any time.

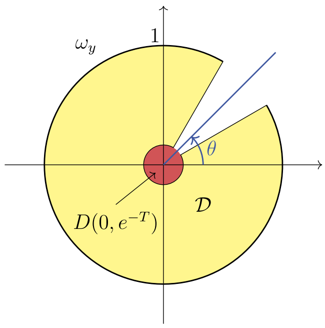

Fig. 1. In yellow, the domain $\mathcal{D}$, in red, the disk $D(0, \mathrm{e}^{-T})$. The thick outer circular arc is the subset ω_y of $\mathbb{T} \simeq \mathbb{U}$. The controllability of the model operator $|D|$ on ω_y in time T would imply the control of the $L^2(D(0, \mathrm{e}^{-T}))$-norm of polynomials by their L^2 norm on $\mathcal{D}$.

In this case, the (unbounded) operator $-\partial_x^2 + (nx)^2$ on L^2 is perfectly known: its first eigenvalue is n and the associated eigenfunction[2] is $v_n(x) = \mathrm{e}^{-nx^2/2}$. So the functions Φ_n defined by $\Phi_n(x, y) = \mathrm{e}^{-nx^2/2}\mathrm{e}^{\mathrm{i}ny}$ are eigenfunctions of the operator $\partial_x^2 + x^2\partial_y^2$ with respective eigenvalue n.

Let us write the associated observability inequality on the solutions to the Grushin equation of the form $\sum a_n \mathrm{e}^{-nt}\Phi_n(x, y)$, where the sum has finite support[3]:

$$\sum \left(\frac{\pi}{n}\right)^{1/2} |a_n|^2 \mathrm{e}^{-2nT} \le C \int_{x\in\mathbb{R}} \left(\int_{\substack{t\in[0,T]\\ y\in\omega_y}} \left| \sum a_n \mathrm{e}^{n\left(-\frac{x^2}{2} - t + \mathrm{i}y\right)} \right|^2 \mathrm{d}t\,\mathrm{d}y \right) \mathrm{d}x. \tag{10}$$

The first difference between this observability inequality, that we try to disprove, and the observability inequality of the toy model (8), is the factor $(\pi/n)^{1/2}$, but it is not a real problem. The main difference is the presence of another variable: x. For each x, the term $\mathrm{e}^{-nx^2/2}$ acts as a contraction of $\mathcal{D}$, so we make the change of variable that takes into account this contraction $z_x = \mathrm{e}^{-x^2/2 - t + \mathrm{i}y}$. We have $\mathrm{d}t\,\mathrm{d}y = |z_x|^{-2}\,\mathrm{d}\lambda(z_x)$, and this change of variables sends $(0, T) \times \omega_y$ to $\mathrm{e}^{-x^2/2}\mathcal{D}$, with, as in the toy model, $\mathcal{D} = \{\mathrm{e}^{-T} < |z| < 1, \arg(z) \in \omega_y\}$:

$$\sum \left(\frac{\pi}{n}\right)^{1/2} |a_n|^2 \mathrm{e}^{-2nT} \le C \int_{x\in\mathbb{R}} \int_{\mathrm{e}^{-x^2/2}\mathcal{D}} \left| \sum a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)\,\mathrm{d}x.$$

We have seen in the toy model that for all polynomials $f(z) = \sum_{n\ge1} a_n z^n$ with $f(0) = 0$ that $\int_{D(0,\mathrm{e}^{-T})} |f(z)|^2 |z|^{-2}\,\mathrm{d}\lambda(z) = \pi \sum \frac{1}{n}|a_n|^2 \mathrm{e}^{-2nT}$, which is smaller than the left-hand side of the observability inequality (10), up to a constant $\sqrt{\pi}$. So, as in the toy model, this would imply that for all polynomials f:

$$\int_{D(0,\mathrm{e}^{-T})} |f(z)|^2\,\mathrm{d}\lambda(z) \le \sqrt{\pi} C \int_{x\in\mathbb{R}} \int_{\mathrm{e}^{-x^2/2}\mathcal{D}} |f(z)|^2\,\mathrm{d}\lambda(z)\,\mathrm{d}x. \tag{11}$$

We want to apply the same method as the one used in the toy model to disprove this inequality, but we have to be a little careful: the right-hand side exhibits an integral over $\mathrm{e}^{-x^2/2}\mathcal{D}$, and as x tends to infinity, 0 becomes arbitrarily close to the integration set. So, instead of choosing a sequence of polynomials that blows up at $z = 0$, we choose one that blows up away from 0 and from every $\mathrm{e}^{-x^2/2}\mathcal{D}$. More precisely, we choose $\theta \notin \overline{\omega_y}$, $z_0 = \mathrm{e}^{\mathrm{i}\theta - 2T}$, and f_k a sequence of polynomials that converges to $z \mapsto (z - z_0)^{-1}$ uniformly on every compact of $\mathbb{C} \setminus (z_0[1, +\infty[)$ (see Fig. 2).

With the same argument as in the toy model, we know that the left-hand side $\int_{D(0,\mathrm{e}^{-T})} |f_k(z)|^2\,\mathrm{d}\lambda(z)$ tends to infinity as k tends to infinity. As for the right-hand side, since $z \mapsto (z - z_0)^{-1}$ is bounded in $\bigcup_x \mathrm{e}^{-x^2/2}\mathcal{D} = \{0 < |z| < 1, \arg(z) \in \omega_y\}$, f_k is bounded on $\mathrm{e}^{-x^2/2}\mathcal{D}$ uniformly in $x \in \mathbb{R}$ and $k \in \mathbb{N}$ by some M. So, the right-hand side satisfies[4]:

[2] We choose to normalize the eigenfunction so that $v_n(0) = 1$ instead of $|v_n|_{L^2(\mathbb{R})} = 1$, that way, the proof will be slightly easier. Note that with this choice of normalization, we have $|v_n|_{L^2(\mathbb{R})} = (\pi/n)^{1/4}$.

[3] We could extend all the following estimates by density to some functional spaces, but we will not need to, as we did not need to extend the estimate (8) to other functions than polynomials.

[4] Let us remind that λ is the Lebesgue measure on $\mathbb{C}$, so, for $A \subset \mathbb{C}$ measurable, $\lambda(A)$ is the area of A.

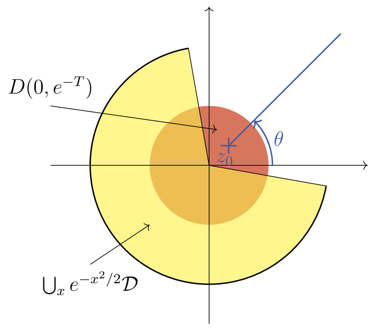

Fig. 2. The equivalent of Fig. 1 for the Grushin equation. Again in red, the disk $D(0,\mathrm{e}^{-T})$, and in yellow the union of $\mathrm{e}^{-x^2/2}\mathcal{D}$, which is the "pacman" $\{0<|z|<1, \arg(z)\in\omega_y\}$. We choose a sequence of polynomials that converges to $z\mapsto(z-z_0)^{-1}$ away from the blue half-line.

$$\begin{aligned}\int_{x\in\mathbb{R}}\int_{\mathrm{e}^{-x^2/2}\mathcal{D}} |f(z)|^2\,\mathrm{d}\lambda(z)\,\mathrm{d}x &\le \int_{x\in\mathbb{R}}\int_{\mathrm{e}^{-x^2/2}\mathcal{D}} M^2\,\mathrm{d}\lambda(z)\,\mathrm{d}x\\ &\le \int_{x\in\mathbb{R}}\lambda(\mathrm{e}^{-x^2/2}\mathcal{D})M^2\,\mathrm{d}x\\ &\le \int_{x\in\mathbb{R}}\pi\mathrm{e}^{-x^2}M^2\,\mathrm{d}x\\ &\le \pi^{3/2}M^2.\end{aligned}$$

We have proved that the left-hand side of inequality (11) applied to $f=f_k$ tends to infinity as k tends to infinity while its right-hand side stays bounded; thus, this inequality is false, and the Grushin equation for $(x,y)\in\mathbb{R}\times\mathbb{T}$ is never null-controllable in $\omega=\mathbb{R}\times(\mathbb{T}\setminus[a,b])$. □

2.3. The case of the Grushin equation on $(-1,1)\times\mathbb{T}$

Here we show the main theorem. In comparison with the previous case, we have two difficulties: λ_n is not exactly n, and $v_n(x)$ is not exactly $\mathrm{e}^{-nx^2/2}$. Let us write the observability inequality[5] on $\sum a_n\mathrm{e}^{-\lambda_n t}v_n(x)\mathrm{e}^{\mathrm{i}ny}$, where $\lambda_n=n+\rho_n$:

$$\sum|v_n|^2_{L^2(-1,1)}|a_n|^2\mathrm{e}^{-2\lambda_n T}\le C\int_{\substack{t\in[0,T]\\x\in(-1,1)\\y\in\omega_y}}\left|\sum a_n v_n(x)\mathrm{e}^{n(-t+\mathrm{i}y)}\mathrm{e}^{-\rho_n t}\right|^2\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x. \tag{12}$$

As in the previous two cases, the first step is to relate this inequality to an estimate on polynomials (Proposition 7).

Proposition 7. *Let $U=\{0<|z|<1,\arg(z)\in\omega_y\}$, let $\delta>0$ and $U^\delta=\{z\in\mathbb{C},\mathrm{distance}(z,U)<\delta\}$ (see Fig. 3).*
The observability inequality of the Grushin equation implies that there exists $C'>0$ and an integer N such that, for all polynomials $f(z)=\sum_{n>N}a_nz^n$ with at least the N first derivatives vanishing at zero,[6]

$$|f|_{L^2(D(0,\mathrm{e}^{-T}))}\le C'|f|_{L^\infty(U^\delta)}. \tag{13}$$

Proof. About the left-hand side of the observability inequality (12), we remark that it is almost the same as in the toy model. Indeed, if $a_0=0$, we have seen in the proof of the non-null controllability of the toy model that $\int_{D(0,\mathrm{e}^{-T})}|\sum a_nz^{n-1}|^2\,\mathrm{d}\lambda(z)=\sum\frac{\pi}{n}|a_n|^2\mathrm{e}^{-2nT}$. And since $|v_n|^2_{L^2(-1,1)}$ is greater than $cn^{-1/2}$ for some $c>0$ (see Lemma 21 for a proof), we have:

$$\int_{D(0,\mathrm{e}^{-T})}\left|\sum a_nz^{n-1}\right|^2\mathrm{d}\lambda(z)\le\pi c^{-1}\sum|v_n|^2_{L^2(-1,1)}|a_n|^2\mathrm{e}^{-2nT}.$$

---

[5] As in the previous cases, all the sums are supposed with finite support. We could extend by density all the inequalities that follow, but we will not need to.

[6] This condition is not really needed, but it makes some theorems less cumbersome to state.

Moreover, reminding that $\lambda_n = n + \rho_n$, we know that (ρ_n) is bounded (see Theorem 22 or [5, section 3.3] for a simpler proof). So, $\mathrm{e}^{-2nT} \le \mathrm{e}^{2\sup_k(\rho_k)T}\mathrm{e}^{-2\lambda_n T}$. We use that to bound the right-hand side of the previous inequality:

$$\int_{D(0,\mathrm{e}^{-T})} \left|\sum a_n z^{n-1}\right|^2 \mathrm{d}\lambda(z) \le \pi c^{-1}\mathrm{e}^{2\sup_k(\rho_k)T} \sum |v_n|^2_{L^2(-1,1)} |a_n|^2 \mathrm{e}^{-2\lambda_n T}. \tag{14}$$

We now want to bound from above the right-hand side of the observability inequality (12) by $C'\left|\sum a_n z^{n-1}\right|^2_{L^\infty(U^\delta)}$ for some C'. We make the change of variables $z = \mathrm{e}^{-t+\mathrm{i}y}$:

$$\int_{\substack{t\in[0,T]\\ x\in(-1,1)\\ y\in\omega_y}} \left|\sum a_n v_n(x)\mathrm{e}^{n(-t+\mathrm{i}y)}\mathrm{e}^{-\rho_n t}\right|^2 \mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x = \int_{x\in(-1,1)} \left(\int_{z\in\mathcal{D}} \left|\sum a_n v_n(x) z^{n-1}|z|^{\rho_n}\right|^2 \mathrm{d}\lambda(z)\right)\mathrm{d}x. \tag{15}$$

As in the case of the Grushin equation over $\mathbb{R}\times\mathbb{T}$ studied in the previous section, there is a multiplication by $v_n(x)$. But this time, the action of this multiplication is a little more complicated than just a contraction by a factor $\mathrm{e}^{-x^2/2}$. The other difficulty is the factor $\mathrm{e}^{-\rho_n t} = |z|^{\rho_n}$, which does not seem to be a big issue at a first glance, as it is close to 1; but since it is not holomorphic, it is actually the biggest issue we are facing. To be able adapt the method used in the previous cases, we need to somehow estimate the sum $|\sum v_n(x)|z|^{\rho_n} a_n z^n|_{L^2(\mathcal{D})}$ by an appropriate norm of $\sum a_n z^n$. The Theorem 18 hinted in the outline gives us such an estimate, with the spectral analysis of Section 4 giving us the required hypotheses. More precisely, we prove in Section 4.4 the following lemma.

