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Topology/Dynamical systems

On dynamics of the Sierpiński carpet

Sur la dynamique du tapis de Sierpiński

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ABSTRACT

We prove that the Sierpiński curve admits a homeomorphism with strong mixing properties. We also prove that the constructed example does not have Bowen's specification property.

RÉSUMÉ

Nous montrons que la courbe de Sierpiński admet un homéomorphisme ayant des propriétés de mélange fortes. Nous montrons également que l'application construite n'a pas la propriété de spécification de Bowen.

1. Introduction

The aim of this note is to exhibit a homeomorphism of the Sierpiński curve (known as the planar universal curve or Sierpiński carpet) with some strong mixing properties. In 1993, Aarts and Oversteegen proved that the Sierpiński curve admits a transitive homeomorphism [1], answering a question of Gottschalk. They also showed that it does not admit a minimal one. Earlier, in 1991, Kato proved that the Sierpiński curve does not admit expansive homeomorphisms [13]. In [3], Biś, Nakayama and Walczak proved that the Sierpiński curve admits a homeomorphism with positive entropy. They also showed that it admits a minimal group action (by [1] it cannot be a $\mathbb{Z}$-action). There has been quite a lot of interest in dynamical properties of the planar universal curve, also due to its occurrence as Julia sets of various complex maps (see, e.g., [9]). Nonetheless, we were unable to find any examples in the literature that would explicitly show homeomorphisms of the Sierpiński curve with chaosity beyond Devaney chaos. The writing of the note was also motivated by some recent questions. During the Workshop on Dynamical Systems and Continuum Theory, held at the University of Vienna in June 2015, the following question was raised.

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Question 1.1. *Suppose that a 1-dimensional continuum X admits a mixing homeomorphism. Must X be $\frac{1}{n}$-indecomposable for some n?*

Recall that a continuum X is $\frac{1}{n}$-*indecomposable* if, given n mutually disjoint subcontinua of X, at least one of them must have empty interior in X. Note that the Sierpiński curve is not $\frac{1}{n}$-indecomposable for any $n \in \mathbb{N}$. This is because it is locally connected, so every point has an arbitrarily small connected neighborhood.

Theorem 1.2. *The Sierpiński curve S admits a homeomorphism $H: S \to S$ such that:*

(1) *H has a fully supported measure μ, such that (H, μ) is Bernoulli,*
(2) *H has a dense set of periodic points,*
(3) *H does not have the specification property.*

Since every Bernoulli measure is strongly mixing, and μ in Theorem 1.2 is fully supported, we immediately obtain the following result, answering Question 1.1 in the negative.

Corollary 1.3. *The Sierpiński curve S admits a topologically mixing homeomorphism with dense set of periodic points.*

Our example is quite simple; however, it relies on many nontrivial facts from topology and ergodic theory. In principle, the general strategy is very similar to the one in [1], but the starting point is a bit different. We start with an Anosov torus diffeomorphism, which allows us to say much more about the dynamics of the constructed map. By the arguments in the proof of Theorem 1.2, it seems very likely that the Aarts–Oversteegen technique [1], which we employ here, will never lead to a map with the specification property. This motivates the following natural question.

Question 1.4. *Does the Sierpiński curve admit a homeomorphism with the specification property?*

2. Preliminaries

By a dynamical system (X, T) we mean a compact metric space (X, d) with a continuous map $T: X \to X$. We identify $\mathbb{T}^2$ with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.

A *Sierpiński curve* is any set of the form $\mathbb{S}^2 \setminus \bigcup_{i=1}^{\infty} \operatorname{int} D_i$ where

(S1) each D_i is a disc and $D_i \cap D_j = \emptyset$ for $i \neq j$,
(S2) $\{D_i\}_{i=1}^{\infty}$ is a null sequence, i.e. the diameters of D_i tend to zero, as $i \to \infty$,
(S3) $\bigcup_{i=1}^{\infty} D_i$ is dense in $\mathbb{S}^2$.

Whyburn [17] proved that Sierpiński's curve does not depend on the choice of the sequence of discs $\{D_i\}_{i=1}^{\infty}$, that is, any two Sierpiński curves are homeomorphic.

