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A nonlinear shell model of Koiter's type

Un modèle non linéaire de coques de type Koiter

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ABSTRACT

We define a new two-dimensional nonlinear shell model "of Koiter's type" that can be used for the modeling of any type of shell and boundary conditions and for which we establish an existence theorem. The model uses a specific three-dimensional stored energy function of Ogden's type that satisfies all the assumptions of John Ball's fundamental existence theorem in three-dimensional nonlinear elasticity and that is adapted here to the modeling of thin nonlinearly elastic shells by means of specific deformations that are quadratic with respect to the transverse variable.

RÉSUMÉ

Nous définissons un nouveau modèle bidimensionnel non linéaire de coques « de type Koiter » qui peut être utilisé pour la modélisation de tout type de coque et de conditions aux limites et pour lequel nous établissons un théorème d'existence. Ce modèle utilise une densité d'énergie de type Ogden satisfaisant toutes les hypothèses du théorème d'existence fondamental de John Ball en élasticité tridimensionnelle non linéaire et qui est adaptée ici à la modélisation des coques non linéairement élastiques minces au moyen de déformations particulières, qui sont quadratiques en la variable transverse.

1. Preliminaries

Greek indices and exponents vary in the set $\{1, 2\}$, Latin indices and exponents vary in the set $\{1, 2, 3\}$, and the summation convention for repeated indices and exponents is used in conjunction with these rules. Boldface letters are used to designate vector and matrix fields.

The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. The inner product, the exterior product, and the norm, in $\mathbb{E}^3$ are respectively denoted $\cdot$, $\wedge$, and $|\cdot|$. The space of real $n \times n$ matrices is denoted $\mathbb{M}^n$ and the Frobenius norm in $\mathbb{M}^n$ is denoted $\|\cdot\|$. A matrix in $\mathbb{M}^n$ with components g_{ij} is denoted (g_{ij}), the first index (here i) indicating the row in the matrix.

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A *domain* in $\mathbb{R}^2$ is a bounded and connected open subset of $\mathbb{R}^2$ whose boundary is Lipschitz-continuous in the sense of Nečas [15].

Given an open subset ω of $\mathbb{R}^2$, a finite dimensional real space $\mathbb{Y}$, any $p \geq 1$ and any integer $m \geq 0$, the notation $\mathcal{C}^m(\overline{\omega}; \mathbb{Y})$, resp. $W^{m,p}(\omega; \mathbb{Y})$, denotes the space of $\mathbb{Y}$-valued fields with components in $\mathcal{C}^m(\overline{\omega})$, resp. in the Sobolev space $W^{m,p}(\omega)$.

Given an open subset ω of $\mathbb{R}^2$, we let $y = (y_\alpha)$ denote a generic point in ω and we let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. An *immersion* from $\overline{\omega}$ into $\mathbb{E}^3$ is a mapping $\boldsymbol{\psi} \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$ such that the two vector fields $\partial_\alpha \boldsymbol{\psi} : \overline{\omega} \to \mathbb{E}^3$ are linearly independent at each point of $\overline{\omega}$. The image $\boldsymbol{\psi}(\overline{\omega})$ of $\overline{\omega}$ by $\boldsymbol{\psi}$ is a *surface* (with boundary) in $\mathbb{E}^3$.

Given an immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$\boldsymbol{a}_3(\boldsymbol{\psi}) := \frac{\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}}{|\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|},\ a_{\alpha\beta}(\boldsymbol{\psi}) := \partial_\alpha \boldsymbol{\psi} \cdot \partial_\beta \boldsymbol{\psi},\ b_{\alpha\beta}(\boldsymbol{\psi}) := \partial_{\alpha\beta} \boldsymbol{\psi} \cdot \boldsymbol{a}_3(\boldsymbol{\psi}),\ \text{and}\ a(\boldsymbol{\psi}) := \det(a_{\alpha\beta}(\boldsymbol{\psi}));$$

the functions $a_{\alpha\beta}(\boldsymbol{\psi})$ and $b_{\alpha\beta}(\boldsymbol{\psi})$ respectively denote the covariant components of the *first fundamental form* and those of the *second fundamental form* along the surface $\boldsymbol{\psi}(\overline{\omega})$.

Given an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ considered as fixed, we let (for brevity)

$$\boldsymbol{a}_3 := \boldsymbol{a}_3(\boldsymbol{\theta}), \quad a_{\alpha\beta} := a_{\alpha\beta}(\boldsymbol{\theta}), \quad b_{\alpha\beta} := b_{\alpha\beta}(\boldsymbol{\theta}), \ \text{ and } a := a(\boldsymbol{\theta}),$$

and, given any arbitrary immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$G_{\alpha\beta}(\boldsymbol{\psi}) := \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\psi}) - a_{\alpha\beta}) \ \text{ and } \ R_{\alpha\beta}(\boldsymbol{\psi}) := b_{\alpha\beta}(\boldsymbol{\psi}) - b_{\alpha\beta}$$

respectively denote the covariant components of the *change of metric tensor* and those of the *change of curvature tensor* between the surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $\boldsymbol{\psi}(\overline{\omega})$.