Lemma 8. *There exists an integer N and $C_2 > 0$ such that for every $x \in (-1,1)$, z and ζ in $\mathcal{D}$, and every polynomial $\sum_{n>N} a_n z^n$ with derivatives up to order N vanishing at 0:*

$$\left|\sum v_n(x) a_n z^{n-1}|\zeta|^{\rho_n}\right| \le C_2 \left|\sum a_n z^{n-1}\right|_{L^\infty(U^\delta)}.$$

Applying the above lemma for $z = \zeta$, and assuming that $a_n = 0$ when $n \le N$, we have for every $z \in \mathcal{D}$:

$$\left|\sum v_n(x) a_n z^{n-1}|z|^{\rho_n}\right| \le C_2 \left|\sum a_n z^{n-1}\right|_{L^\infty(U^\delta)}$$

so, the right-hand side of the observability inequality satisfies:

$$\begin{aligned}
&\int_{x\in(-1,1)} \left(\int_{\substack{t\in[0,T]\\ y\in\omega_y}} \left|\sum a_n v_n(x)\mathrm{e}^{n(-t+\mathrm{i}y)}\mathrm{e}^{-\rho_n t}\right|^2 \mathrm{d}t\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{x\in(-1,1)} \left(\int_{z\in\mathcal{D}} \left|\sum a_n v_n(x) z^{n-1}|z|^{\rho_n}\right|^2 \mathrm{d}\lambda(z)\right)\mathrm{d}x && \text{(Eq. (15))}\\
&\le \int_{x\in(-1,1)} \left(\int_{z\in\mathcal{D}} C_2^2 \left|\sum a_{n+1} z^n\right|^2_{L^\infty(U^\delta)} \mathrm{d}\lambda(z)\right)\mathrm{d}x && \text{(previous lemma)}\\
&\le 2C_2^2\pi \left|\sum a_{n+1} z^n\right|^2_{L^\infty(U^\delta)} && (\mathrm{area}(\mathcal{D}) \le \pi).
\end{aligned}$$

So, together with Eq. (14) on the left-hand side of the observability inequality, we have proved that the observability inequality implies that for all polynomials $f = \sum_{n\ge N} a_n z^{n-1}$:

$$|f|^2_{L^2(D(0,\mathrm{e}^{-T}))} \le 2\pi^2 \mathrm{e}^{2\sup_k(\rho_k)T} c^{-1} C C_2^2 |f|^2_{L^\infty(U^\delta)}. \qquad \square$$

We can find a counterexample of the inequality of the previous proposition exactly in the same way as we disproved the null controllability of the Grushin equation over $\mathbb{R}\times\mathbb{T}$.

Proof of Theorem 2. First we choose $0 < \delta < \mathrm{e}^{-T}$, so that $D(0,\mathrm{e}^{-T}) \not\subset U^\delta$. We also choose θ non-adherent to ω_y, and $z_0 = r\mathrm{e}^{\mathrm{i}\theta}$ with $r \in (\delta, \mathrm{e}^{-T})$ (so that $z_0 \in D(0,\mathrm{e}^{-T})$ but $z_0 \notin \overline{U^\delta}$, see Fig. 3). Then we choose $\tilde{f}_k$ a sequence of polynomials that converges uniformly on every compact subset of $\mathbb{C} \setminus z_0[1,+\infty)$ to $z \mapsto (z-z_0)^{-1}$. Finally, to satisfy the condition of

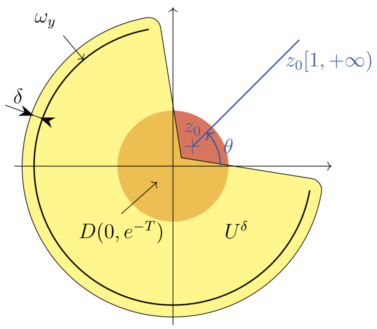

Fig. 3. In yellow, the domain U^δ, in red, the disk $D(0,\mathrm{e}^{-T})$ and in blue, the point z_0 and the half-line $z_0[1,+\infty)$. Since f_k converges to $z \mapsto z^{N+1}(z-z_0)^{-1}$ away from the blue line, the L^∞ norm of f_k over U^δ is bounded, as long as $\delta < \mathrm{distance}(z_0,\mathcal{D})$.

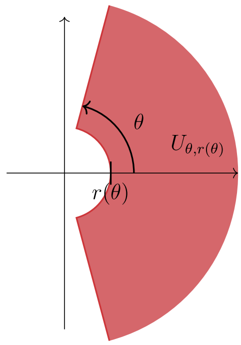

Fig. 4. An example of a set $U_{\theta,r(\theta)}$, whose union for $0<\theta<\pi/2$ is the domain of definition of functions in $S(r)$. The angle θ is allowed to be arbitrarily close to $\pi/2$, but then, the radius $r(\theta)$ of the disk we avoid can grow arbitrarily fast.

"enough vanishing derivatives at 0" of the previous proposition, we chose $f_k(z) = z^{N+1}\tilde{f}_k(z)$ with N given by the previous proposition. This sequence tends to $z \mapsto z^{N+1}(z-z_0)^{-1}$.

Then, again by Fatou's lemma, $|f_k|_{L^2(D(0,\mathrm{e}^{-T}))} \to +\infty$ as $k \to +\infty$, and since $z \mapsto z^{N+1}(z-z_0)^{-1}$ is bounded on U^δ, f_k is uniformly bounded on U^δ. Therefore, the inequality $|f_k|_{L^2(D(0,\mathrm{e}^{-T}))} \le C|f_k|_{L^\infty(U^\delta)}$ is false for k large enough, and according to the previous theorem, so is the observability inequality. $\square$

3. Estimates for the holomorphy default operators

3.1. Symbols

In this section and the following, we study some operators on polynomials of the form $\sum a_n z^n \mapsto \sum \gamma_n a_n z^n$. Since these operators make the link between the holomorphy of the solution to the toy model (in the variable $z = \mathrm{e}^{-t+\mathrm{i}y}$) and the solutions to the real Grushin equation (see Lemma 8), we will call them *holomorphy default operators*. We will also call the sequence (γ_n) the *symbol* of the operator.

Our main goal is the proof of some estimates on those holomorphy default operators, in the form of Theorem 18. As a first step, we define the space of symbols we are interested in, and prove some simple facts about this space.

Definition 9. Let $r : (0,\pi/2) \to \mathbb{R}_+$ be a non-decreasing function, and for θ in $(0,\pi/2)$, let $U_{\theta,r(\theta)} = \{|z| > r(\theta), |\arg(z)| < \theta\}$ (see Fig. 4). We note $S(r)$ the set of functions γ from the union of the $U_{\theta,r(\theta)}$ to $\mathbb{C}$ which are holomorphic and have sub-exponential growth on each $U_{\theta,r(\theta)}$, i.e. for each $\theta \in (0,\pi/2)$ and $\epsilon > 0$, we have $p_{\theta,\epsilon}(\gamma) := \sup_{z\in U_{\theta,r(\theta)}} |\gamma(z)\mathrm{e}^{-\epsilon|z|}| < +\infty$. We endow $S(r)$ with the topology defined by the seminorms $p_{\theta,\epsilon}$ for all $\theta \in (0,\pi/2)$ and $\epsilon > 0$.

From now on, when we write $S(r)$, it is implicitly assumed that r is a non-decreasing function for $(0,\pi/2)$ to $\mathbb{R}_+$.

Example 10.

- Every bounded holomorphic function on the half plane $\{\Re(z) \ge 0\}$ is in $S(0)$. For instance, $z \mapsto \mathrm{e}^{-z}$ is in $S(0)$.
- Every polynomial is in $S(0)$.
- For all $s > 0$, $z \mapsto z^s$ is in $S(0)$.

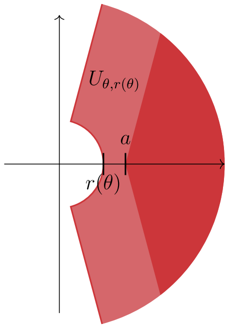

Fig. 5. If $a > r(\theta)$, then the set $\{a+z, |\arg(z)| < \theta\}$ (in darker red) is a subset of $U_{\theta,r(\pi)}$.

- More generally, if γ is holomorphic on every domain $U_{\theta,r(\theta)}$ and has at most polynomial growth on those domains, γ is in $S(r)$.

Remark 11. The only values of a symbol $\gamma \in S(r)$ we are actually interested in are the values $\gamma(n)$ at the integers; the other values do not appear in the operator $H_\gamma : \sum a_n z^n \mapsto \sum \gamma(n) a_n z^n$. However, the holomorphic hypothesis, and hence the other values of γ, is quite essential for the proof of the estimate in Theorem 18. It mainly appears to justify a change of integration path in the integral $\hat{\gamma}(\zeta) = \int_0^{+\infty} \gamma(x) \mathrm{e}^{-\mathrm{i}x\zeta}\,\mathrm{d}x$ (see Propositions 14 and 15).

Even if they do not seem to play any role in the operator H_γ, the very fact that the values of γ at non-integers exist impose some structure to the values $\gamma(n)$ at the integers. A structure we unfortunately have not been able to express in a different, more manageable way.

Let us remind that if Ω is an open subset of $\mathbb{C}$, $\mathcal{O}(\Omega)$ is the space of holomorphic functions in Ω with the topology of uniform convergence in every compact subset of Ω.

Proposition 12. *The space $S(r)$ enjoys the following properties:*

- *the topology of $S(r)$ is stronger than the topology of uniform convergence on every compact;*
- *for all compact K of $\bigcup U_{\theta,r(\theta)}$, and all $j \in \mathbb{N}$, the seminorm $\gamma \mapsto |\gamma^{(j)}|_{L^\infty(K)}$ is continuous on $S(r)$;*
- *for all z_0 in the domain of definition of γ, the punctual evaluation at z_0, i.e. $\gamma \mapsto \gamma(z_0)$, is continuous on $S(r)$;*
- *the application $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$ is continuous from $S(r) \times S(r)$ to $S(r)$.*

Proof.

- Let K be a compact subset of $\bigcup_{0<\theta<\pi/2} U_{\theta,r(\theta)}$. By the Borel–Lebesgue property, there is a finite number of θ in $(0, \pi/2)$, say $\theta_1, \ldots, \theta_k$ such that $K \subset \bigcup_{j=1}^k U_{\theta_k,r(\theta_k)}$. By noting $R = \sup_{z\in K} |z|$, we then have $|u|_{L^\infty(K)} \le \sup_{1\le j\le k} p_{\theta_j,1}(\gamma)\mathrm{e}^R$. This proves the first fact.
- We remind that if Ω is an open subset of $\mathbb{C}$, j is a natural number and K a compact subset of Ω then Cauchy's integral formula implies that the seminorm on $\mathcal{O}(\Omega)$: $f \mapsto |f^{(j)}|_{L^\infty(K)}$ is continuous. Thus, the second point is a consequence of the first one.
- Since $\{z_0\}$ is compact, the third point is a direct consequence of the second point (or the first).
- In order to prove the fourth point, we write for $z \in U_{\theta,r(\theta)}$: $|\gamma_1(z)\gamma_2(z)| \le p_{\theta,\epsilon/2}(\gamma_1) p_{\theta,\epsilon/2}(\gamma_2) \mathrm{e}^{\epsilon|z|}$, so $p_{\theta,\epsilon}(\gamma_1\gamma_2) \le p_{\theta,\epsilon/2}(\gamma_1) p_{\theta,\epsilon/2}(\gamma_2)$. $\square$

Proposition 13. *We have the following continuous injections between spaces $S(r)$:*

- *if $r' \ge r$, then denoting $U' = \bigcup U_{\theta,r'(\theta)}$, the restriction map $\gamma \in S(r) \mapsto \gamma_{|U'} \in S(r')$ is continuous;*
- *let θ_0 in $(0, \pi/2)$ and $a > r(\theta_0)$. Define $r'(\theta)$ by $r'(\theta) = 0$ if $|\theta| < \theta_0$ and $r'(\theta) = r(\theta)$ otherwise. Then $\gamma \in S(r) \mapsto \gamma(\cdot + a) \in S(r')$ is continuous.*

Proof.

