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Geometry

Compact simple Lie groups admitting left-invariant Einstein metrics that are not geodesic orbit ☆

Groupes de Lie simples compacts admettant des métriques d'Einstein invariantes à gauche, dont une géodésique n'est pas une orbite

Huibin Chen, Zhiqi Chen, Shaoqiang Deng

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China

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ABSTRACT

In this article, we prove that the compact simple Lie groups $\mathrm{SU}(n)$ for $n \geq 6$, $\mathrm{SO}(n)$ for $n \geq 7$, $\mathrm{Sp}(n)$ for $n \geq 3$, $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, and F_4 admit left-invariant Einstein metrics that are not geodesic orbit. This gives a positive answer to an open problem recently posed by Nikonorov.

RÉSUMÉ

Dans cet article, nous démontrons que les groupes simples compacts $SU(n)$ pour $n \geq 6$, $SO(n)$ pour $n \geq 7$, $Sp(n)$ pour $n \geq 3$, E_6, E_7, E_8 et F_4 admettent des métriques d'Einstein invariantes à gauche, dont une géodésique maximale n'est pas une orbite d'un sous-groupe à un paramètre du groupe des isométries complet. Ceci fournit une réponse positive à un problème récemment posé par Nikonorov.

1. Introduction

The purpose of this short note is to give a positive answer to an open problem recently posed by Nikonorov. In his paper [10], Nikonorov proved that there exists a left-invariant Einstein metric on the compact simple Lie group G_2 that is not a geodesic orbit metric. This metric is the first non-naturally reductive left-invariant Einstein metric on G_2 discovered by I. Chrysikos and Y. Sakane in [6]. Recall that a Riemannian metric on a connected manifold M is said to be a geodesic orbit metric if any maximal geodesic of the metric is the orbit of a one-parameter subgroup of the full group of isometries (in this case, the Riemannian manifold is called a geodesic orbit space). It is well known that any naturally reductive metric must

☆ Supported by NSFC (No. 11671212, 51535008) of China.
E-mail addresses: chenhuibin@mail.nankai.edu.cn (H. Chen), chenzhiqi@nankai.edu.cn (Z. Chen), dengsq@nankai.edu.cn (S. Deng).

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be geodesic orbit, but the converse is not true. Recently, many interesting results have been established on non-naturally reductive homogeneous Einstein metrics. It is therefore a natural problem to study homogeneous Einstein metrics that are not geodesic orbit.

The following problem is posed in [10].

Problem 1.1. Is there any other compact simple Lie group admitting a left-invariant Einstein metric that is not geodesic orbit?

The main result of this short note is the following.

Theorem 1.2. *The compact simple Lie groups* $\mathrm{SU}(n)$ *for* $n \geq 6$, $\mathrm{SO}(n)$ *for* $n \geq 7$, $\mathrm{Sp}(n)$ *for* $n \geq 3$, $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ *and* F_4 *admit left-invariant Einstein metrics that are not geodesic orbit.*

2. Some known results and generalized Wallach spaces

In this section, we will recall a sufficient and necessary condition in [10] for a left-invariant metric on a compact Lie group to be a geodesic orbit metric. We will also give some results on generalized Wallach spaces. We first recall a result of [7] on the characterization of geodesic orbit metrics.

Lemma 2.1 *([7]). Let M be a homogeneous Riemannian manifold and G the identity component of the full group of isometries. Write $M = G/H$, where H is the isotropic subgroup of G at $x \in M$, and suppose the Lie algebra of G has a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}\, H$, and $\mathfrak{m}$ is the orthogonal complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an $\mathrm{Ad}H$-invariant inner product on $\mathfrak{g}$. Then M is a geodesic orbit space if and only if, for any $X \in \mathfrak{m}$, there exits $Z \in \mathfrak{h}$ such that $([X + Z, Y]_{\mathfrak{m}}, X) = 0$ for all $Y \in \mathfrak{m}$.

In [10], the author obtained a sufficient and necessary condition for a left-invariant Riemannian metric on a compact Lie group to be a geodesic orbit metric.