Given a domain $\omega \subset \mathbb{R}^2$, a "small" parameter $\varepsilon > 0$, and an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$\Omega := \omega \times]-\varepsilon, \varepsilon[\ \text{ and } \ \boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$$

denote the extension of the mapping $\boldsymbol{\theta}$ defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ at each } y \in \overline{\omega} \text{ and each } x_3 \in [-\varepsilon, \varepsilon].$$

Then one can show that, if $\varepsilon > 0$ is small enough, $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ and $\boldsymbol{\Theta}$ is injective over $\overline{\Omega}$; cf. Theorem 4.1-1 in [5].

We let $x = (x_i)$ with $(x_\alpha) = (y_\alpha) \in \overline{\omega}$ and $x_3 \in [-\varepsilon, \varepsilon]$ denote a generic point in $\overline{\Omega}$, we let $\partial_i := \partial/\partial x_i$, and we let

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ and } (g^{kl}) := (g_{ij})^{-1}$$

respectively denote the covariant and contravariant components of the *metric tensor field* associated with the mapping $\boldsymbol{\Theta}$.

Finally, we let

$$A^{ijkl} := \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}) \ \text{ in } \overline{\Omega},$$

and

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \ \text{ in } \overline{\omega},$$

respectively denote the contravariant components of the *three-dimensional*, and *two-dimensional*, *elasticity tensor* associated with an elastic material with *Lamé constants* λ and μ. If $3\lambda + 2\mu > 0$ and $\mu > 0$, both tensors are uniformly positive-definite, in the sense that there exist two constants $C_0 > 0$ and $c_0 > 0$ depending on λ and μ such that

$$C_0 \sum_{i,j} (t_{ij})^2 \leq A^{ijkl}(x)\, t_{kl} t_{ij} \ \text{ for all } x \in \overline{\Omega} \text{ and all symmetric } 3 \times 3 \text{ tensors } (t_{ij}),$$

and

$$c_0 \sum_{\alpha,\beta} (s_{\alpha\beta})^2 \leq a^{\alpha\beta\sigma\tau}(y)\, s_{\sigma\tau} s_{\alpha\beta} \ \text{ for all } y \in \overline{\omega} \text{ and all symmetric } 2 \times 2 \text{ tensors } (s_{\alpha\beta});$$

(cf. Theorems 3.9-1 and 4.4-1 in [5]). Note, however, that the Lamé constants of all known elastic materials satisfy the stronger assumptions $\lambda > 0$ and $\mu > 0$.

2. A brief review of the mathematical modeling of nonlinearly elastic shells

A *nonlinearly elastic shell* is a three-dimensional elastic body whose *reference configuration* is of the form

$$\mathbf{\Theta}(\overline{\Omega}) \text{ with } \Omega := \omega \times]-\varepsilon, \varepsilon[,$$

where $\varepsilon > 0$ and $\mathbf{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ is defined in terms of an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ as in Sect. 1. Then $S = \boldsymbol{\theta}(\overline{\omega})$ and $2\varepsilon > 0$ respectively designate the *middle surface* and the *thickness* of the shell. Note that we assume for simplicity that the thickness of the shell is constant.

The *deformation* undergone by such a shell subjected to applied *body* and *surface forces* and to specific *boundary conditions* can be computed by using either a *three-dimensional model*, i.e. where *the shell is considered as a three-dimensional body*, or a *two-dimensional model*, i.e. where *the shell is assimilated to its middle surface*.

The objective of this Note is to propose a *new two-dimensional nonlinear shell model* that has the advantage over the existing ones that the associated minimization problem has at least one solution for any type of geometry of the middle surface and any type of boundary conditions (Theorem 4 below), while being at the same time "formally asymptotically equivalent" (in a specific sense; cf. Theorem 3) to the classical *nonlinear Koiter model*, which is one of the most commonly used nonlinear shell models, but for which no existence theorem is available in the literature.

To begin with, we give a brief description of two available nonlinear shell models. For a detailed account, see, e.g., [3,4] and the references therein.

The *three-dimensional nonlinear shell model* asserts that the deformation of a shell with $(\hat{\Omega})^-$ as its reference configuration should minimize the *total energy*

$$\hat{\mathcal{I}}(\hat{\mathbf{\Psi}}) := \int_{\hat{\Omega}} \hat{W}(\hat{x}, \hat{\nabla}\hat{\mathbf{\Psi}}(\hat{x}))\, \mathrm{d}\hat{x} - \hat{\mathcal{L}}(\hat{\mathbf{\Psi}}),$$

where

$$\hat{\Omega} := \mathbf{\Theta}(\Omega)$$

and $\hat{W} : (\hat{\Omega})^- \times \mathbb{M}^3_+ \to \mathbb{R}$ denotes the *stored energy function* of the elastic material constituting the shell and $\hat{\mathcal{L}}$ is a linear functional that takes into account the applied forces, over a set of *admissible deformations* (a precise definition of this set will be given later; for the time being, we simply mention that such deformations must be *orientation-preserving*, in the sense that $\det \hat{\nabla}\hat{\mathbf{\Psi}}(\hat{x}) > 0$ at each $\hat{x} \in (\hat{\Omega})^-$).