- For readability, let us write $U_\theta = U_{\theta,r(\theta)}$ and $U'_\theta = U_{\theta,r'(\theta)}$. To prove the first point, we simply remark that $r' \ge r$ implies $U'_\theta \subset U_\theta$, so we have: $|\gamma(z)\mathrm{e}^{-\epsilon|z|}|_{L^\infty(U'_\theta)} \le |\gamma(z)\mathrm{e}^{-\epsilon|z|}|_{L^\infty(U_\theta)}$.
- Looking at Fig. 5 should convince us that it makes sense when looking at the domain of definition (we let the careful reader check it formally). The continuity is a consequence of: $|\gamma(z+a)\mathrm{e}^{-\epsilon|z|}| \le \mathrm{e}^{\epsilon a}|\gamma(z+a)\mathrm{e}^{-\epsilon|z+a|}|$. $\square$

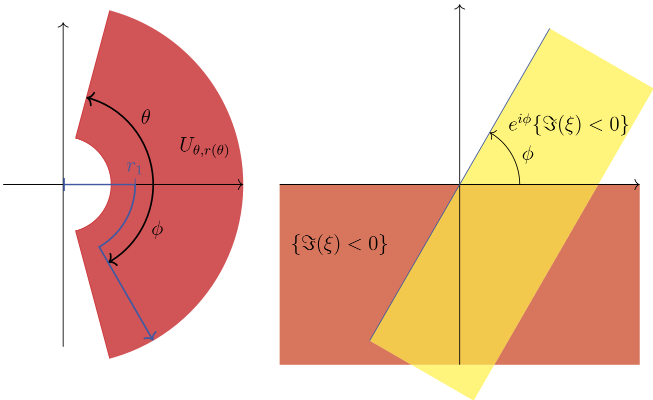

Fig. 6. In the left figure: in red, a part of the domain of definition of γ, and in blue, an integration path that allows us to extend $\hat{\gamma}$. In the right figure: in red, the a priori domain of definition of $\hat{\gamma}$, in yellow, the domain we extend $\hat{\gamma}$ to, when choosing the blue integration path of the left figure.

3.2. *Fourier transform of a symbol and convolution kernel*

The proof of the main estimate on holomorphy default operators relies on Poisson's summation formula applied to the sum $\sum \gamma(n)z^n$. In order to do that, we need some information on the Fourier transform of γ, the first of which being the existence of it.

We suppose in this subsection that for some θ_0 in $(0,\pi/2)$, $r(\theta_0)=0$ (so that $r(\theta)=0$ for $0<\theta\leq\theta_0$). Then we define the Fourier transform $\hat{\gamma}$ of γ for ξ with negative imaginary part by $\hat{\gamma}(\xi)=\int_0^{+\infty}\gamma(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x$. We first prove that this Fourier transform can be extended on a bigger domain than the lower half-plane, then, assuming some regularity at 0, we prove an estimate on it.

Proposition 14. *Let γ in $S(r)$. The Fourier transform $\hat{\gamma}$ of γ, which is holomorphic on $\{\Im(\xi)<0\}$, can be holomorphically extended on $\mathbb{C}\setminus\mathrm{i}[0,+\infty)$.*

Proof. Let ϕ in $(0,\pi/2)$, let θ in $(\phi,\pi/2)$ and $r_1>r(\theta)$. We make a change of contour in the integral defining $\hat{\gamma}(\xi)$: let c the path $[0,r_1]\cup\{r_1\mathrm{e}^{\mathrm{i}\varphi},-\phi\leq\varphi\leq 0\}\cup\mathrm{e}^{-\mathrm{i}\phi}[r_1,+\infty)$ (see Fig. 6). We have for ξ in $\{\Im(\xi)<0\}\cap\mathrm{e}^{\mathrm{i}\phi}\{\Im(\xi)<0\}$:

$$\begin{aligned}\hat{\gamma}(\xi)&=\int_0^{+\infty}\gamma(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x\\&=\int_c\gamma(z)\mathrm{e}^{-\mathrm{i}z\xi}\,\mathrm{d}z\\&=\int_0^{r_1}\gamma(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x+\int_0^{-\phi}\gamma(r_1\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}\mathrm{e}^{\mathrm{i}t}\xi}\mathrm{i}r_1\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t+\int_{r_1}^{+\infty}\gamma(\mathrm{e}^{-\mathrm{i}\phi}x)\mathrm{e}^{-\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\xi x}\mathrm{e}^{-\mathrm{i}\phi}\,\mathrm{d}x.\end{aligned}$$

The first two terms can be extended holomorphically on $\mathbb{C}$, while the third can be extended holomorphically on $\mathrm{e}^{\mathrm{i}\phi}\{\Im(\xi)<0\}$. So, taking $\phi\to\pi/2$, $\hat{\gamma}$ can be extended holomorphically on $\{\Im(\xi)<0\}\cup\mathrm{i}\{\Im(\xi)<0\}$. By taking the path c' the symmetric of c with respect to the real line, $\hat{\gamma}$ can also be extended holomorphically on $\{\Im(\xi)<0\}\cup-\mathrm{i}\{\Im(\xi)<0\}$. $\square$

Proposition 15. *Let $\epsilon>0$. There exists $C>0$ and $\eta>0$ such that for all γ in $S(r)$ satisfying $p(\gamma):=\sup_{|z|<1,|\arg(z)|<\theta_0}\frac{|\gamma(z)|}{|z|}<+\infty$ and for all ξ in $\{-\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta},r>\epsilon,|\theta|<2\theta_0\}$ (see Fig. 7):*

$$|\hat{\gamma}(\xi)|\leq C(p(\gamma)+p_{\theta_0,\eta}(\gamma))|\xi|^{-2}.$$

Proof. The proof is mostly redoing the calculation of the proof of the previous proposition, but this time using the increased regularity (at 0) to get the stated estimate.

Let $\xi=-\mathrm{i}r\mathrm{e}^{\mathrm{i}2\theta}$ with $|\theta|<\theta_0$ and $r>\epsilon$. Thanks to the proof of the previous proposition, we have $\hat{\gamma}(\xi)=\int_{\mathrm{e}^{-\mathrm{i}\theta}\mathbb{R}_+}\gamma(z)\mathrm{e}^{-\mathrm{i}z\xi}\,\mathrm{d}z=\mathrm{e}^{-\mathrm{i}\theta}\int_0^{+\infty}\gamma(\mathrm{e}^{-\mathrm{i}\theta}x)\mathrm{e}^{-\mathrm{e}^{\mathrm{i}\theta}xr}\,\mathrm{d}x$. We then write $|\hat{\gamma}(\xi)|\leq\int_0^1 p(\gamma)x\mathrm{e}^{-\cos(\theta)rx}\mathrm{d}x+\int_1^{+\infty}p_{\theta_0,\eta}(\gamma)\mathrm{e}^{\eta x}\mathrm{e}^{-\cos(\theta)rx}\mathrm{d}x$, which is true for all $\eta>0$.

For the first term of the right-hand side, we make the change of variables $x'=\cos(\theta)rx$, so that $\int_0^1 x\mathrm{e}^{-\cos(\theta)rx}\,\mathrm{d}x=\frac{1}{(r\cos(\theta))^2}\int_0^{r\cos(\theta)}x'\mathrm{e}^{-x'}\,\mathrm{d}x\leq\frac{1}{(r\cos(\theta))^2}\Gamma(2)$.

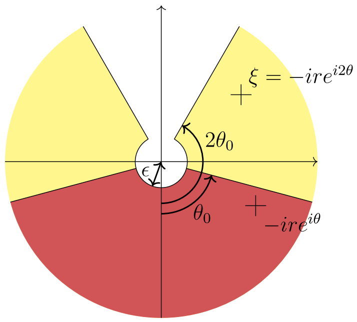

Fig. 7. The sub-exponential growth of γ gives us an estimate on $\hat{\gamma}$ on the red domain, and a change of integration path allows us to extend this estimate on the yellow domain.

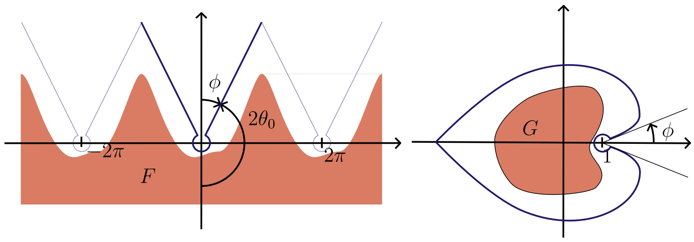

Fig. 8. Left figure: in red, the domain F, in plain blue, the boundary of $\{-ire^{i\theta}, r > \epsilon, |\theta| \le 2\theta_0\}$ and in dotted blue, the boundary of the 2π-periodic version of the previous domain. Right figure: in red, the domain $G = e^{-iF}$, and in blue, the boundary of $\{e^{\xi}, |\xi| > \epsilon, |\arg(\xi)| > \phi\}$.

For the second term of the right-hand side, we have $\int_1^{+\infty} e^{x(\eta-\cos(\theta)r)}\,dx = \frac{1}{\cos(\theta)r-\eta}e^{\eta-\cos(\theta)r}$ as long as $\eta < \cos(\theta)r$. We then choose $\eta = \frac{\epsilon}{2}\cos(\theta_0)$ so that $|\theta| < \theta_0$ and $r > \epsilon$ implies $\cos(\theta)r - \eta > \cos(\theta_0)\epsilon - \frac{\epsilon}{2}\cos(\theta_0) = \frac{1}{2}\cos(\theta_0)\epsilon$. So $\int_1^{+\infty} e^{x(\eta-\cos(\theta)r)}\,dx \le \frac{2}{\epsilon\cos(\theta_0)}e^{\eta}e^{-r\cos(\theta_0)}$. So, writing $c = \sup_{t>0}(t^2e^{-t})$ and $C_2 = \frac{2ce^{\eta}}{\epsilon\cos(\theta_0)}$, we have $\int_0^{+\infty} e^{x(\eta-\cos(\theta)r)}\,dx \le C_2\frac{1}{(r\cos(\theta_0))^2}$.

Combining these two inequalities, we have:

$$|\hat{\gamma}(\xi)| \le (\Gamma(2)p(\gamma) + C_2 p_{\theta_0,\eta}(\gamma))\frac{1}{\cos(\theta_0)^2}r^{-2}. \qquad \square$$

With the previous two properties, we can prove the main tool for establishing estimates on holomorphy default operators (Proposition 16).

Proposition 16. *Let γ in $S(r)$ and K_γ the function defined by $K_\gamma(z) = \sum\gamma(n)z^n$. Then K_γ admits a holomorphic extension to $\mathbb{C}\setminus[1,+\infty[$. Moreover, the map $\gamma \in S(r) \mapsto K_\gamma \in \mathcal{O}(\mathbb{C}\setminus[1,+\infty[)$ is continuous.*

Remark 17. This theorem was already essentially proved by Lindelöf [18] in the special case $r(\theta) = \frac{r_0}{\cos(\theta)}$, that is, when the domain of definition of γ is the half-plane $\{\Re(z) > r_0\}$, and the case of a general r was proved by Arakelyan [2]. Our method is different than in the previous two references, and, most importantly, we prove the continuity of the analytic continuation with respect to the topology of $S(r)$.

Proof. Let G be a connected relatively compact open subset of $\mathbb{C}\setminus[1,+\infty)$. We suppose without loss of generality that G intersects the unit disk $D(0,1)$. We want to show that K_γ can be extended to a bounded holomorphic function on G, and that this extension depends continuously on γ for the topology of uniform convergence on G.

First, we reduce the problem to a case where we can use the previous estimate on the Fourier transform of symbols, by defining $\tilde{\gamma}(z) = \gamma(z+n_1) - \gamma(n_1)$, with n_1 large enough. We can explicitly compute K_γ from $K_{\tilde{\gamma}}$, so we focus on the latter, and apply Poisson summation formula to the sum defining $K_{\tilde{\gamma}}$, the estimate on $\hat{\tilde{\gamma}}$ allowing us to holomorphically extend the sum.

