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On properties and applications of (p,q)-extended τ-hypergeometric functions

Sur les propriétés et applications des fonctions τ-hypergéométriques (p,q)-étendues

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ABSTRACT

We introduce the (p,q)-extended τ-hypergeometric and confluent hypergeometric functions along with their integral representations. We also present closed integral expressions for the Mathieu-type $\boldsymbol{a}$-series and for the associated alternating versions whose terms contain the (p,q)-extended τ-hypergeometric functions with related contiguous functional relations.

RÉSUMÉ

Nous introduisons les fonctions τ-hypergéométriques et hypergéométriques confluentes (p,q)-étendues, avec leurs représentations intégrales. Nous présentons également des formules intégrales closes pour les $\boldsymbol{a}$-séries de type Mathieu et les versions alternées associées, dont les termes contiennent les fonctions τ-hypergéométriques (p,q)-étendues, avec les relations fonctionnelles de contiguïté.

1. Introduction and motivation

In the recent years, a series of papers have been published by many authors, including Pogány either alone and/or with his co-workers Srivastava and Tomovski [12–17], in which special general Mathieu-type series and their alternating variants have been considered, whose terms contain various special functions, for example, the Gauss hypergeometric function ${}_2F_1$, the generalized hypergeometric function ${}_pF_q$, Meijer G-functions, and so on. The derived results concern, among others,

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closed integral form expressions for the considered series and bilateral bounding inequalities. Recently, extensions, generalizations and unifications of various special functions of (p, q)-variant, and in turn, when $p = q$ the p-variant, have been studied widely together with the set of related higher transcendental hypergeometric-type special functions by several authors; consult, for instance, [2–4,6,9,10]). In particular, Choi et al. [5] introduced and studied the (p, q)-extended Beta, the (p, q)-extended hypergeometric, and the (p, q)-extended confluent hypergeometric functions in the following manner:

$$\mathrm{B}(x, y; p, q) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t\,, \tag{1.1}$$

when $\min\{\Re(x), \Re(y)\} > 0$; $\min\{\Re(p), \Re(q)\} \geq 0$, and by means of (1.1),

$$F_{p,q}(a, b; c; z) = \sum_{n\geq 0} (a)_n \frac{\mathrm{B}(b+n,\, c-b;\; p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!} \qquad |z| < 1;\; \Re(c) > \Re(b) > 0\,, \tag{1.2}$$

and

$$\Phi_{p,q}(b; c; z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+n,\, c-b;\; p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!} \qquad \Re(c) > \Re(b) > 0\,. \tag{1.3}$$

Here we remark that the definition (1.1) is a special case of the definition in [18, Eq. (6.1)]. Related properties, various integral representations, differentiation formulæ, Mellin transform, recurrence relations, summations are also given in [5]. On the other hand, τ-extension of hypergeometric and confluent hypergeometric functions have been introduced by Virchenko [19,20] (also see [7]), and studied recently by Parmar [11].

Inspired by certain recent extensions of the various special functions of (p, q)-variants, we introduce (p, q)-extended τ-hypergeometric and confluent hypergeometric functions along with their integral representations. We also present contiguous functional relations by closed-form integral expressions for the Mathieu-type $\boldsymbol{a}$-series and for the associated alternating versions whose terms contain the (p, q)-extended τ-hypergeometric function.

2. (p, q)-extended τ-hypergeometric functions

In this section, we introduce and investigate the (p, q)-extended τ-hypergeometric function and (p, q)-extended τ-confluent hypergeometric function by means of the (p, q)-extended beta function as follows:

$$R^{\tau}_{p,q}(a, b; c; z) = \sum_{n\geq 0} (a)_n \frac{\mathrm{B}(b+\tau n,\, c-b;\; p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!}, \tag{2.1}$$

where $\min\{\Re(p), \Re(q)\} > 0$, $\tau \geq 0$, $|z| < 1$ while $\Re(c) > \Re(b) > 0$ when $p = 0 = q$, and