We will assume here that the reference configuration $(\hat{\Omega})^-$ of the shell is a *natural state* (i.e. stress free) and that the elastic material constituting the shell is *isotropic, homogeneous, and satisfies the axiom of frame-indifference*. Then one can show (cf., e.g., [3, Theorem 4.5-1]) that, in this case, the following Taylor expansion must hold at each $\hat{x} \in (\hat{\Omega})^-$ and each $\boldsymbol{F} \in \mathbb{M}^3_+$:

$$\hat{W}(\hat{x}, \boldsymbol{F}) = \frac{\lambda}{2}(\operatorname{tr} \boldsymbol{E})^2 + \mu \|\boldsymbol{E}\|^2 + \mathcal{O}(\|\boldsymbol{E}\|^3) \text{ with } \boldsymbol{E} := \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}),$$

where $\lambda > 0$ and $\mu > 0$ are the *Lamé constants* of the given elastic material.

While the Lamé constants are generally known for each elastic material, the remainder $\mathcal{O}(\|\boldsymbol{E}\|^3)$ is not. As a consequence, several competing expressions of $\hat{W}$ exist in the literature. Among the various examples of stored energy functions available in the literature, of particular relevance to this paper is that proposed by Ciarlet and Geymonat in [7], viz.,

$$\hat{W}(\boldsymbol{F}) := a\,\|\boldsymbol{F}\|^2 + b\,\|\mathbf{Cof}\boldsymbol{F}\|^2 + c\,(\det \boldsymbol{F})^2 - d\,\log(\det \boldsymbol{F}) + e \text{ at each } \boldsymbol{F} \in \mathbb{M}^3_+,$$

where the constants $a > 0$, $b > 0$, $c > 0$, $d > 0$, and $e \in \mathbb{R}$, are chosen in terms of two given Lamé constants $\lambda > 0$ and $\mu > 0$ in such a way that the principal part of $\hat{W}$ with respect to $\boldsymbol{E} := (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})/2$ is *"governed" by the Lamé constants* λ *and* μ, according to the above formula for the Taylor expansion of $\hat{W}(\hat{x}, \boldsymbol{F})$. This stored energy function is in addition *polyconvex* and *becomes infinite as* $\det \boldsymbol{F} \to 0^+$, so that the corresponding *total energy* $\hat{\mathcal{I}}$ *possesses at least a minimizer* in a specific set of admissible deformations, according to the landmark existence theorem of Ball [1].

The *two-dimensional nonlinear shell model of W.T. Koiter* asserts that the *deformation of the middle surface* of a shell with $\mathbf{\Theta}(\overline{\Omega})$ as its reference configuration is a sufficiently regular *immersion* $\boldsymbol{\psi} : \overline{\omega} \to \mathbb{E}^3$ that minimizes the *total energy* (cf. [13])

$$\mathrm{j_K}(\boldsymbol{\psi}) := \frac{\varepsilon}{2}\int_{\omega} \Big\{ a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\psi}) G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^2}{3} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\psi}) R_{\alpha\beta}(\boldsymbol{\psi}) \Big\} \sqrt{a}\, \mathrm{d}y - \ell(\boldsymbol{\psi}),$$

where the functions $a^{\alpha\beta\sigma\tau}$, $G_{\alpha\beta}(\boldsymbol{\psi})$, $R_{\alpha\beta}(\boldsymbol{\psi})$, and $\sqrt{a}$, are those defined in Sect. 1, and

$$\ell(\boldsymbol{\psi}) := \hat{\mathcal{L}}(\mathbf{\Psi}_{\mathrm{KL}} \circ \mathbf{\Theta}^{-1}),$$

where $\hat{\mathcal{L}}$ is the linear functional appearing in the expression of the total energy $\hat{\mathcal{I}}$ above and $\boldsymbol{\Psi}_{\rm KL}$ is the *Kirchhoff–Love deformation associated with* $\boldsymbol{\psi}$, viz., the mapping $\boldsymbol{\Psi}_{\rm KL} : \overline{\Omega} \to \mathbb{E}^3$ defined by

$$\boldsymbol{\Psi}_{\rm KL}(\cdot, x_3) := \boldsymbol{\psi} + x_3 \boldsymbol{a}_3(\boldsymbol{\psi}) \text{ in } \overline{\omega} \text{ at each } x_3 \in [-\varepsilon, \varepsilon].$$

The two-dimensional nonlinear shell model of W.T. Koiter is one of the most commonly used two-dimensional nonlinear shell models in computational mechanics, in spite of the fact that it has not yet been justified by an existence theorem (and most likely will never be).

Our objective here is to define a *two-dimensional nonlinear shell model* that is *similar to Koiter's model* (Theorem 3), while being in addition justified by an *existence theorem* (Theorem 4).

It is worthwhile mentioning that the two-dimensional nonlinear shell model defined in this paper *can be used for any type of shell*, without any restriction on the geometry of the surface or on the boundary conditions, by contrast with the two-dimensional nonlinear shell models that have been justified so far in the literature by existence theorems: the *membrane-dominated model* (see Le Dret & Raoult [14]), the *flexural-dominated model* (see Ciarlet & Coutand [6] and Friesecke, James, Mora & Müller [10]), and the two-dimensional model of Koiter's type for spherical and "almost spherical shells" obtained by R. Bunoiu and the authors of the present paper [2,8].