Choice of n_1. The Poisson summation formula will involve terms of the form $\hat{\tilde{\gamma}}(i\ln(\zeta)+2\pi k)$, so we let $F = \{\xi \in \mathbb{C}, e^{-i\xi} \in G\}$. For all ζ in $\mathbb{C}$, $\zeta \in G$ is equivalent to $i\ln(\zeta) \in F$, whatever the determination of the logarithm.

Since G is relatively compact on $\mathbb{C}\setminus[1,+\infty)$, there exists ϕ in $(0,\pi)$ and $\epsilon > 0$ such that $G \subset \mathbb{C}\setminus\{e^{\xi}, |\xi| \le \epsilon$ or $|\arg(\xi)| \le \phi\}$. Then, noting $\theta_0 = \frac{1}{2}(\pi-\phi)$, F is a subset of $\{-ire^{i\theta}, r > \epsilon, |\theta| < 2\theta_0\}$ (see Fig. 8). Let n_1 be a natural number greater than $r(\theta_0)$, for instance $n_1 = \lfloor r(\theta_0) \rfloor + 1$, let $\tilde{r} : (0, \pi/2) \to \mathbb{R}_+$ be defined by $\tilde{r}(\theta) = 0$ for $0 < \theta \leq \theta_0$ and $\tilde{r}(\theta) = r(\theta)$ for $\theta > \theta_0$, and let $\tilde{\gamma}$ be defined by $\tilde{\gamma}(z) = \gamma(z + n_1) - \gamma(n_1)$.

According to the second point of Proposition 13, $\tilde{\gamma}$ is in $S(\tilde{r})$ and depends continuously on γ. Moreover, we have, for z in $\{|z| < 1, |\arg(z)| < \theta_0\}$, $|\tilde{\gamma}(z)| \leq \sup_{t \in [n_1, z+n_1]} |\gamma'(t)| |z|$, so, if we define $p(\tilde{\gamma})$ as in the previous proposition by $p(\tilde{\gamma}) = \sup_{|z|<1, |\arg(z)|<\theta_0} \frac{|\tilde{\gamma}(z)|}{|z|}$, we have $p(\tilde{\gamma}) \leq \sup_{|z| \leq 1, |\arg(z)| \leq \theta_0} |\gamma'(z + n_1)|$, which is finite since the subset $\{z + n_1, |z| \leq 1, |\arg(z)| \leq \theta_0\}$ is compact in $\bigcup U_\theta$, and thanks to the second point of Proposition 12, $\gamma \mapsto p(\tilde{\gamma})$ is continuous.

Relation between K_γ and $K_{\tilde{\gamma}}$. We have for all ζ in the unit disk:

$$
\begin{aligned}
K_\gamma(\zeta) &= \sum_{n > r(0)} \gamma(n) \zeta^n \\
&= \sum_{r(0) < n < n_1} \gamma(n) \zeta^n + \zeta^{n_1} \left(\gamma(n_1) \sum_{n \geq 0} \zeta^n + \sum_{n \geq 0} \tilde{\gamma}(n) \zeta^n \right) \\
&= \sum_{r(0) < n < n_1} \gamma(n) \zeta^n + \gamma(n_1) \frac{\zeta^{n_1}}{1 - \zeta} + \zeta^{n_1} K_{\tilde{\gamma}}(\zeta).
\end{aligned} \tag{16}
$$

So, if we prove that $K_{\tilde{\gamma}}$ extends holomorphically to G and that the extension depends continuously on $\tilde{\gamma}$ in the topology of uniform convergence on G, we will have proved the same for K_γ.

Poisson summation formula and holomorphic extension. We have by definition of $K_{\tilde{\gamma}}$, for all $|\zeta| < 1$: $K_{\tilde{\gamma}}(\zeta) = \sum_{n > 0} \tilde{\gamma}(n) \zeta^n$. So, the Poisson summation formula implies that, for all $|\zeta| < 1$,

$$
K_{\tilde{\gamma}}(\zeta) = 2\pi \sum_{k \in \mathbb{Z}} \widehat{\zeta^x \tilde{\gamma}(x)}(2\pi k) = 2\pi \sum_{k \in \mathbb{Z}} \hat{\tilde{\gamma}}(\mathrm{i} \ln \zeta + 2\pi k).
$$

Let us recall that $F = \{\xi, \mathrm{e}^{-\mathrm{i}\xi} \in G\}$ is a subset of $\{-\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}, r > \epsilon, |\theta| < 2\theta_0\}$, and let us remark that it is a 2π-periodic domain, so if z is in F, then for all $k \in \mathbb{Z}$, $|z + 2\pi k| > \epsilon$. So the estimate of Fourier transform of symbols (Proposition 15) implies that the sum $k_{\tilde{\gamma}}(z) := \sum_{k \in \mathbb{Z}} \hat{\tilde{\gamma}}(z + 2\pi k)$ converges, and satisfies $|k_{\tilde{\gamma}}(z)| \leq C(p(\tilde{\gamma}) + p_{\theta_0,\eta}(\tilde{\gamma})) \sum_{k \in \mathbb{Z}} |z + 2\pi k|^{-2} \leq C'_\epsilon(p(\tilde{\gamma}) + p_{\theta_0,\eta}(\tilde{\gamma}))$. Moreover, this sum converges uniformly in $z \in F$, so the limit function $k_{\tilde{\gamma}}$ is holomorphic, and depends continuously on $\tilde{\gamma} \in S(\tilde{r})$.

Since we have $K_{\tilde{\gamma}}(\zeta) = k_{\tilde{\gamma}}(\mathrm{i} \ln \zeta)$, $K_{\tilde{\gamma}}$ extends holomorphically on $G \setminus [0, +\infty)$. But we already knew that $K_{\tilde{\gamma}}$ is holomorphic on the unit disk, so $K_{\tilde{\gamma}}$ is holomorphic in G. Moreover, since $k_{\tilde{\gamma}}$ depends continuously on $\tilde{\gamma}$, $K_{\tilde{\gamma}}$ also depends continuously on $\tilde{\gamma}$. This completes the proof of the proposition. □

3.3. Proof of the estimate for the holomorphy default operators

Before stating the estimates for holomorphy default operators, let us define a few notations. Let r be a non-decreasing function from $(0, \pi/2)$ to $\mathbb{R}_+$. We note $r(0) = \inf_{\theta \in (0,\pi/2)} r(\theta)$. Let $\mathcal{O}_{r(0)}$ be the closed subspace of $\mathcal{O}(\mathbb{C})$ of entire functions of the form $\sum_{n > r(0)} a_n z^n$, i.e. $\mathcal{O}_{r(0)} = \{f \in \mathcal{O}(\mathbb{C}), \forall 0 \leq j \leq r(0), f^{(j)}(0) = 0\}$. If we endow $\mathcal{O}_{r(0)}$ with the $L^\infty(U)$ norm for some open bounded subset U of $\mathbb{C}$, we will note this space $\mathcal{O}^\infty_{r(0)}(U)$.

Theorem 18. *Let $r : (0, \pi/2) \to \mathbb{R}_+$ be a non-decreasing function. Let γ in $S(r)$ and H_γ the operator on polynomials with $\lfloor r(0) \rfloor$ vanishing derivatives at* 0*, defined by:*

$$
H_\gamma\left(\sum_{n > r(0)} a_n z^n \right) = \sum_{n > r(0)} \gamma(n) a_n z^n.
$$

Let U be an open bounded domain, star shaped with respect to 0*. Let $\delta > 0$ and $U^\delta = \{z \in \mathbb{C}, \operatorname{distance}(z, U) < \delta\}$. Then there exists $C > 0$ such that, for all polynomials f with vanishing derivatives of order up to $\lfloor r(0) \rfloor$,*

$$
|H_\gamma(f)|_{L^\infty(U)} \leq C |f|_{L^\infty(U^\delta)}.
$$

Moreover, the constant C above can be chosen continuously in $\gamma \in S(r)$: the map $\gamma \in S(r) \mapsto H_\gamma$ is continuous from $S(r)$ to $\mathcal{L}\big(\mathcal{O}^\infty_{r(0)}(U^\delta), \mathcal{O}^\infty_{r(0)}(U)\big)$.

Before proving the theorem, let us remark that the sub-exponential growth of γ implies that H_γ do maps $\mathcal{O}_{r(0)}$ to $\mathcal{O}_{r(0)}$, so the theorem actually makes sense.

Proof. Let $R>0$ large enough so that $\bar U \subset D(0,R)$. If $f=\sum a_n z^n$ is an entire function, we have $a_n = \frac{1}{2i\pi}\oint_{\partial D(0,R)} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta$, so, for z in U, we have:

$$H_\gamma(f)(z) = \sum_n \gamma_n \frac{1}{2i\pi} \oint_{\partial D(0,R)} \frac{f(\zeta)}{\zeta^{n+1}} z^n \,d\zeta = \oint_{\partial D(0,R)} \frac{1}{2i\pi\zeta} K_\gamma\left(\frac{z}{\zeta}\right) f(\zeta)\,d\zeta.$$

We want to change the integration path for one that is closer to U. For any closed curve c around U, since U is star-shaped with respect to 0, for any $z\in U$ and $\zeta \in c$, we never have $z/\zeta \in [1,+\infty)$. So, the subset $\{z/\zeta, z\in U, \zeta\in c\}$ is a compact subset of $\mathbb{C}\setminus[1,+\infty)$, and according to the previous proposition, $M_c(\gamma) := \sup_{z\in U,\zeta\in c}\left|K_\gamma\left(\frac{z}{\zeta}\right)\right|$ is finite and depends continuously on $\gamma \in S(r)$.

So, we have for $z\in U$, $|H_\gamma(z)| \le \sup_{\zeta\in c}\frac{1}{2\pi|\zeta|} M_c(\gamma)\sup_c |f|$. Since we can choose c as close as we want to U, this proves the theorem. □

Remark 19. Actually, the theorem we proved is the following: if (γ_n) is a sequence of complex numbers such that the entire series $\sum_{n\ge 0}\gamma_n\zeta^n$ has non-zero convergence radius and that $K_\gamma(\zeta) := \sum_{n\ge 0}\gamma_n\zeta^n$ admits a holomorphic extension on $\mathbb{C}\setminus[1,+\infty)$, then, for all domain U satisfying the hypotheses of the theorem, and for all $\delta>0$, there exists $C>0$ such that, for all entire functions f, $|H_\gamma(f)|_{L^\infty(U)} \le C|f|_{L^\infty(U^\delta)}$. Moreover, C can be chosen continuously in K_γ (for the topology of uniform convergence on every compact).

4. Spectral analysis of the Fourier components

4.1. Introduction

In this section, we prove estimates on the first eigenvalue λ_α of $-\partial_x^2 + (\alpha x)^2$ on $(-1,1)$ with Dirichlet boundary conditions, and on its associated eigenfunction. Let us recall some facts already mentioned by K. Beauchard, P. Cannarsa, and R. Guglielmi [5], which are proved thanks to Sturm–Liouville's theory:

Proposition 20. *Let α be a real number. The (unbounded) operator $P_\alpha = -\partial_x^2 + (\alpha x)^2$ on L^2 (with domain $H_0^1(-1,1)\cap H^2(-1,1)$) admits an orthonormal basis $(v_{\alpha k})_{k\ge 0}$ of eigenvectors, with the associated eigenvalues sequence $(\lambda_{\alpha k})_{k\ge 0}$ being non-decreasing and tending to $+\infty$ as $k\to+\infty$. Moreover, the first eigenvalue $\lambda_\alpha = \lambda_{\alpha 0}$ is simple, greater than $|\alpha|$, and we have $\lambda_\alpha \sim_{\alpha\to+\infty} \alpha$. Finally, the associated eigenvector $v_\alpha = v_{\alpha 0}$ is even, positive on $(-1,1)$, and non-increasing on $[0,1)$.*

These properties are also linked to the scaling $x = y/\sqrt{\alpha}$. Indeed, if we define $\tilde v_\alpha$ by $\tilde v_\alpha(y) = v_\alpha(y/\sqrt{\alpha})$, $\tilde v_\alpha$ satisfies $-\tilde v_\alpha'' + y^2\tilde v_\alpha = \frac{\lambda_\alpha}{\alpha}\tilde v_\alpha$, a fact that we will use extensively in all the proofs in this section. As an example of this scaling, we can already prove the following lemma, which was used to get a lower bound on the left-hand side of the observability inequality in Proposition 7 (Lemma 21).