Theorem 2.2 *([10]). A simple compact Lie group G with a left-invariant Riemannian metric ρ is a geodesic orbit space if and only if there is a closed connected subgroup K of G such that for any $X \in \mathfrak{g}$ there exists $W \in \mathfrak{k}$ such that for any $Y \in \mathfrak{g}$ the equality $([X + W, Y], X) = 0$ holds or, equivalently, $[A(X), X + W] = 0$, where $A : \mathfrak{g} \to \mathfrak{g}$ is a metric endomorphism and $\mathfrak{g}, \mathfrak{k}$ are the Lie algebras of Lie groups G, K, respectively.*

We now recall the definition of generalized Wallach spaces. Let G/K be a reductive homogeneous space, where G is a semi-simple compact connected Lie group, K is a connected closed subgroup of G, and $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras, respectively. If $\mathfrak{m}$, the tangent space of G/K at $o = \pi(e)$, can be decomposed into three $\mathrm{ad}(\mathfrak{k})$-invariant irreducible summands pairwise orthogonal with respect to B as:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

such that $[\mathfrak{m}_i, \mathfrak{m}_i] \in \mathfrak{k}$ for $i \in \{1, 2, 3\}$ and $[\mathfrak{m}_i, \mathfrak{m}_j] \in \mathfrak{m}_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then G/K is called a generalized Wallach space.

In [5] and [10], the authors gave a complete classification of generalized Wallach spaces with G simple. Based on this result, the authors in [3] obtained some Einstein metrics arising from generalized Wallach spaces. We now recall some results in [3].

Let $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{m}_{p+1} \oplus \mathfrak{m}_{p+2} \oplus \mathfrak{m}_{p+3} = (\mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p) \oplus (\mathfrak{k}_{p+1} \oplus \mathfrak{k}_{p+2} \oplus \mathfrak{k}_{p+3})$. We assume that $\dim_{\mathbb{R}} \mathfrak{k}_0 \leq 1$ and the ideals $\mathfrak{k}_i$ are mutually non-isomorphic for $i = 1, \cdots, p$. We consider the following inner product on $\mathfrak{g}$:

$$\langle\ ,\ \rangle = u_0 B(\ ,\ )|_{\mathfrak{k}_0} + \cdots + u_p B(\ ,\ )|_{\mathfrak{k}_p} + u_{p+1} B(\ ,\ )|_{\mathfrak{k}_{p+1}} + u_{p+2} B(\ ,\ )|_{\mathfrak{k}_{p+2}} + u_{p+3} B(\ ,\ )|_{\mathfrak{k}_{p+3}}, \tag{2.1}$$

where $B(\ ,\ )$ is the negative Killing form of $\mathfrak{g}$ and $u_i \in \mathbb{R}^+$ for all $0 \neq i \neq p+3$.

Denote $d_i = \dim_{\mathbb{R}} \mathfrak{k}_i$ and let $\{e^i_\alpha\}_{\alpha=1}^{d_i}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$, in the sense that $e^i_\alpha \in \mathfrak{k}_i$ and α is the number of basis in $\mathfrak{k}_i$. Let $A^\gamma_{\alpha,\beta} = B([e^i_\alpha, e^j_\beta], e^k_\gamma)$, equivalently, $A^\gamma_{\alpha,\beta}$ are determined uniquely by the identity $[e^i_\alpha, e^j_\beta] = \sum_\gamma A^\gamma_{\alpha,\beta} e^k_\gamma$. Set

$$(ijk) := \left[{i \atop j\,k} \right] = \sum (A^\gamma_{\alpha,\beta})^2,$$

Table 1
Number of non-naturally reductive left-invariant Einstein metrics on exceptional simple Lie group G arising from generalized Wallach spaces.

G	Types	K	$p+q$	$N_{\text{non-nn}}$
F_4	F_4-I	SO(8)	$1+3$	1 [4]
	F_4-II	$SU(2)\times SU(2)\times SO(5)$	$3+3$	3 [3]
E_6	E_6-III	$SU(2)\times Sp(3)$	$2+3$	4 [3]
	E_6-II	$U(1)\times SU(2)\times SU(2)\times SU(4)$	$0+3+3$	7 [3]
E_7	E_7-I	$SU(2)\times SU(2)\times SU(2)\times SO(8)$	$4+3$	7 [3]
	E_7-III	SO(8)	$1+3$	1 [11]
	E_7-II	$U(1)\times SU(2)\times SU(6)$	$0+2+3$	6 [3]
E_8	E_8-I	$SU(2)\times SU(2)\times SO(12)$	$3+3$	11 [3]
	E_8-II	$\mathrm{Ad}(SO(8)\times SO(8))$	$2+3$	2 [3]

where the sum is taken over all indices α,β,γ with $e^i_\alpha\in\mathfrak{k}_i, e^j_\beta\in\mathfrak{k}_j$, and $e^k_\gamma\in\mathfrak{k}_k$. Then (ijk) is independent of the choice for the B-orthonormal basis of $\mathfrak{k}_i,\mathfrak{k}_j,\mathfrak{k}_k$, and is symmetric with respect to the three indices, i.e. $(ijk)=(jik)=(jki)$.