Details of those proofs that are only briefly sketched here will appear in a forthcoming paper [9].

3. A new stored energy function

The starting point of the definition of our new nonlinear shell model (Sect. 4) is the following *stored energy function*, which is different from, but of the same type as, the stored energy function proposed in [7]: both are polyconvex and the principal part of their Taylor expansions in terms of $\boldsymbol{E} := (\boldsymbol{F}^{\mathsf T}\boldsymbol{F} - \boldsymbol{I})/2$ is similarly governed by the Lamé constants λ and μ.

Theorem 1. *Given constants λ and μ that satisfy $3\lambda + \mu > 0$ and $\mu > 0$, define the function $G :]0, \infty[\to \mathbb{R}$ by*

$$G(t) = (\lambda - \frac{2\mu}{3})\sqrt{t} - (\lambda + \frac{\mu}{3}) \log \sqrt{t} - (\lambda + \frac{\mu}{12}) \text{ for each } t > 0.$$

Then the stored energy function $\hat{W} : \mathbb{M}^3_+ \to \mathbb{R}$ defined by

$$\hat{W}(\boldsymbol{F}) := \frac{\mu}{12}\big(\mathrm{tr}(\boldsymbol{F}^{\mathsf T}\boldsymbol{F})\big)^2 + G(\det(\boldsymbol{F}^{\mathsf T}\boldsymbol{F})) \text{ for each } \boldsymbol{F} \in \mathbb{M}^3_+,$$

has the following properties:

(i) *$\hat{W}$ satisfies the axiom of frame indifference:*

$$\hat{W}(\boldsymbol{R}\boldsymbol{F}) = \hat{W}(\boldsymbol{F}) \text{ for all } \boldsymbol{F} \in \mathbb{M}^3_+ \text{ and all } \boldsymbol{R} \in \mathbb{O}^3_+;$$

(ii) *$\hat{W}$ becomes infinite under infinite compression or dilatation, in the sense that*

$$\lim_{\det \boldsymbol{F} \to 0^+} \hat{W}(\boldsymbol{F}) = +\infty \text{ and } \lim_{\det \boldsymbol{F} \to +\infty} \hat{W}(\boldsymbol{F}) = +\infty;$$

(iii) *$\hat{W}$ is coercive, in the sense that*

$$\hat{W}(\boldsymbol{F}) \geq \mu\Big(\frac{1}{16}\|\boldsymbol{F}\|^4 - 16\Big) \text{ for all } \boldsymbol{F} \in \mathbb{M}^3_+;$$

(iv) *the principal part of $\hat{W}$ with respect to the strain tensor $\boldsymbol{E}$ is governed by the Lamé constants λ and μ, in the sense that*

$$\hat{W}(\boldsymbol{F}) = \frac{\lambda}{2}(\mathrm{tr}\, \boldsymbol{E})^2 + \mu\|\boldsymbol{E}\|^2 + \mathcal{O}(\|\boldsymbol{E}\|^3), \text{ where } \boldsymbol{E} := \frac{1}{2}(\boldsymbol{F}^{\mathsf T}\boldsymbol{F} - \boldsymbol{I});$$

(v) *$\hat{W}$ is polyconvex, in the sense that*

$$\hat{W}(\boldsymbol{F}) = \mathbb{W}(\boldsymbol{F}, \det \boldsymbol{F}) \text{ for all } \boldsymbol{F} \in \mathbb{M}^3_+,$$

where $\mathbb{W} : \mathbb{M}^3 \times]0, \infty[\to \mathbb{R}$ is the convex function defined by

$$\mathbb{W}(\boldsymbol{F}, t) := \frac{\mu}{12}\|\boldsymbol{F}\|^4 + (\lambda - \frac{2\mu}{3})t - (\lambda + \frac{\mu}{3}) \log t - (\lambda + \frac{\mu}{12}) \text{ for all } (\boldsymbol{F}, t) \in \mathbb{M}^3 \times]0, \infty[. \qquad \square$$

Sketch of the proof. The proofs of parts (i), (ii), and (v) are straightforward.

Part (iii) is proved by combining the following estimates, satisfied by all matrices $\boldsymbol{F} \in \mathbb{M}^3_+$:

$$\hat{W}(\boldsymbol{F}) = \frac{\mu}{12}\|\boldsymbol{F}\|^4 + G((\det \boldsymbol{F})^2) \geq \frac{\mu}{12}\|\boldsymbol{F}\|^4 - \mu \det \boldsymbol{F} + \frac{\mu}{4}$$

and

$$\det \boldsymbol{F} \leq \frac{1}{3\sqrt{3}}\|\boldsymbol{F}\|^3.$$

Part (iv) is a straightforward consequence of the following relations, satisfied by all matrices $\boldsymbol{F} \in \mathbb{M}^3_+$:

$$\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I} + 2\boldsymbol{E}, \quad \operatorname{tr}(\boldsymbol{F}^T\boldsymbol{F}) = 3 + 2\operatorname{tr}\boldsymbol{E},$$
$$\det(\boldsymbol{F}^T\boldsymbol{F}) = 1 + 2\operatorname{tr}\boldsymbol{E} + 2(\operatorname{tr}\boldsymbol{E})^2 - 2\operatorname{tr}(\boldsymbol{E}^2) + 8\det\boldsymbol{E}.$$