Lemma 21. *If we normalize v_n by $v_n(0)=1$ instead of $|v_n|_{L^2(-1,1)}=1$, there exists $c>0$ such that for all $n\ge 1$, $|v_n|_{L^2(-1,1)} \ge cn^{-1/4}$.*

Proof. Let us note $\tilde v_n(y) = v_n(y/\sqrt{n})$, which is the solution to the Cauchy problem $-\tilde v_n'' + y^2\tilde v_n = \frac{\lambda_n}{n}\tilde v_n$, $\tilde v_n(0)=1$, $\tilde v_n'(0)=0$. Moreover, $\tilde v_n(\pm\sqrt{n})=0$. Since $\lambda_n\sim n$, $\tilde v_n$ converges to the solution $\tilde v$ to $-\tilde v'' + y^2\tilde v = \tilde v$, $\tilde v(0)=1$, $\tilde v'(0)=0$, that is, $\tilde v(y) = e^{-y^2/2}$, this convergence being uniform on every compact subsets of $\mathbb{R}$.

So, $\int_{-1}^1 \tilde v_n(y)^2\,dy \xrightarrow[n\to+\infty]{} \int_{-1}^1 e^{-y^2}\,dy$, and we have $c := \inf_n \int_{-1}^1 \tilde v_n(y)^2\,dy > 0$. By the change of variables $x = y/\sqrt{n}$, we have:

$$\int_{-1/\sqrt{n}}^{1/\sqrt{n}} v_n(x)^2\,dx = \frac{1}{\sqrt{n}}\int_{-1}^{1}\tilde v_n(y)^2\,dy \ge \frac{c}{\sqrt{n}}$$

and since $\int_{-1}^1 v_n(x)^2\,dx \ge \int_{-1/\sqrt{n}}^{1/\sqrt{n}} v_n(x)^2\,dx$, this proves the lemma. □

4.2. Exponential estimate of the first eigenvalue

In this subsection, we still normalize v_α so that $v_\alpha(0)=1$ instead of normalizing it in $L^2(-1,1)$. The main result of this section is about refining the estimates $\lambda_n \sim n$.

Theorem 22. *There exists a non-decreasing function $r:(0,\pi/2)\to\mathbb{R}_+$ and a function γ in $S(r)$ (see Definition 9) such that, for all reals $\alpha > r(0)$, $\lambda_\alpha = \alpha + \gamma(\alpha)e^{-\alpha}$.*

Remark 23.

- This is a semi-classical problem with $h=\frac{1}{\alpha}$. The asymptotic expansion of λ_α was already known for α real (see for instance [15]), but the estimate in our result is also valid for α complex, which was not known before (as far as the author knows).
- We will also prove that for all θ in $(0,\pi/2)$:

$$\gamma(\alpha) \underset{\substack{|\alpha|\to\infty \\ \alpha\in U_{\theta,r(\theta)}}}{\sim} 4\pi^{-1/2}\alpha^{3/2}.$$

A careful examination of the proof even shows that we have an asymptotic expansion of the form $\gamma(\alpha)=\sum_{k\ge 0} a_k\alpha^{3/2-k}$, this expansion being valid in each $U_{\theta,r(\theta)}$, and where the a_k can be in principle computed explicitly.

Proof. The proof is in three steps. We first explicitly solve the equation satisfied by v_α for $\alpha>0$, expressing the solution as an integral on some complex path. Then, writing the boundary condition for this explicit solution constitutes an implicit equation satisfied by α and $\rho_\alpha=\lambda_\alpha-\alpha$, this equation still making sense if α is complex with positive real part. We use Newton's method to solve this implicit equation, with the stationary phase theorem providing the necessary estimates for Newton's method to converge. Finally, the stationary phase theorem also implies an equivalent of the solution that Newton's method gives us, which will allow us to conclude.

Explicit solution to the equation satisfied by $\boldsymbol{v_\alpha}$. Let us recall that v_α satisfies $-v''_\alpha+(\alpha x)^2v_\alpha=\lambda_\alpha v_\alpha$. We have, by choice of normalization, $v_\alpha(0)=1$, and since v_α is even $v'_\alpha(0)=0$. Let w_α be defined by $v_\alpha=\mathrm{e}^{-\alpha x^2/2}w_\alpha$. By developing the derivatives, we have: $-w''_\alpha+2\alpha x w'_\alpha=(\lambda_\alpha-\alpha)w_\alpha$. Finally, we make the change of variables $x=y/\sqrt{\alpha}$, so that $\tilde w_\alpha(y)=w_\alpha(y/\sqrt{\alpha})$ satisfies $-\tilde w''_\alpha+2y\tilde w'_\alpha=(\frac{1}{\alpha}\lambda_\alpha-1)\tilde w_\alpha$ as well as $\tilde w_\alpha(0)=1$, $\tilde w'_\alpha(0)=0$, $\tilde w_\alpha(\sqrt{\alpha})=0$. So, for all real $\tilde\rho$, we consider the ordinary differential equation:

$$\begin{aligned} &-\tilde w''+2x\tilde w'-\tilde\rho\tilde w=0\\ &\quad\tilde w(0)=1,\ \tilde w'(0)=0. \end{aligned} \tag{17}$$

Let $g(z)=\mathrm{e}^{-z^2/4-(1+\tilde\rho/2)\ln(z)}$ (with the logarithm chosen so that $\ln(1)=0$ and ln is continuous on the path we will integrate g on). This function satisfies $-z^2g-2(zg)'-\tilde\rho g=0$ on any simply connected domain of $\mathbb{C}^\star$. Let Γ_+ and Γ_- be paths in $\mathbb{C}^\star$ from $-\infty$ to ∞ going above and below 0 respectively. For instance, we can take $\Gamma_+=(-\infty,-\epsilon]\cup\{\epsilon\mathrm{e}^{\mathrm{i}(\pi-\theta)},0\le\theta\le\pi\}\cup[\epsilon,+\infty)$ and $\Gamma_-=(-\infty,-\epsilon]\cup\{\epsilon\mathrm{e}^{\mathrm{i}\theta},-\pi\le\theta\le 0\}\cup[\epsilon,+\infty)$ for some $\epsilon>0$. Then, by integration by parts, the functions $\tilde w_+$ and $\tilde w_-$ defined by $\tilde w_\pm(y)=\int_{\Gamma_\pm}g(z)\mathrm{e}^{-yz}\,\mathrm{d}z$ are solutions to the equation $-\tilde w''+2y\tilde w'=\tilde\rho\tilde w$. When $\rho<2$, these solutions satisfy:

$$\begin{aligned} \tilde w'_\pm(0)&=-\int_{\Gamma_\pm} z\mathrm{e}^{z^2/4-(1+\tilde\rho/2)\ln(z)}\,\mathrm{d}z\\ &=-(1+\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}\tilde\rho})\int_0^{+\infty}\mathrm{e}^{-x^2/4-\tilde\rho/2\ln(x)}\,\mathrm{d}x. \end{aligned}$$

Finally, when $\rho<2$, the solution to Eq. (17) is, up to a constant,

$$\tilde w(y)=\left(\left(1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\tilde\rho}\right)\int_{\Gamma_+}-\left(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\tilde\rho}\right)\int_{\Gamma_-}\right)\exp\left(-yz-\frac{z^2}{4}-\left(1+\frac{\tilde\rho}{2}\right)\ln z\right)\mathrm{d}z,$$

where we have defined $(a_+\int_{\Gamma_+}+a_-\int_{\Gamma_-})f(s)\,\mathrm{d}s=a_+\int_{\Gamma_+}f(s)\,\mathrm{d}s+a_-\int_{\Gamma_-}f(s)\,\mathrm{d}s$.

Implicit equation and Newton's method. In the case where $\tilde\rho=\tilde\rho(\alpha):=\frac{1}{\alpha}\lambda_\alpha-1$, the above solution is up to a constant $\tilde w_\alpha$, so α and $\tilde\rho(\alpha)$ satisfy $\tilde w(\sqrt{\alpha})=0$ when $\tilde\rho=\tilde\rho(\alpha)$. So, let us specify in the above solution $y=\sqrt{\alpha}$ and make the change of variables/change of integration path $z=\sqrt{\alpha}s$, and write $-yz-z^2/4=-\alpha(1+s/2)^2+\alpha$:

$$\tilde w(\sqrt{\alpha})=\sqrt{\alpha}\mathrm{e}^{\alpha}\left(\left(1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\tilde\rho}\right)\int_{\Gamma_+}-\left(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\tilde\rho}\right)\int_{\Gamma_-}\right)\exp\left(-\alpha\left(1+\frac{s}{2}\right)^2-\left(1+\frac{\tilde\rho}{2}\right)\ln s\right)\mathrm{d}s.$$

So, letting $\Phi(\rho,\alpha)$ be defined by:

$$\Phi(\rho,\alpha)=\left((1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\rho})\int\limits_{\Gamma_+}-(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\rho})\int\limits_{\Gamma_-}\right)\exp\left(-\alpha\left(1+\frac{s}{2}\right)^2-\left(1+\frac{\rho}{2}\right)\ln s\right)\mathrm{d}s,$$

and assuming $\tilde{\rho}(\alpha)<2$, we have $\Phi(\tilde{\rho}(\alpha),\alpha)=0$. When $|\rho|<2$, we even have the equivalence between $\Phi(\rho,\alpha)=0$ and $\alpha(1+\rho)$ being an eigenvalue of $-\partial_x^2+(\alpha x)^2$ on $(-1,1)$.

Note that the equation $\Phi(\rho,\alpha)=0$ still makes sense if we take α with positive real part, and, as stated previously, we want to solve it with Newton's method (Theorem 30). In order to prove the convergence of Newton's method on suitable sets (i.e. for each $\theta\in(0,\pi/2)$, a set $U_{\theta,r(\theta)}$), we need to estimate $(\partial_\rho\Phi)^{-1}$, $\partial_\rho^2\Phi$, and $\Phi(0,\alpha)$; in particular, we will show that the latter decays faster than the two former as $|\alpha|$ tends to $+\infty$.

By differentiating under the integral we have:

$$\partial_\rho\Phi(\rho,\alpha)=\mathrm{i}\frac{\pi}{2}\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\rho}\int\limits_{\Gamma_+}+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\rho}\int\limits_{\Gamma_-}\right)\exp\left(-\alpha\left(1+\frac{s}{2}\right)^2-\left(1+\frac{\rho}{2}\right)\ln s\right)\mathrm{d}s$$

$$-\frac{1}{2}\left((1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\rho})\int\limits_{\Gamma_+}-(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\rho})\int\limits_{\Gamma_-}\right)\exp\left(-\alpha\left(1+\frac{s}{2}\right)^2-\left(1+\frac{\rho}{2}\right)\ln s\right)\ln s\,\mathrm{d}s$$

so, by the stationary phase theorem, with the only critical point being -2 (see Proposition 29):

$$\begin{aligned}\partial_\rho\Phi(\rho,\alpha)=&\sqrt{\frac{\pi}{\alpha}}\Bigg(\mathrm{i}\frac{\pi}{2}\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\rho}\mathrm{e}^{-(1+\rho/2)(\ln 2+\mathrm{i}\pi)}+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\rho}\mathrm{e}^{-(1+\rho/2)(\ln 2-\mathrm{i}\pi)}\right)\\&-\frac{1}{2}((1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\rho})\mathrm{e}^{-(1+\rho/2)(\ln 2+\mathrm{i}\pi)}(\ln 2+\mathrm{i}\pi)\\&-(1+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\rho})\mathrm{e}^{-(1+\rho/2)(\ln 2-\mathrm{i}\pi)}(\ln 2-\mathrm{i}\pi))\Bigg)+\mathcal{O}_{\alpha\in U_{\theta,1}}(|\alpha|^{-3/2})\\=&\,\mathrm{i}\sqrt{\frac{\pi}{\alpha}}2^{-(1+\frac{\rho}{2})}\left(\pi\cos\left(\frac{\pi\rho}{2}\right)-\ln(2)\sin\left(\frac{\pi\rho}{2}\right)\right)+\mathcal{O}_{\alpha\in U_{\theta,1}}(|\alpha|^{-3/2}),\end{aligned}\tag{18}$$

the $\mathcal{O}$ being uniform in $|\rho|\le 1$.