$$\Phi^{\tau}_{p,q}(b; c; z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+\tau n,\, c-b;\; p, q)}{\mathrm{B}(b,\, c-b)} \frac{z^n}{n!}, \tag{2.2}$$

with the parameter range and domain being $\min\{\Re(p), \Re(q)\} > 0$, $\tau \geq 0$, and if $p = 0 = q$ it is $\Re(c) > \Re(b) > 0$, respectively. The case $p = 0 = q$ reduces for series to Virchenko's τ-hypergeometric function [20] and τ-confluent hypergeometric function [19]:

$${}_2R^{\tau}_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n\geq 0} (a)_n \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}.$$

Here $\tau > 0, \Re(a) > 0, \Re(c) > \Re(b) > 0;\, |z| < 1$ and

$${}_1\Phi^{\tau}_1(b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n\geq 0} \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}, \qquad \tau > 0,\, \Re(c) > \Re(b) > 0,$$

respectively. Further, definitions (2.1) and (2.2) reduce to (1.2) and (1.3), when specifying $\tau = 1$.

We begin the exposition of our main results by presenting a set of Laplace integral representations for (p, q)-extended τ-hypergeometric function.

Theorem 1. *For all* $\min\{\Re(p), \Re(q)\} > 0$, $\tau \geq 0$; $\Re(z) < 1$ *or* $\Re(a) > 0$ *when* $p = 0 = q$ *the following Laplace-type integral representation holds true:*

$$R^{\tau}_{p,q}(a, b; c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty} \mathrm{e}^{-zt} t^{a-1}\, \Phi^{\tau}_{p,q}(b; c; zt)\, \mathrm{d}t.$$

Proof. Using the definition of the Pochhammer symbol $(a)_n$ in (2.1), by considering the Gamma function integral

$$\Gamma(\eta)\, \xi^{-\eta} = \int_0^\infty \mathrm{e}^{-\xi t} t^{\eta-1}\, \mathrm{d}t, \qquad \min\{\Re(\xi), \Re(\eta)\} > 0 \tag{2.3}$$

and (2.2), we are led to the desired result. □

Theorem 2. *For all* $\min\{\Re(p), \Re(q)\} > 0,\ \tau \ge 0;\ |\arg(1-z)| < \pi$*; and* $\Re(c) > \Re(b) > 0$ *when* $p = 0 = q$*, we have the following Euler-type integral representation:*

$$R^\tau_{p,q}(a,b;c;z) = \frac{1}{\mathrm{B}(b,\, c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt^\tau)^{-a}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t.$$

Proof. By the definition (1.1) of the (p,q)-extended Beta applied in (2.1) and interchanging summation and integration, we conclude

$$R^\tau_{p,q}(a,b;c;z) = \frac{1}{\mathrm{B}(b,\, c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}} \sum_{n\ge 0} (a)_n \frac{(zt^\tau)^n}{n!}\, \mathrm{d}t.$$

Employing the generalized binomial expansion

$$(1-zt^\tau)^{-a} = \sum_{n\ge 0} (a)_n \frac{(zt^\tau)^n}{n!},$$

which obviously converges for all $|z| < 1, \tau \ge 0$, we finish the proof. □

A similar consideration gives the integral form of $\Phi^\tau_{p,q}$, *viz.*

Theorem 3. *For all* $p, q \in \mathbb{C} \setminus \{0\}$, $\min\{\Re(p), \Re(q)\} > 0,\ \tau \ge 0$*, and all* $b, c \in \mathbb{C}, \Re(c) > \Re(b) > 0$ *when* $p = 0 = q$ *we have the integral expression*

$$\Phi^\tau_{p,q}(b;c;z) = \frac{1}{\mathrm{B}(b,\, c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\, \mathrm{e}^{zt^\tau-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t.$$