Using the two involutions of $\mathfrak{g}$, the authors calculate all the numbers (ijk) for the Lie algebra under consideration. A direct conclusion is that if $(ijk)\neq 0$, then $[\mathfrak{k}_i,\mathfrak{k}_j]\neq 0$.

Remark 2.3. Although there are some cases in our discussion with isomorphic ideals in $\mathfrak{k}$, it is easy to see that they are non-equivalent $\mathrm{Ad}(K)$-modules. Combining this fact with Theorem 3.18 of [5], we conclude that the Ricci curvature is still diagonal.

3. Proof of Theorem 1.2

In this section, we will show that the compact simple Lie groups $SU(n)$ for $n\geq 6$, $SO(n)$ for $n\geq 7$, $Sp(n)$ for $n\geq 3$, E_6, E_7, E_8 and F_4 admit left-invariant Einstein metrics that are not geodesic orbit. Combined with some known results in the literature, this gives a proof of the main theorem of this paper.

We first prove the following theorem.

Theorem 3.1. *Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}_0\oplus\cdots\oplus\mathfrak{k}_p\oplus\mathfrak{k}_{p+1}\oplus\mathfrak{k}_{p+2}\oplus\mathfrak{k}_{p+3}$ be the B-orthogonal decomposition arising from generalized Wallach spaces. Then G is equipped with the left-invariant Riemannian metrics generated by the inner products (2.1) with $u_{p+i}>0, i=1,2,3$, and at least two of them, not being equal, are not geodesic orbit spaces.*

Proof. Suppose that $(\mathfrak{g},\langle\ ,\ \rangle)$ generates a geodesic orbit space. Then, by Theorem 2.2, for any $X_i\in\mathfrak{k}_{p+i}, i=1,2,3$, there exists $W\in\mathfrak{k}$ such that $[u_{p+i}X_i+u_{p+j}X_j, X_i+X_j+W]=0$, for $i\neq j$. Thus

$$(u_{p+i}-u_{p+j})[X_i,X_j]+[u_{p+i}X_i+u_{p+j}X_j,W]=0.$$

By the structure of generalized Wallach spaces, there exist $X_i\in\mathfrak{k}_{p+i}$ and $X_j\in\mathfrak{k}_{p+j}$ such that $[X_i,X_j]\neq 0$. Since $(ijk)\neq 0$ (see Table 1 in [9]), we have

$$u_{p+i}=u_{p+j},$$

for any $i\neq j$, that is,

$$u_{p+1}=u_{p+2}=u_{p+3}. \qquad \square$$

Let us recall some known results from the literature about Einstein metrics on compact simple Lie groups related to the decomposition of generalized Wallach spaces. In [8], Mori obtained some non-naturally reductive Einstein metrics on compact simple Lie groups $SU(n)$ for $n\geq 6$. All these metrics satisfy the condition $u_{p+1}\neq u_{p+2}=u_{p+3}=1$. In 2015, the authors of [1,2] obtained non-naturally reductive Einstein metrics on $SO(n)$ for $n\geq 7$ and $Sp(n)$ for $n\geq 3$ of the form (2.1), such that at least two of u_{p+1},u_{p+2},u_{p+3} are not equal. More recently, in [3], [4] and [11], the authors found a large number of non-naturally reductive Einstein metrics on all the compact simple exceptional Lie groups (except G_2) of the form (2.1). The results in [3], [4] and [11] can be summarized in the Table 1.

In this table, we use the notations in [5] to represent the type of generalized Wallach space, $N_{\text{non-nn}}$ represents the number of non-naturally reductive Einstein metrics on G, and p,q coincides with the indices in the decomposition $\mathfrak{g}=\mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_p\oplus\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_q$, where in fact $q=3$ for all types and $0+p+q$ means that there is a center of dimension 1 in $\mathfrak{k}$.

Now Theorem 1.2 follows from Theorem 2.2 and the above results.

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