If $\rho_{\max}>0$ is chosen so that for all $|\rho|\le\rho_{\max}$, $|\pi\cos(\pi\rho/2)-\ln(2)\sin(\pi\rho/2)|\ge\frac{\pi}{2}$, there exists $C'_\theta>0$ and $r(\theta)>0$ such that for all $|\rho|<\rho_{\max}$ and α in $U_{\theta,r(\theta)}$, $|(\partial_\rho\Phi(\rho,\alpha))^{-1}|<C'_\theta\sqrt{|\alpha|}$.

Similarly, $\partial_\rho^2\Phi(\rho,\alpha)$ can be expressed in terms of $\int_{\Gamma_\pm}\exp(-\alpha(1+s/2)^2-(1+\rho/2)\ln(s))\ln(s)^m\,\mathrm{d}s$ with $m\in\{0,1,2\}$, so, by the stationary phase theorem, increasing $r(\theta)$ if need be, there exists $C_\theta>0$ such that, for all $|\rho|<\rho_{\max}$ and α in $U_{\theta,r(\theta)}$, $|\partial_\rho^2\Phi(\rho,\alpha)|<C_\theta\frac{1}{\sqrt{|\alpha|}}$.

Now, by explicitly writing the integrals defining Φ, we have for all α with $\Re(\alpha)>0$:

$$\begin{aligned}\Phi(0,\alpha)&=2\int\limits_{\Gamma_+}\exp\left(-\alpha(1+\frac{s}{2})^2\right)\frac{1}{s}\,\mathrm{d}s-2\int\limits_{\Gamma_-}\exp\left(-\alpha(1+\frac{s}{2})^2\right)\frac{1}{s}\,\mathrm{d}s\\&=2\lim_{\epsilon\to 0}\int_{\pi}^{-\pi}\exp\left(-\alpha(1+\frac{1}{2}\epsilon\mathrm{e}^{\mathrm{i}\theta})^2\right)\mathrm{i}\,\mathrm{d}\theta\\&=-4\mathrm{i}\pi\mathrm{e}^{-\alpha},\end{aligned}$$

so, increasing again $r(\theta)$ if necessary, we have for all α in $U_{\theta,r(\theta)}$:

$$|\Phi(0,\alpha)|\le\min((2C_\theta {C'_\theta}^2)^{-1},\frac{1}{5}{C'_\theta}^{-1})|\alpha|^{-1/2}.$$

Then according to Theorem 30, with $R=\rho_{\max}/10$, $C_1=C_\theta|\alpha|^{-1/2}$, $C_2=C'_\theta|\alpha|^{1/2}$ and with starting point $z_0=0$, the sequence $(\tilde{\rho}_n(\alpha))$ defined by $\tilde{\rho}_0(\alpha)=0$, $\tilde{\rho}_{n+1}(\alpha)=\tilde{\rho}_n(\alpha)-\partial_\rho\Phi(\tilde{\rho}_n(\alpha),\alpha)^{-1}\Phi(\tilde{\rho}_n(\alpha),\alpha)$ converges and the limit $\tilde{\rho}_\infty(\alpha)$ satisfies $|\tilde{\rho}_\infty(\alpha)-\tilde{\rho}_k(\alpha)|\le C|A\sqrt{\alpha}\mathrm{e}^{-\alpha}|^{2^k}$ for some $C>0$ and $A>0$.

Equivalent of the solution and conclusion. Let us first prove that $\tilde{\rho}_\infty$ is holomorphic. By induction, every $\tilde{\rho}_k$ is holomorphic, and the estimate $|\tilde{\rho}_\infty(\alpha)-\tilde{\rho}_k(\alpha)|\le C|A\sqrt{\alpha}\mathrm{e}^{-\alpha}|^{2^k}$ shows that $\tilde{\rho}_k$ converges uniformly in $U_{\theta,r(\theta)}$ (provided that $r(\theta)>0$), so $\tilde{\rho}_\infty$ is also holomorphic.

Now let us compute an equivalent of $\tilde\rho_\infty$. According to the previous estimate with $k=1$, we have $\tilde\rho_\infty(\alpha) = -\partial_\rho\Phi(0,\alpha)^{-1}\Phi(0,\alpha) + \mathcal{O}(\mathrm{e}^{-2\alpha})$ for $\alpha \in U_{\theta,r(\theta)}$. Thanks to the stationary phase theorem, or more specifically Eq. (18), we have: $\partial_\rho\Phi(0,\alpha) = \mathrm{i}\pi^{3/2}\alpha^{-1/2} + \mathcal{O}_{\alpha\in U_{\theta,r(\theta)}}(|\alpha|^{-3/2})$, and $\Phi(0,\alpha) = -4\mathrm{i}\pi\mathrm{e}^{-\alpha}$. So, we have: $\tilde\rho_\infty(\alpha) = 4\pi^{-1/2}\alpha^{1/2}\mathrm{e}^{-\alpha}(1 + \mathcal{O}(|\alpha|^{-1})) + \mathcal{O}(\mathrm{e}^{-2\alpha})$, and since $\mathrm{e}^{-2\alpha} = \mathcal{O}(|\alpha|^{-1/2}\mathrm{e}^{-\alpha})$ for $\alpha \in U_{\theta,r(\theta)}$, we finally have $\tilde\rho_\infty(\alpha) \sim 4\pi^{-1/2}\alpha^{1/2}\mathrm{e}^{-\alpha}$ for $\alpha \in U_{\theta,r(\theta)}$.

We still have to check that for α real, $\tilde\rho_\infty(\alpha)$ is equal to $\tilde\rho_\alpha$ (let us remind that $\lambda_\alpha = \alpha(1+\tilde\rho_\alpha)$). According to Eq. (18), we have for some $C''_\theta > 0$ and for all $|\rho| \le \rho_{\max}$ and $\alpha \in U_{\theta,r(\theta)}$: $\Im(\partial_\rho\phi(\rho,\alpha)) \ge C''_\theta/\sqrt{|\alpha|}$. So for $|\rho| < \rho_{\max}$ and $\alpha \in U_{\theta,r(\theta)}$, $|\Phi(\rho,\alpha)| = |\Phi(\rho,\alpha) - \Phi(\tilde\rho_\infty(\alpha))| \ge C''_\theta|\rho - \tilde\rho_\infty(\alpha)|/\sqrt{|\alpha|}$. So for α real big enough, $\tilde\rho_\infty(\alpha)$ is the smallest non-negative zero of $\Phi(\cdot,\alpha)$.

So $\tilde\rho_\infty$ is the smallest positive eigenvalue of $-\partial_x^2 + (\alpha x)^2$, and since these eigenvalues are all positive (Proposition 20), we actually have $\lambda_\alpha = \alpha(\tilde\rho_\infty(\alpha)+1)$. □

Remark 24. This proof is the one we are not (yet?) able to carry if we replace in the Grushin equation (2) the potential x^2 by $a(x) = x^2 + x^3 b(x)$, where b is any non-null analytic function. Indeed, the proof above relies on an exact integral representation of the solution to $-v'' + a(x)v = \lambda v$, which is impossible in general if $b \ne 0$.

4.3. Agmon estimate for the first eigenfunction

Thanks to the Theorem 22, we can define λ_α for $\alpha \in \bigcup U_{\theta,r(\theta)}$ by $\lambda_\alpha = \alpha + \gamma(\alpha)\mathrm{e}^{-\alpha}$, and v_α as the solution to $-v''_\alpha + (\alpha x)^2 v_\alpha = \lambda_\alpha v_\alpha$, $v_\alpha(0) = 1$, $v'(\alpha) = 0$. As a solution to an ordinary differential equation that depends analytically on a parameter α, $v_\alpha(x)$ depends analytically on α, and we have thus $v_\alpha(\pm 1) = 0$. We now prove some estimates on $v_\alpha(x)$, in the form of the following proposition.

Proposition 25. *Let $1 \ge \epsilon > 0$ and ${}^\epsilon w(x)(\alpha) = \mathrm{e}^{\alpha(1-\epsilon)x^2/2} v_\alpha(x)$. There exists a non-decreasing function $r : (0,\pi/2) \to \mathbb{R}_+$ such that ${}^\epsilon w$ is bounded from $[-1,1]$ to $S(r)$.*

Remark 26. The boundedness of ${}^\epsilon w$ in the statement of Proposition 25 is to be understood as the boundedness of the subset $\{{}^\epsilon w(x), x \in [-1,1]\}$ of $S(r)$, which is equivalent to the fact that, for all seminorms $p_{\epsilon',\theta}$, the set $\{p_{\epsilon',\theta}({}^\epsilon w(x)), x \in [-1,1]\}$ is a bounded set of $\mathbb{R}$.

Proof. This is mostly a complicated way of stating Agmon's estimate (see for instance Agmon's initial work [1] or Helffer and Sjöstrand's article [16], the latter being closer to what we are doing).

Let $\theta_0 \in (0,\pi/2)$. We will prove that there exists $C > 0$ and $r'(\theta_0)$ such that, for all $\alpha \in U_{\theta_0,r'(\theta_0)}$ and $x \in (-1,1)$, ${}^\epsilon w(\alpha)(x) \le C|\alpha|^{3/4}$, which is enough to prove the stated proposition. In this proof, we will just note w instead of ${}^\epsilon w$, and for convenience, we will note $w_\alpha(x)$ instead of $w(x)(\alpha)$.

For all α, we have: $-w''_\alpha + 2\alpha(1-\epsilon)xw'_\alpha + \big((1-(1-\epsilon)^2)(\alpha x)^2 - \epsilon\alpha - \rho(\alpha)\big)w_\alpha = 0$. Let us write $\alpha = \frac{1}{h}\mathrm{e}^{\mathrm{i}\theta}$, $\delta^2 = 1-(1-\epsilon)^2$, and multiply the previous equation by $h^2\mathrm{e}^{-\mathrm{i}\theta}\bar w_\alpha$. We get:

$$-h^2\mathrm{e}^{-\mathrm{i}\theta}w''_\alpha\bar w_\alpha + 2h(1-\epsilon)xw'_\alpha\bar w_\alpha + \big(\mathrm{e}^{\mathrm{i}\theta}\delta^2x^2 - h(\epsilon + h\mathrm{e}^{-\mathrm{i}\theta}\rho(\alpha))\big)|w_\alpha(x)|^2 = 0.$$

By integration by parts, we have $-\int_{-1}^1 w''_\alpha(x)\bar w_\alpha(x)\,\mathrm{d}x = \int_{-1}^1 |w'_\alpha(x)|^2\,\mathrm{d}x$, and, since $2\Re(w'_\alpha\bar w_\alpha) = \frac{\mathrm{d}}{\mathrm{d}x}|w_\alpha|^2$, we have $2\int_{-1}^1 x\Re(w'_\alpha(x)\bar w_\alpha(x))\,\mathrm{d}x = -\int_{-1}^1 |w_\alpha(x)|^2\,\mathrm{d}x$, so integrating the equation and taking the real part, we get the Agmon estimate, valid for all α such that $\rho(\alpha)$ is defined:

$$h^2\int_{-1}^{1}|w'_\alpha(x)|^2\,\mathrm{d}x + \int_{-1}^{1}\left(\delta^2x^2 - h\frac{1+h\Re(\mathrm{e}^{-\mathrm{i}\theta}\rho(\alpha))}{\cos(\theta)}\right)|w_\alpha(x)|^2\,\mathrm{d}x = 0. \tag{19}$$

The final ingredient we need to conclude is a comparison between w_α and $\mathrm{e}^{-\epsilon\alpha x^2/2}$, which will give us a control of the L^2 norm of w_α on sets of the form $(-R|\alpha|^{-1/2}, R|\alpha|^{-1/2})$. The function $\tilde w$ defined by $\tilde w(z) = \mathrm{e}^{-\epsilon z^2/2}$ satisfies $-\tilde w'' + 2(1-\epsilon)z\tilde w' + \delta^2z^2\tilde w - \epsilon\tilde w = 0$ for z in $\mathbb{C}$, so the solution $\tilde w_\rho$ of $-\tilde w''_\rho + 2(1-\epsilon)z\tilde w'_\rho + \delta^2z^2\tilde w_\rho = (\epsilon+\rho)\tilde w_\rho$ tends to $\mathrm{e}^{-\epsilon z^2/2}$ in $L^2(D(0,R))$ as ρ tends to 0. So, for all $R > 0$, there exists $\rho_{\max}$ such that for $|\rho| \le \rho_{\max}$, $|\tilde w_\rho - \mathrm{e}^{-\epsilon z^2/2}|_{L^2(D(0,R))} \le 1$. But $w_\alpha(x) = \tilde w_{\rho(\alpha)/\alpha}(\sqrt{\alpha}x)$, so, if $\rho(\alpha)/\alpha \le \rho_{\max}$, we have $|w_\alpha - \mathrm{e}^{-\epsilon\alpha x^2/2}|_{L^2(|x|\le R/\sqrt{|\alpha|})} \le |\alpha|^{-1/2}$. So, there exists $r'(\theta_0) \ge r(\theta_0)$ such that, for all α in $U_{\theta_0,r'(\theta_0)}$,

$$|w_\alpha|_{L^2(-R/\sqrt{|\alpha|},R/\sqrt{|\alpha|})} \le C_{\epsilon,\theta_0}|\alpha|^{-1/4}. \tag{20}$$

Let $E = \{x \in (-1,1), \delta^2x^2 - 2h/\cos(\theta_0) \le 0\} = \{|x| \le \sqrt{2}/(\delta\sqrt{\cos(\theta_0)})|\alpha|^{-1/2}\}$ and $\alpha = \frac{1}{h}\mathrm{e}^{\mathrm{i}\theta}$ in $U_{\theta_0,r'(\theta_0)}$. We have $|h\Re(\mathrm{e}^{-\mathrm{i}\theta}\rho(\alpha))| \le 1$ and $|\theta| < \theta_0$, so, for x in $[-1,1]\setminus E$, $\delta^2x^2 - h(1+h\Re(\mathrm{e}^{-\mathrm{i}\theta}\rho(\alpha)))/\cos(\theta) > 0$. Thus, thanks to Agmon's estimate (19), $h^2|w'_\alpha|^2_{L^2(-1,1)} \le C'_{\theta_0}|w_\alpha|^2_{L^2(E)}$. But, thanks to inequality (20), we have $|w_\alpha|^2_{L^2(E)} \le C_{\epsilon,\theta_0}h^{1/2}$, so $|w'_\alpha|^2_{L^2(-1,1)} \le C'_{\epsilon,\theta_0}h^{-3/2}$.