Remark 1. The assumptions $3\lambda + \mu > 0$ and $\mu > 0$ made in Theorem 1 are thus slightly stronger than those used for establishing the uniform positive-definiteness of the three-dimensional and two-dimensional elasticity tensors, viz., $3\lambda + 2\mu > 0$ and $\mu > 0$ (cf. Sect. 1); but they are significantly weaker than those that were made in [7], viz., $\lambda > 0$ and $\mu > 0$, however. □

The next theorem, together with Theorem 1(iv), shows that the stored energy function defined in Theorem 1 is indeed an alternative to the stored energy function proposed in [7].

Theorem 2. *Let $\hat{\Omega}$ be a domain in $\mathbb{R}^3$, let $\hat{\Gamma}_0$ be a non-empty relatively open subset of the boundary of $\hat{\Omega}$, let $\hat{W} : \mathbb{M}^3_+ \to \mathbb{R}$ be the stored energy function defined in Theorem 1, let $\hat{\mathcal{L}} : W^{1,4}(\hat{\Omega}; \mathbb{E}^3) \to \mathbb{R}$ be a continuous linear form, and let $\hat{\boldsymbol{\Phi}}_0 \in W^{1,4}(\hat{\Omega}; \mathbb{E}^3)$ be a mapping that satisfies $\det \hat{\boldsymbol{\nabla}}\hat{\boldsymbol{\Phi}}_0 > 0$ a.e. in $\hat{\Omega}$.*

Define the subset $U(\hat{\Omega}; \mathbb{E}^3)$ of $W^{1,4}(\hat{\Omega}; \mathbb{E}^3)$ and the functional $\hat{\mathcal{J}} : U(\hat{\Omega}; \mathbb{E}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$U(\hat{\Omega}; \mathbb{E}^3) := \{\hat{\boldsymbol{\Psi}} \in W^{1,4}(\hat{\Omega}; \mathbb{E}^3);\ \det \hat{\boldsymbol{\nabla}}\hat{\boldsymbol{\Psi}} > 0 \text{ a.e. in } \hat{\Omega} \text{ and } \hat{\boldsymbol{\Psi}} = \hat{\boldsymbol{\Phi}}_0 \text{ on } \hat{\Gamma}_0\},$$
$$\hat{\mathcal{J}}(\hat{\boldsymbol{\Psi}}) := \int_{\hat{\Omega}} \hat{W}(\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{\Psi}})\,\mathrm{d}\hat{x} - \hat{\mathcal{L}}(\hat{\boldsymbol{\Psi}}) \text{ at each } \hat{\boldsymbol{\Psi}} \in U(\hat{\Omega}; \mathbb{E}^3).$$

Then there exists $\hat{\boldsymbol{\Psi}}^* \in U(\hat{\Omega}; \mathbb{E}^3)$ such that

$$\hat{\mathcal{J}}(\hat{\boldsymbol{\Psi}}^*) = \inf_{\hat{\boldsymbol{\Psi}} \in U(\hat{\Omega}; \mathbb{E}^3)} \hat{\mathcal{J}}(\hat{\boldsymbol{\Psi}}). \qquad \square$$

Sketch of the proof. It is easy to see that

$$\|\mathbf{Cof}\boldsymbol{F}\| \leq \|\boldsymbol{F}\|^2 \text{ and } \det \boldsymbol{F} \leq \frac{1}{3\sqrt{3}}\|\boldsymbol{F}\|^3 \text{ for each } \boldsymbol{F} \in \mathbb{M}^3_+.$$

Then we infer from the coercivity property of $\hat{W}$ established in Theorem 1(iii) that

$$\hat{W}(\boldsymbol{F}) \geq \frac{\mu}{34}\Big(\|\boldsymbol{F}\|^4 + \|\mathbf{Cof}\boldsymbol{F}\|^2 + |\det \boldsymbol{F}|^{4/3}\Big) - 16\mu \text{ for each } \boldsymbol{F} \in \mathbb{M}^3_+.$$

Together with the other properties of $\hat{W}$ established in Theorem 1, this shows that the functional $\hat{\mathcal{J}}$ and set $U(\hat{\Omega}; \mathbb{E}^3)$ satisfy all the assumptions of John Ball's fundamental existence theorem (see [1]), which thus ensures the existence of a minimizer $\hat{\boldsymbol{\Psi}}^*$ of $\hat{\mathcal{J}}$ over the set $U(\hat{\Omega}; \mathbb{E}^3)$. □

We now establish that the stored energy function of Theorem 1, in addition to being suited for modeling *three-dimensional nonlinear elastic bodies*, is also equally well suited for modeling *two-dimensional nonlinearly elastic shells* in the following sense: it shows that the two-dimensional strain energy obtained from $\hat{W}$ by integrating across the thickness the corresponding strain energy restricted to a specific class of deformations *coincides "to within the first order" with Koiter's strain energy*.