3. On Mathieu-type series built by $R^\tau_{p,q}$

Extending the Mathieu-type series studied in [12] by imposing the $R^\tau_{p,q}(a,b;c;z)$ input–kernel in the summands, we define the Mathieu-type **a**-series $\mathfrak{R}_{\lambda,\eta}$ and its alternating variant $\widetilde{\mathfrak{R}}_{\lambda,\eta}$ in the form of series

$$\mathfrak{R}_{\lambda,\eta}(R^\tau_{p,q};\boldsymbol{a};r) := \sum_{n\ge 1} \frac{R^\tau_{p,q}\,(\lambda,\, b;\, c;\, -\frac{r^2}{a_n})}{a_n^\lambda (a_n + r^2)^\eta},$$

and

$$\widetilde{\mathfrak{R}}_{\lambda,\eta}(\,R^\tau_{p,q};\boldsymbol{a};r) := \sum_{n\ge 1} \frac{(-1)^{n-1} R^\tau_{p,q}\,(\lambda,\, b;\, c;\, -\frac{r^2}{a_n})}{a_n^\lambda (a_n + r^2)^\eta},$$

being in both series the parameters' range $\tau \ge 0;\ \lambda, \eta, r > 0$. Now we establish a contiguous integral form expressions for the series $\mathfrak{R}_{\lambda,\eta}(R^\tau_{p,q};\boldsymbol{a};r)$ and $\widetilde{\mathfrak{R}}_{\lambda,\eta}(R^\tau_{p,q};\boldsymbol{a};r)$ with respect to parameters λ, η. We note that the function $z \mapsto R^\tau_{p,q}$ (the Laplace transform treated in Theorem 1) is homogeneous of degree $-a$, that is,

$$R^\tau_{p,q}\,(a,b;c;\omega z) = \omega^{-a}\, R^\tau_{p,q}\,(a,b;c;z)\,, \qquad \omega \in \mathbb{R}. \tag{3.1}$$

Theorem 4. *Let $\lambda > 0$, $\eta > 0$, $r > 0$ and suppose the real sequence $\boldsymbol{a} = (a_n)_{n\geq 1}$ is monotone increasing and tends to ∞. Then for $\tau \geq 0$, and $\min\{\Re(p), \Re(q)\} \geq 0$, we have*

$$\begin{aligned}\mathfrak{R}_{\lambda,\eta}(R^{\tau}_{p,q}; \boldsymbol{a}; r) &= \lambda\, \mathscr{I}^{\tau}_{p,q}(\lambda+1, \eta) + \eta\, \mathscr{I}^{\tau}_{p,q}(\lambda, \eta+1) \\ \widetilde{\mathfrak{R}}_{\lambda,\eta}(R^{\tau}_{p,q}; \boldsymbol{a}; r) &= \lambda\, \widetilde{\mathscr{I}}^{\tau}_{p,q}(\lambda+1, \eta) + \eta\, \widetilde{\mathscr{I}}^{\tau}_{p,q}(\lambda, \eta+1)\,,\end{aligned} \tag{3.2}$$

where

$$\mathscr{I}^{\tau}_{p,q}(\lambda,\eta) = \int_{a_1}^{\infty} \frac{R^{\tau}_{p,q}\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})[a^{-1}(x)]}{x^{\lambda}(x+r^2)^{\eta}}\, \mathrm{d}x$$

$$\widetilde{\mathscr{I}}^{\tau}_{p,q}(\lambda,\eta) = \int_{a_1}^{\infty} \frac{R^{\tau}_{p,q}\,(\lambda,\, b;\, c;\, -\frac{r^2}{x})\, \sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^{\lambda}(x+r^2)^{\eta}}\, \mathrm{d}x$$

and $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an increasing function such that $a(x)|_{x\in\mathbb{N}} = \boldsymbol{a}$, $a^{-1}(x)$ denotes the inverse of $a(x)$ and $[a^{-1}(x)]$ stands for the integer part of the quantity $a^{-1}(x)$.