Finally, for all x in $(-1,1)$, we have: $|w_\alpha(x) - w_\alpha(0)| \le |w'_\alpha|_{L^1(-1,1)}$, so thanks to Hölder's inequality, $|w_\alpha(x) - w_\alpha(0)| \le \sqrt{2}|w'_\alpha|_{L^2(-1,1)} \le \sqrt{2C'_{\epsilon,\theta_0}}h^{-3/4}$. $\square$

4.4. Proof of Lemma 8

We prove here Lemma 8. To bound from above $\left|\sum v_n(x)a_n z^{n-1}|\zeta|^{\rho_n}\right|$, the idea is to apply Theorem 18, with Theorems 22 and 25 providing the required hypotheses.

First, in order to apply Theorem 18, we define some symbols. Let $\gamma \in S(r_1)$ obtained by Theorem 22, and $v : (-1,1) \to S(r_2)$ the function obtained by Proposition 25 with $\epsilon = 1$. By taking $r = \max(r_1, r_2)$, we can assume that $\gamma \in S(r)$ and that v take its values in $S(r)$. This v is still bounded (see Proposition 13, first item). Finally, for $\zeta \in \mathcal{D}$ and $x \in (-1,1)$, let $\gamma_{\zeta,x}$ defined by[7]:

$$\gamma_{\zeta,x}(\alpha) = v(x)(\alpha)|\zeta|^{\rho(\alpha)},$$

so that:

$$\sum_{n>r(0)} v_n(x)a_n z^{n-1}|\zeta|^{\rho_n} = \frac{1}{z}H_{\gamma_{\zeta,x}}\left(\sum_{n>r(0)} a_n z^n\right). \tag{21}$$

We then show that the family $(\gamma_{\zeta,x})_{\zeta\in\mathcal{D},x\in(-1,1)}$ is in $S(r)$, and is bounded. We already know that $(v(x))_{x\in(-1,1)}$ is a bounded family in $S(r)$. Since the multiplication is continuous in $S(r)$ (Proposition 12), to prove that $(\gamma_{\zeta,x})$ is a bounded family, it is enough to prove that $(|\zeta|^\rho)_{\zeta\in\mathcal{D}}$ is a bounded family of $S(r)$.

Since $\rho(\alpha) = \mathrm{e}^{-\alpha}\gamma(\alpha)$ with γ having sub-exponential growth (by definition of $S(r)$), $|\rho(\alpha)|$ is bounded on every $U_{\theta,r(\theta)}$ by some c_θ. So, we have for $\zeta \in \mathcal{D}$ and $\alpha \in U_{\theta,r(\theta)}$:

$$||\zeta|^{\rho(\alpha)}| \le \mathrm{e}^{-\ln|\zeta|c_\theta} \le \mathrm{e}^{Tc_\theta}.$$

So $|\zeta|^{\rho(\alpha)}$ is bounded for $\alpha \in U_{\theta,r(\theta)}$, and in particular has sub-exponential growth. Since ρ is holomorphic, so is $\alpha \mapsto |\zeta|^{\rho(\alpha)}$, thus, $\alpha \mapsto |\zeta|^{\rho(\alpha)}$ is in $S(r)$. Moreover, the bound $||\zeta|^{\rho(\alpha)}| \le \mathrm{e}^{Tc_\theta}$ is uniform in $\zeta \in \mathcal{D}$, so $(|\zeta|^\rho)_{\zeta\in\mathcal{D}}$ is a bounded family of $S(r)$.

We have proved $(\gamma_{\zeta,x})$ is a bounded family of $S(r)$, so according to the estimate on holomorphy default operators (Theorem 18), if V is a bounded domain that is star-shaped with respect to 0, for any $\delta' > 0$, there exists $C > 0$ independent of ζ, x, such that:

$$\left|\sum_{n>r(0)} \gamma_{\zeta,x}(n)a_n z^n\right|_{L^\infty(V)} \le C\left|\sum_{n>r(0)} a_n z^n\right|_{L^\infty(V^{\delta'})}.$$

We cannot apply this estimate directly with $U = V$ since $0 \notin U$, but we can choose V and δ' such that $U \subset V$ and $V^{\delta'} \subset U^\delta$ (for instance, $\delta' = \delta/2$ and $V = U^{\delta'}$): there exists $C > 0$ independent of $\zeta \in \mathcal{D}$ and $x \in (-1,1)$:

$$\left|\sum_{n>r(0)} \gamma_{\zeta,x}(n)a_n z^n\right|_{L^\infty(U)} \le C\left|\sum_{n>r(0)} a_n z^n\right|_{L^\infty(U^\delta)}.$$

So, thanks to Eq. (21):

$$\left|\sum_{n>r(0)} v_n(x)a_n z^{n-1}|\zeta|^{\rho_n}\right|_{L^\infty(\mathcal{D})} \le C\mathrm{e}^T\left|\sum_{n>r(0)} a_n z^n\right|_{L^\infty(U^\delta)} \le C\mathrm{e}^T\left|\sum_{n>r(0)} a_n z^{n-1}\right|_{L^\infty(U^\delta)}. \qquad \square$$

5. Conclusion and open problems

We proved the non-null controllability of the Grushin equation on some special control domain, and if we combine our result with the previous ones [5,7], all of the following situations can happen, depending on the control domain ω:

- the Grushin equation is controllable in any time, for instance if $\omega = (0,a) \times (0,1)$;
- the Grushin equation is controllable in large time, but not in small time, for instance if $\omega = (a,b) \times (0,1)$ with $0 < a < b$;
- the Grushin equation is never controllable, for instance if $\omega = (-1,1) \times ((0,1) \setminus [a,b])$ with $0 \le a < b \le 1$.

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[7] We remind that $\mathcal{D} = \{\mathrm{e}^{-T} < |z| < 1, \arg(z) \in \omega_y\}$, see Section 2.1 and Fig. 1. Also, $U = \{0 < |z| < 1, \arg(z) \in \omega_y\}$ and $U^\delta = \{z, \mathrm{distance}(z, U) \in \omega_y\}$.

A pattern that seems to appear in the controllability of degenerate parabolic equations is that the controllability holds in any time when the degeneracy is weak, and never holds when the degeneracy is strong. Our result may indicate that obtaining general results on the controllability of parabolic equation degenerating inside the domain will be difficult in the critical case, i.e. when the degeneracy is neither strong nor weak.

On the Grushin equation, null-controllability is still an open problem for domains that do not fall into one of the three domain types we described before. Also, the controllability of higher-dimension Grushin equations, for $x \in (-1,1)$, $y \in \mathbb{T}^n$, on $\omega = (-1,1) \times \omega_y$ is still an open problem (the case where $\omega = (a,b) \times \mathbb{T}^n$ is mentioned in [5]).

Another question we might ask is whether regular initial conditions can be steered to 0, as it happens for the Grushin equation when the control domain is two symmetric vertical bands [7]. The answer is negative (Proposition 27).

Proposition 27. *Let $T > 0$ and ω as in the main theorem (Theorem 2). For $\alpha > 0$, let $\mathcal{A}_\alpha = \{\sum a_n(x)\mathrm{e}^{iny}, \sum |a_n|^2_{L^2(-1,1)}\mathrm{e}^{2\alpha n} < +\infty\}$. Then, for every $\alpha > 0$, there exists an initial condition f_α in $\mathcal{A}_\alpha$ that cannot be steered in time T to 0 by means of L^2 controls localized in ω.*

Proof. According to Theorem 2, there exists an initial condition $f_0 = \sum a_n v_n(x)\mathrm{e}^{iny}$ in $L^2(\Omega)$ that cannot be steered to 0 by a L^2 control localized on ω in time $T+\alpha$. Let $f(t,x,y)$ be the solution to the Grushin equation (2) with f_0 as the initial condition, and let $f_\alpha(x,y) = f(\alpha,x,y)$. Then, since $\lambda_n = n + o(1)$, $f_\alpha(x,y) = \sum a_n v_n(x)\mathrm{e}^{-\alpha\lambda_n}\mathrm{e}^{iny}$ is in $\mathcal{A}_\alpha$, and if it could be steered to 0 in time T, then, f_0 could be steered to 0 in time $T+\alpha$. $\square$

From this proposition, we could ask if there is even one non-null initial condition that can be steered to 0. For the moment, it is unknown, but we conjecture that there is none.

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Appendix A. The stationary phase theorem

We prove here the following theorem:

Theorem 28. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R})$, $N \in \mathbb{N}$ and $\alpha \in \{\alpha \neq 0, \Re(\alpha) \geq 0\}$:*

$$\int_{\mathbb{R}} \mathrm{e}^{-\alpha x^2/2} u(x)\,\mathrm{d}x = \sum_{k=0}^{N-1} \frac{\sqrt{2\pi}}{2^k k! \alpha^{k+1/2}} u^{(2k)}(0) + S_N(u,\alpha)$$

where α^s is defined to be $\mathrm{e}^{s\ln\alpha}$, with the principal determination of the logarithm, and $S_N(u,\alpha)$ satisfying:

$$|S_N(u,\alpha)| \leq \frac{C}{2^N N! |\alpha|^{N+1/2}} \sum_{k=0}^{2} \|u^{(2N+k)}\|_{L^1}.$$

Proof. Since the proof is essentially the same as the one provided by Martinez [19, theorem 2.6.1] for the case α purely imaginary, we just give the main ideas.

We define the Fourier transform of u in the Schwartz space by $\mathcal{F}(u)(\xi) = \int_{\mathbb{R}} u(x)\mathrm{e}^{-ix\xi}\,\mathrm{d}x$. Then, the Fourier transform of $x \mapsto \mathrm{e}^{-\alpha x^2/2}$ is $\mathcal{F}(\mathrm{e}^{-\alpha x^2/2})(\xi) = \sqrt{\frac{2\pi}{\alpha}}\mathrm{e}^{-\xi^2/2\alpha}$ (this is standard when $\Re(\alpha) > 0$, and by taking the limit in $\mathcal{S}'(\mathbb{R})$ for $\alpha + \epsilon \to \alpha$ when α is purely imaginary). So, we have:

$$\int_{\mathbb{R}} \mathrm{e}^{-\alpha x^2/2} u(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi\alpha}} \int_{\mathbb{R}} \mathrm{e}^{-\xi^2/(2\alpha)} \mathcal{F}(u)(\xi)\,\mathrm{d}\xi.$$

Then, writing

$$\mathrm{e}^{-\xi^2/(2\alpha)} = \sum_{k\geq 0} \frac{(-1)^k}{(2\alpha)^k k!}\xi^{2k} = \sum_{k<N} \frac{(-1)^k}{(2\alpha)^k k!}\xi^{2k} + R_N(\xi,\alpha),$$

with, according to Taylor's formula,

$$|R_N(\xi,\alpha)| \leq \frac{\xi^{2N}}{2^N |\alpha|^N N!},$$

we have

$$\int_{\mathbb{R}} \mathrm{e}^{-\alpha x^2/2} u(x)\,\mathrm{d}x = \sum_{k<N} \frac{\sqrt{2\pi}}{2^k k! \alpha^{k+1/2}} u^{(2k)}(0) + S_N(u,\alpha)$$

with $S_N(u,\alpha) = \int_{\mathbb{R}} R_N(\xi,\alpha)\mathcal{F}u(\xi)\,\mathrm{d}\xi$, which satisfies the estimate stated in the theorem.