Theorem 3. *Given $\varepsilon > 0$, an open subset ω of $\mathbb{R}^2$, and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, let*

$$\Omega := \omega \times \,]-\varepsilon, \varepsilon[\text{ and } \hat{\Omega} := \boldsymbol{\Theta}(\Omega),$$

where the immersion $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ is defined as in Sect. 1 in terms of $\boldsymbol{\theta}$.

Given any immersion $\boldsymbol{\psi} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, *define the mapping* $\tilde{\boldsymbol{\Psi}}_{\text{KL}} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$ *by*

$$\tilde{\boldsymbol{\Psi}}_{\text{KL}}(\cdot, x_3) = \boldsymbol{\psi} + x_3\Big(1 - \frac{\lambda\, a^{\alpha\beta}}{\lambda + 2\mu}\big(G_{\alpha\beta}(\boldsymbol{\psi}) - \frac{x_3}{2} R_{\alpha\beta}(\boldsymbol{\psi})\big)\Big)\boldsymbol{a}_3(\boldsymbol{\psi}) \text{ in } \overline{\omega} \text{ at each } x_3 \in [\varepsilon, \varepsilon].$$

Let $\hat{W} : \mathbb{M}^3_+ \to \mathbb{R}$ *be the stored energy function defined in* Theorem 1. *Then*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \hat{W}(\hat{\nabla}(\tilde{\boldsymbol{\Psi}}_{\text{KL}} \circ \boldsymbol{\Theta}^{-1})) \circ \boldsymbol{\Theta}\, \mathrm{d}x_3 = \frac{1}{4}\Big\{a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\psi}) G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^2}{3} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\psi}) R_{\alpha\beta}(\boldsymbol{\psi})\Big\} + \delta(\varepsilon, \boldsymbol{\psi}) \text{ in } \overline{\omega},$$

where $\delta(\varepsilon, \boldsymbol{\psi}) \in \mathcal{C}^0(\overline{\omega})$ *is of a higher order than the term* $\Big\{a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\psi}) G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^2}{3} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\psi}) R_{\alpha\beta}(\boldsymbol{\psi})\Big\}$ *above, in the sense that it either depends at least cubically on* $G_{\alpha\beta}(\boldsymbol{\psi})$ *and* $\varepsilon\, R_{\alpha\beta}(\boldsymbol{\psi})$, *or quadratically, but then with a multiplicative factor of* ε *to a power* ≥ 1. □

Sketch of the proof. The proof is based on the following three observations.

First, given any mapping $\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$, Theorem 1(iv) implies that

$$\hat{W}(\hat{\nabla}(\boldsymbol{\Phi} \circ \boldsymbol{\Theta}^{-1})) \circ \boldsymbol{\Theta} = \frac{1}{2}\Sigma^{ij}(\boldsymbol{\Phi}) E_{ij}(\boldsymbol{\Phi}) + \delta_1((E_{ij}(\boldsymbol{\Phi}))) \text{ in } \overline{\Omega},$$

where

$$\Sigma^{ij}(\boldsymbol{\Phi}) := A^{ijkl} E_{kl}(\boldsymbol{\Phi}) \text{ and } E_{ij}(\boldsymbol{\Phi}) := \frac{1}{2}(\partial_i \boldsymbol{\Phi} \cdot \partial_j \boldsymbol{\Phi} - \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}),$$

and δ_1 is a function that depends on the matrix field $(E_{ij}(\boldsymbol{\Phi}))$ at least cubically.

Second, given any immersion $\boldsymbol{\psi} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, we infer from the specific definitions of the mapping $\tilde{\boldsymbol{\Psi}}_{\text{KL}}$ in terms of $\boldsymbol{\psi}$ given in the statement of the theorem, and of the functions A^{ijkl} and $a^{\alpha\beta\sigma\tau}$ given in terms of $\boldsymbol{\theta}$ in Sect. 1, that

$$E_{\alpha\beta}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) = \Big(G_{\alpha\beta}(\boldsymbol{\psi}) - x_3\, R_{\alpha\beta}(\boldsymbol{\psi})\Big) + \delta_2(x_3, \boldsymbol{\psi}) \text{ in } \overline{\Omega},$$
$$\Sigma^{\alpha\beta}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) = \frac{1}{2} a^{\alpha\beta\sigma\tau}\Big(G_{\sigma\tau}(\boldsymbol{\psi}) - x_3 R_{\sigma\tau}(\boldsymbol{\psi})\Big) + \delta_3(x_3, \boldsymbol{\psi}) \text{ in } \overline{\Omega},$$
$$\Sigma^{3i}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) = \Sigma^{i3}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) = \delta_4(x_3, \boldsymbol{\psi}) \text{ in } \overline{\Omega},$$

where δ_2, δ_3, and δ_4, are functions that either depend at least cubically on the matrix fields $(G_{\alpha\beta}(\boldsymbol{\psi}))$ and $(x_3\, R_{\alpha\beta}(\boldsymbol{\psi}))$, or quadratically, but then with a multiplicative factor of x_3 to a power ≥ 1 (recall that $|x_3| \leq \varepsilon$ and that ε is the half-thickness of the shell, which may be chosen as small as we please). Consequently, there exists a function δ_5 satisfying the same properties as the functions $\delta_2, \delta_3, \delta_4$ above such that

$$\Sigma^{ij}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) E_{ij}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}) = \frac{1}{2} a^{\alpha\beta\sigma\tau}\Big(G_{\sigma\tau}(\boldsymbol{\psi}) - x_3\, R_{\sigma\tau}(\boldsymbol{\psi})\Big)\Big(G_{\alpha\beta}(\boldsymbol{\psi}) - x_3\, R_{\alpha\beta}(\boldsymbol{\psi})\Big) + \delta_5(x_3, \boldsymbol{\psi}) \text{ in } \overline{\Omega}.$$