Proof. Taking $\xi = a_n + r^2$ in the familiar gamma formula (2.3), after rearrangement by specifying $\omega = -r^2$, $z = a_n$, in (3.1), the function $\mathfrak{R}_{\lambda,\eta}(R^{\tau}_{p,q}; \boldsymbol{a}; r)$ becomes

$$\mathfrak{R}_{\lambda,\eta}(R^{\tau}_{p,q}; \boldsymbol{a}; r) = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-r^2 s} \frac{t^{\lambda-1}s^{\eta-1}}{\Gamma(\lambda)\Gamma(\eta)} \sum_{n\geq 1} \mathrm{e}^{-a_n(t+s)}\, \Phi^{\tau}_{p,q}(b; c; -r^2 t)\, \mathrm{d}t\, \mathrm{d}s\,.$$

Using the Cahen formula for Dirichlet series [1, p. 97], [8, p. 11, Theorem 11],

$$\sum_{n\geq 1} \mu_n \mathrm{e}^{-a_n x} = x \int_0^{\infty} \mathrm{e}^{-xt} \sum_{n:\, a_n \leq t} \mu_n\, \mathrm{d}t, \qquad \Re(x) > 0\,,$$

according to the technique developed in [16], we obtain

$$\mathcal{D}_a(u) = \sum_{n\geq 1} \mathrm{e}^{-a_n u} = u \int_{a_1}^{\infty} \mathrm{e}^{-ux}[a^{-1}(x)]\, \mathrm{d}x\,,$$

which results, for $u = t + s$, in

$$\begin{aligned}\mathfrak{R}_{\lambda,\eta}(R^{\tau}_{p,q}; \boldsymbol{a}; r) = \frac{1}{\Gamma(\lambda)\Gamma(\eta)} &\int_0^{\infty}\int_0^{\infty}\int_{a_1}^{\infty} \mathrm{e}^{-(r^2+x)s-tx}(t+s)t^{\lambda-1}s^{\eta-1}[a^{-1}(x)] \\ &\times\ \Phi^{\tau}_{p,q}(b; c; -r^2 t)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x =: \mathcal{I}_t + \mathcal{I}_s\,,\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_t &= \frac{1}{\Gamma(\eta)} \int_0^{\infty} \left(\int_{a_1}^{\infty} \left(\int_0^{\infty} \frac{\mathrm{e}^{-xt}t^{\lambda}}{\Gamma(\lambda)}\, \Phi^{\tau}_{p,q}(b; c; -r^2 t)\, \mathrm{d}t \right) \mathrm{e}^{-xs}[a^{-1}(x)]\, \mathrm{d}x \right) \mathrm{e}^{-r^2 s}s^{\eta-1}\, \mathrm{d}s \\ &= \frac{\lambda}{\Gamma(\eta)} \int_{a_1}^{\infty} \left(\int_0^{\infty} \mathrm{e}^{-(x+r^2)s}s^{\eta-1}\, \mathrm{d}s \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}}\, R^{\tau}_{p,q}\left(\lambda+1,\, b;\, c;\, -\frac{r^2}{x}\right) \mathrm{d}x \\ &= \lambda \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\lambda+1}(x+r^2)^{\eta}}\, R^{\tau}_{p,q}\left(\lambda+1,\, b;\, c;\, -\frac{r^2}{x}\right) \mathrm{d}x = \lambda\, \mathscr{I}^{\tau}_{p,q}(\lambda+1, \eta),\end{aligned}$$

and in similar way follows $\mathcal{I}_s = \eta\, \mathscr{I}^{\tau}_{p,q}(\lambda, \eta+1)$. These give (3.2).

The Cahen integral form of the alternating Dirichlet series $\widetilde{\mathcal{D}}_a(x)$ reads [16]:

$$\widetilde{\mathcal{D}}_a(v) = \sum_{n\geq 1}(-1)^{n-1}\mathrm{e}^{-a_n v} = v \int_{a_1}^{\infty} \mathrm{e}^{-vx}\sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right) \mathrm{d}x.$$

The rest is obvious by following the lines of establishing (3.2). □

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