We refer to Martinez's proof in the already mentioned book [19] for more details on the computations. □

Here is the particular case of the stationary phase theorem we will need.

Proposition 29. *Let $\Gamma_{+,\epsilon}$ be the path $(-\infty,\epsilon] \cup \{\epsilon \mathrm{e}^{\mathrm{i}(\pi-\theta)}, 0 \le \theta \le \pi\} \cup [\epsilon,+\infty)$. Let θ in $(0,\pi/2)$. There exists $C_\theta > 0$ such that for all $\alpha \in U_{\theta,1}$, $\rho \in D(0,1)$ and $m \in \{0,1,2\}$, with $f(s) = \mathrm{e}^{-(1+\rho/2)\ln(s)}(\ln(s))^m$:*

$$\left| \int_{\Gamma_{+,\epsilon}} \mathrm{e}^{-\alpha(1+s/2)^2} f(s)\,\mathrm{d}s - 2\sqrt{\frac{\pi}{\alpha}} f(-2) \right| \le C_\theta |\alpha|^{-3/2}.$$

Proof. We start by choosing $\chi \in C_c^\infty(-3,-1)$ with $\chi = 1$ on $(-5/2,-3/2)$, and we modify slightly the path $\Gamma_{+,\epsilon}$ so that it is a C^∞ path (the result of the integral of course not depending on this modification of $\Gamma_{+,\epsilon}$), for instance, if ϕ is in $C_c^\infty(-\epsilon,\epsilon)$ with $\phi \ge 0$ and $\phi(0) > 0$, we can choose $\Gamma_+(t) = t + 2\mathrm{i}\phi(t)$. Then we write:

$$\int_{\Gamma_{+,\epsilon}} \mathrm{e}^{-\alpha(1+s/2)^2} f(s)\,\mathrm{d}s = \int_{-3}^{-1} \mathrm{e}^{-\alpha(1+t/2)^2} \chi(t) f(t)\,\mathrm{d}s + \int_{\mathbb{R}\setminus(-5/2,-3/2)} \mathrm{e}^{-\alpha(1+\Gamma_+(t)/2)^2} (1-\chi(t)) f(\Gamma_+(t))\,\mathrm{d}t.$$

We can apply the previous theorem to the first term, so we only need to show that for some $C'_\theta > 0$, the second term is bounded by $C'_\theta |\alpha|^{-3/2}$. We note $\varphi(t) = (1+\Gamma_+(t)/2)^2 = (1+t/2)^2 - \phi^2(t) + 2\mathrm{i}\phi(t)(1+t/2)$ whose only critical point is -2, and let L the operator defined by $Lu = \frac{1}{\varphi'}u'$ so that $L\mathrm{e}^{-\alpha\varphi} = -\alpha \mathrm{e}^{-\alpha\varphi}$. So, noting $L^t u = (\frac{1}{\varphi'}u)'$, we have, by integration by parts:

$$\int_{\mathbb{R}\setminus(-5/2,-3/2)} \mathrm{e}^{-\alpha(1+\Gamma_+(t)/2)^2} (1-\chi(t)) f(\Gamma_+(t))\,\mathrm{d}t = \frac{1}{\alpha^2} \int_{\mathbb{R}\setminus(-5/2,-3/2)} \mathrm{e}^{-\alpha\varphi(t)} (L^t)^2((1-\chi) f \circ \Gamma_+)(t)\,\mathrm{d}t$$

Then, writing $|\mathrm{e}^{-\alpha\varphi(t)}| \le \mathrm{e}^{-\Re(\alpha\varphi(t))}$ and $\Re(\alpha\varphi(t)) = \Re(\alpha)((1+t/2)^2 - \phi^2(t)) - 2(1+t/2)\Im(\alpha)\varphi(t))$. If $|\arg(\alpha)| < \theta$, then for some $c_\theta > 0$, $\Re(\alpha) \ge c_\theta|\alpha|$, and if we choose ϕ small enough, we have for some $c'_\theta > 0$ and all $t \notin (-5/2,-3/2)$, $\Re(\alpha\varphi(t)) \ge c'_\theta |\alpha| (1+t/2)^2$. So, for all α in $U_{\theta,1}$:

$$\left| \int_{\mathbb{R}\setminus(-5/2,-3/2)} \mathrm{e}^{-\alpha(1+\Gamma_+(t)/2)^2} (1-\chi(t)) f(\Gamma_+(t))\,\mathrm{d}t \right| \le$$
$$\frac{1}{|\alpha|^2} |(L^t)^2((1-\chi) f \circ \Gamma_+)|_{L^\infty(\mathbb{R}\setminus(-5/2,-3/2))} \int_{\mathbb{R}} \mathrm{e}^{-c'_\theta|\alpha|t^2/4}\,\mathrm{d}t$$

which concludes the proof. □

Appendix B. Newton's method

We prove here that Newton's method can solve equations of the form $\Phi(z) = 0$ by an iterative scheme, assuming the starting point is close enough to a solution. While such a theorem can be stated in Banach spaces, we will only need it in the complex plane.

Theorem 30. *Let $D = D(0,R)$ be a disk in the complex plane. We will note $5D = D(0,5R)$ and $6D = D(0,6R)$. Let $\Phi : 6D \to \mathbb{C}$ be a holomorphic function such that:*

- *for all $z \in D$, $\Phi(z) \in D$;*
- *For all z in $6D$, $\Phi'(z) \ne 0$.*

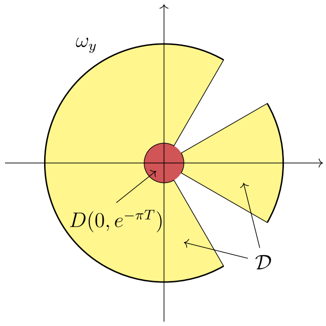

Fig. 9. The domain $\mathcal{D}$ for the Grushin equation in the rectangle. The equivalent of the U of Section 2.3 is $U = \{0 < |z| < 1, |\arg(z)| \in \omega_y\}$. We still cannot control entire functions in $D(0, \mathrm{e}^{-\pi T})$ from their L^2 norm in U^δ if δ is smaller than $\mathrm{e}^{-\pi T}$.

Then, noting $C_1 = \sup_{5D} |\Phi''|$ and $C_2 = \sup_{5D} |\Phi'^{-1}|$ and $A = C_1 C_2^2$, if z_0 is in D and $|\Phi(z_0)| \le \min((2A)^{-1}, 2RC_2^{-1})$, then the sequence (z_n) defined by $z_{n+1} = z_n - \Phi'(z_n)^{-1}\Phi(z_n)$ converges, and the limit z_∞ satisfies $\Phi(z_\infty) = 0$. Moreover, for all $k \ge 0$, $|z_\infty - z_k| \le \frac{2}{C_1 C_2}|A\Phi(z_0)|^{2^k}$.

Proof. Let $\epsilon_0 = |\Phi(z_0)|$. We prove by induction the predicate $P(n): z_n \in 5D$ and $|\Phi(z_n)| \le A^{-1}(A\epsilon_0)^{2^n}$. About the case $n = 0$, we made the hypothesis $z_0 \in D$, while the inequality just reads $|\Phi(z_0)| \le |\Phi(z_0)|$.

Now suppose that $P(k)$ holds for all $k \le n$. Let $v_k = -\Phi'(z_k)^{-1}\Phi(z_k)$, so that $z_{n+1} = z_0 + v_0 + \cdots v_n$. By definition of C_1 and C_2 and the fact that for all $k \le n$, $z_k \in 5D$, $|z_{n+1}| \le |z_0| + \sum_{k=0}^{n} C_2 A^{-1}(A\epsilon_0)^{2^k} \le R + C_2\epsilon_0 \frac{1}{1-A\epsilon_0}$. Since we have by hypothesis $\epsilon_0 \le \min((2A)^{-1}, 2RC_2^{-1})$, we have $z_{n+1} \le R + 2C_2\epsilon_0 \le 5R$, which proves that $z_{n+1} \in 5D$.

In order to prove that $|\Phi(z_{n+1})| \le A^{-1}(A\epsilon_0)^{2^{n+1}}$, we make a Taylor expansion of Φ about z_n: for all $\delta \in \mathbb{C}$ such that $z_n + \delta \in 5D$:

$$|\Phi(z_n + \delta) - \Phi(z_n) - \delta\Phi'(z_n)| \le \frac{1}{2} C_1 |\delta|^2.$$

We then choose δ so that $\Phi(z_n) + \delta\Phi'(z_n) = 0$. With this δ, the previous inequality is $|\Phi(z_{n+1})| \le \frac{1}{2} C_1 |\Phi(z_n)|^2 |\Phi'(z_n)^{-1}|^2$. So, by the definition of C_2 and the induction hypothesis, $|\Phi(z_{n+1})| \le \frac{1}{2} C_1 C_2^2 \left(A^{-1}(A\epsilon_2)^{2^n}\right)^2 = \frac{1}{2} A^{-1}(A\epsilon_0)^{2^{n+1}}$, which ends the proof of the induction.

By the same kind of computations we made in order to prove $z_{n+1} \in 5D$, we have, for $n \ge k$: $|z_n - z_k| \le C_2 \sum_{j=k}^{n-1} A^{-1}(A\epsilon_0)^{2^j} \le C_2 A^{-1}(A\epsilon_0)^{2^k} \frac{1}{1-A\epsilon_0} \le 2C_2 A^{-1}(A\epsilon_0)^{2^j}$. This proves the stated estimate and that (z_n) converges. Since we have $|\Phi(z_n)| \le A^{-1}(A\epsilon_0)^{2^n}$, the limit z_∞ satisfies $\Phi(z_\infty) = 0$. □

Appendix C. The Grushin equation on the rectangle

We look here at the Grushin equation $\partial_t f - \partial_x^2 f - x^2 \partial_y^2 f = \mathbb{1}_\omega$ with $(x, y) \in \Omega = (-1, 1) \times (0, 1)$ and with Dirichlet boundary conditions on $\partial\Omega$. The situation is the same as the Grushin equation on the torus $(-1, 1) \times \mathbb{T}$ (Theorem 31).

Theorem 31. *Let $[a, b]$ be a non-trivial segment of $(0, 1)$, $\omega_y = (0, 1) \setminus [a, b]$, $\omega = (-1, 1) \times \omega_y$ and $T > 0$. The Grushin equation on Ω is not controllable on ω in time T.*

Sketch of the proof. This time, we look for a counterexample of the observability inequality among linear combinations of the eigenfunctions Φ_n defined by $\Phi_n(x, y) = v_{n\pi}(x)\sin(n\pi y)$. Then, writing $\sin(n\pi y) = \frac{1}{2i}(\mathrm{e}^{\mathrm{i}n\pi y} - \mathrm{e}^{-\mathrm{i}n\pi y})$, we have:

$$\int\limits_{\substack{0<t<T \\ -1<x<1 \\ y\in\omega_y}} \left|\sum a_n \mathrm{e}^{-\lambda_{n\pi} t}\Phi_n(x, y)\right|^2 \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$
$$\le \frac{1}{2} \int\limits_{\substack{0<t<T \\ -1<x<1 \\ y\in\omega_y}} \left(\left|\sum a_n \mathrm{e}^{-\lambda_{n\pi} t} v_{n\pi}(x)\mathrm{e}^{\mathrm{i}n\pi y}\right|^2 + \left|\sum a_n \mathrm{e}^{-\lambda_{n\pi} t} v_{n\pi}(x)\mathrm{e}^{-\mathrm{i}n\pi y}\right|^2\right) \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, we can do the same proof as in Section 2.3, but with $\mathcal{D} = \{\mathrm{e}^{-\pi T} < |z| < 1, |\arg(z)| \in \omega_y\}$ (see Fig. 9) and $U = \{0 < |z| < 1, |\arg(z)| \in \omega_y\}$. □

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