Third,

$$\int_{-\varepsilon}^{\varepsilon} a^{\alpha\beta\sigma\tau}\Big(G_{\sigma\tau}(\boldsymbol{\psi}) - x_3\, R_{\sigma\tau}(\boldsymbol{\psi})\Big)\Big(G_{\alpha\beta}(\boldsymbol{\psi}) - x_3\, R_{\alpha\beta}(\boldsymbol{\psi})\Big)\, \mathrm{d}x_3$$
$$= 2\varepsilon\Big\{a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\psi}) G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^2}{3}\, a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\psi}) R_{\alpha\beta}(\boldsymbol{\psi})\Big\} \text{ in } \overline{\omega}. \qquad \square$$

Remark 2. The vector fields $\tilde{\boldsymbol{\Psi}}_{\text{KL}}$ defined in Theorem 3 are *quadratic* with respect to the transverse variable x_3, while the *Kirchhoff–Love deformations* $\boldsymbol{\Psi}_{\text{KL}}$ appearing in the definition of the two-dimensional nonlinear shell model of W.T. Koiter (Sect. 2) are only *affine* with respect to x_3. The reason why Kirchhoff–Love deformations $\boldsymbol{\Psi}_{\text{KL}}$ cannot be used in Theorem 3 instead of the vector fields $\tilde{\boldsymbol{\Psi}}_{\text{KL}}$ is that the normal stress $\Sigma^{33}(\boldsymbol{\Psi}_{\text{KL}})$ associated with a Kirchhoff–Love deformation is not of an order lower than that of the tangential stress tensor field $(\Sigma^{\alpha\beta}(\boldsymbol{\Psi}_{\text{KL}}))$, as it should, according to John [11,12]; indeed,

$$\Sigma^{33}(\boldsymbol{\Psi}_{\text{KL}}) := A^{33kl} E_{kl}(\boldsymbol{\Psi}_{\text{KL}}) = \lambda g^{\sigma\tau} E_{\sigma\tau}(\boldsymbol{\Psi}_{\text{KL}})$$

is of the same order as

$$\Sigma^{\alpha\beta}(\boldsymbol{\Psi}_{\text{KL}}) := A^{\alpha\beta kl} E_{kl}(\boldsymbol{\Psi}_{\text{KL}}) = A^{\alpha\beta\sigma\tau} E_{\sigma\tau}(\boldsymbol{\Psi}_{\text{KL}}) = (\lambda g^{\alpha\beta} g^{\sigma\tau} + 2\mu g^{\alpha\sigma} g^{\beta\tau}) E_{\sigma\tau}(\boldsymbol{\Psi}_{\text{KL}}),$$

while, as shown in the proof above, $\Sigma^{33}(\tilde{\boldsymbol{\Psi}}_{\text{KL}})$ *is of a lower order than* $(\Sigma^{\alpha\beta}(\tilde{\boldsymbol{\Psi}}_{\text{KL}}))$. □

4. A two-dimensional nonlinear model of Koiter's type

We are now in a position to define our *new nonlinear shell model* and to justify this model by an *existence theorem*. Note that this new model, defined in part (ii) of the next theorem, is *two-dimensional* since its unknown is the matrix field $(\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) : \omega \to \mathbb{M}^3$ formed by a vector field $\boldsymbol{\psi} : \omega \to \mathbb{E}^3$ that governs the deformation of the middle surface of the shell and by two *colinear vector fields* $\boldsymbol{\eta} : \omega \to \mathbb{E}^3$ and $\boldsymbol{\zeta} : \omega \to \mathbb{E}^3$ that govern the deformation of the fibers orthogonal to the middle surface of the undeformed shell, all of which being thus defined over the *two-dimensional* domain ω.

Theorem 4. *Given $\varepsilon > 0$, an open subset ω of $\mathbb{R}^2$, and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$, let*

$$\Omega := \omega \times \left]-\varepsilon, \varepsilon\right[\text{ and } \hat{\Omega} := \boldsymbol{\Theta}(\Omega),$$

where the immersion $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ is defined as in Sect. 1 in terms of $\boldsymbol{\theta}$. Let γ_0 be a non-empty relatively open subset of the boundary of ω, let $\hat{\mathcal{L}} : W^{1,4}(\hat{\Omega}; \mathbb{E}^3) \to \mathbb{R}$ be a continuous linear form, and let $G :]0, \infty[\to \mathbb{R}$ be the function defined in Theorem 1.

(i) *Define the subset $U(\Omega; \mathbb{E}^3)$ of $W^{1,4}(\Omega; \mathbb{E}^3)$ and the functional $\mathcal{J} : U(\Omega; \mathbb{E}^3) \to \mathbb{R} \cup \{+\infty\}$ by*

$$U(\Omega; \mathbb{E}^3) := \{\boldsymbol{\Psi} \in W^{1,4}(\Omega; \mathbb{E}^3);\ \det \boldsymbol{\nabla}\boldsymbol{\Psi} > 0 \text{ a.e. in } \Omega \text{ and } \boldsymbol{\Psi} = \boldsymbol{\Theta} \text{ on } \gamma_0 \times \left]-\varepsilon, \varepsilon\right[\},$$

$$\mathcal{J}(\boldsymbol{\Psi}) := \int_\Omega \left\{ \frac{\mu}{12} \left[\operatorname{tr}(C^i_j(\boldsymbol{\Psi})) \right]^2 + G(\det(C^i_j(\boldsymbol{\Psi}))) \right\} \sqrt{g}\, dx - \hat{\mathcal{L}}(\boldsymbol{\Psi} \circ \boldsymbol{\Theta}^{-1}) \text{ at each } \boldsymbol{\Psi} \in U(\Omega; \mathbb{E}^3),$$

where

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta},\ g := \det(g_{ij}),\ (g^{kl}) := (g_{ij})^{-1},\ \text{and } C^i_j(\boldsymbol{\Psi}) := g^{ik}(\partial_k \boldsymbol{\Psi} \cdot \partial_j \boldsymbol{\Psi}).$$

Then there exists $\boldsymbol{\Psi}^* \in U(\Omega; \mathbb{E}^3)$ such that

$$\mathcal{J}(\boldsymbol{\Psi}^*) = \inf_{\boldsymbol{\Psi} \in U(\Omega; \mathbb{E}^3)} \mathcal{J}(\boldsymbol{\Psi}).$$

(ii) *Define the subset $V(\omega; \mathbb{M}^3)$ of $W^{1,4}(\omega; \mathbb{M}^3)$ and the functional $\mathrm{j} : V(\omega; \mathbb{M}^3) \to \mathbb{R} \cup \{+\infty\}$ by*

$$V(\omega; \mathbb{M}^3) := \{(\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in W^{1,4}(\omega; \mathbb{M}^3);\ \boldsymbol{\eta} \wedge \boldsymbol{\zeta} = \mathbf{0} \text{ a.e. in } \omega \text{ and } \boldsymbol{\Psi}_{\eta,\zeta} \in U(\Omega; \mathbb{E}^3)\},$$

$$\mathrm{j}(\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) := \mathcal{J}(\boldsymbol{\Psi}_{\eta,\zeta}) \text{ at each } (\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in V(\omega; \mathbb{M}^3),$$

where

$$\boldsymbol{\Psi}_{\eta,\zeta}(y, x_3) := \boldsymbol{\psi}(y) + x_3 \boldsymbol{\eta}(y) + x_3^2 \boldsymbol{\zeta}(y) \text{ at each } (y, x_3) \in \Omega.$$

Then there exists $(\boldsymbol{\psi}^*, \boldsymbol{\eta}^*, \boldsymbol{\zeta}^*) \in V(\omega; \mathbb{M}^3)$ such that

$$\mathrm{j}(\boldsymbol{\psi}^*, \boldsymbol{\eta}^*, \boldsymbol{\zeta}^*) = \inf_{(\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in V(\omega; \mathbb{M}^3)} \mathrm{j}(\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}). \qquad \square$$

Sketch of the proof. Part (i) of the theorem is a straightforward consequence of Theorem 2 with

$$U(\hat{\Omega}; \mathbb{E}^3) := \{\boldsymbol{\Psi} \circ \boldsymbol{\Theta}^{-1};\ \boldsymbol{\Psi} \in U(\Omega; \mathbb{E}^3)\} \text{ and } \hat{\mathcal{J}}(\hat{\boldsymbol{\Psi}}) := \mathcal{J}(\hat{\boldsymbol{\Psi}} \circ \boldsymbol{\Theta}) \text{ for each } \hat{\boldsymbol{\Psi}} \in U(\hat{\Omega}; \mathbb{E}^3).$$

Part (ii) of the theorem is proved by showing that the set

$$V(\hat{\Omega}; \mathbb{E}^3) := \{\boldsymbol{\Psi}_{\eta,\zeta} \circ \boldsymbol{\Theta}^{-1};\ \boldsymbol{\Psi}_{\eta,\zeta}(\cdot, x_3) := \boldsymbol{\psi} + x_3 \boldsymbol{\eta} + x_3^2 \boldsymbol{\zeta} \text{ in } \omega,\ x_3 \in \left]-\varepsilon, \varepsilon\right[,\ (\boldsymbol{\psi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in V(\omega; \mathbb{M}^3)\}$$

is sequentially weakly closed in the space $W^{1,4}(\hat{\Omega}; \mathbb{E}^3)$ and that the restriction of the above functional $\hat{\mathcal{J}}$ to the set $V(\hat{\Omega}; \mathbb{E}^3)$ is sequentially weakly semi-continuous. To this end, we note that the stored energy function of the functional $\hat{\mathcal{J}}$ is precisely the function $\hat{W}$ defined in Theorem 1, so that we can use John Ball's arguments in [1] to establish the existence of a minimizer. $\square$

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