2.1. Topological notions of mixing

A dynamical system (X, T) is topologically mixing if, for any two nonempty open sets U, V, there is an $N > 0$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. There are many different extensions of the above property to characterize stronger mixing in the system. From the point of view of our work, the following two are very important. It is not hard to see that they imply topological mixing.

In his seminal paper [4], Bowen introduced an important, strong version of mixing, called periodic specification property. Let $T: X \to X$ be a continuous onto map. Following Bowen (cf. [8]), we say that (X, T) has the *specification property* if for any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that, for any integer $s \geq 2$, any s points $y_1, \ldots, y_s \in X$, and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ of $2s$ integers with $j_{m+1} - k_m \geq N$ for $m = 1, \ldots, s-1$, there is a point $x \in X$ such that, for each positive integer $m \leq s$, we have $d(T^i(x), T^i(y_m)) < \varepsilon$ for all $j_m \leq i \leq k_m$. If, in addition, we can select x in such a way that $T^{k_m - j_1 + N}(x) = x$ then (X, T) has the *periodic specification property*. Note that the problem of characterizing the relations between various types of mixing for maps in specified classes of one-dimensional continua is of high interest (e.g., see [11] and references therein).

2.2. Invariant measures

Let X be a compact metric space with metric d and let $M(X)$ be the space of Borel probability measures on X equipped with the *Lévy–Prokhorov metric* ρ defined by

$$\rho(\mu, \nu) = \inf\{\varepsilon : \mu(A) \leq \nu(A^{\varepsilon}) + \varepsilon \text{ for all Borel subsets } A \subset X\},$$

where $A^\delta = \{x : \mathrm{dist}(x, A) < \delta\}$. While the formula defining ρ is not symmetric with respect to μ and ν, it is an old result of Strassen that ρ is in fact a symmetric function (see Section 2.3 in [12] and comments therein). The topology induced by ρ coincides with the weak*-topology on $M(X)$. It is also well known that $(M(X), \rho)$ is a compact metric space. For a dynamical system (X, T), we denote by $M_T(X)$ the set of all T-invariant measures from $M(X)$. For more details on the Lévy–Prokhorov metric and the weak*-topology, the reader is referred to [12], and basic properties related to invariant measures (ergodicity, strong mixing, Bernoulli shift) can be found in [16].

2.3. Quasi-hyperbolic toral automorphisms

Let A be a 2×2 matrix with integer entries such that $|\det A| = 1$. Then A^{-1} also has integer entries, and so A induces a homeomorphism of the 2-dimensional torus $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ by $F(x) = Ax (\mathrm{mod}\ 1)$, e.g., see [6] for more details. Since $|\det A| = 1$, every toral automorphism preserves the Lebesgue measure. It is known that the periodic points of an ergodic toral automorphism are exactly those with rational coordinates (see [8, Proposition 24.7]). It was first proved by Adler and Weiss for $\mathbb{T}^2$, and then extended by Katznelson to each $\mathbb{T}^n$, that if a toral automorphism is ergodic with respect to the Lebesgue measure, then it is measure-theoretically conjugate to a Bernoulli shift (e.g., see [8, Theorem 24.6]). Following Lind [14], we say that F is *quasi-hyperbolic* if A does not have roots of unity as eigenvalues. In dimension 2, every quasi-hyperbolic automorphism must be hyperbolic, that is, it does not have eigenvalues on the unit circle, and has periodic specification property [14].

2.4. Branched covering from $\mathbb{T}^2$ to $\mathbb{S}^2$

Take a quotient of $\mathbb{T}^2$ by the relation J, which identifies (x, y) with $(-x, -y)$. The relation J induces a branched covering map $\pi\colon \mathbb{T}^2 \to \mathbb{S}^2$ (see, e.g., [16, p. 140]), which is 2-to-1 except at four branch points in $\mathbb{T}^2$ given by $\mathcal{C} = \{(0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2)\}$. Since the relation J is preserved by any toral automorphism, for every toral automorphism F, we obtain a factor map $G\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that $G \circ \pi = \pi \circ F$. Note that if $x \notin \mathcal{C} \cup F^{-1}(\mathcal{C})$, then there is an open neighborhood U of x such that U has at most one element of any equivalence class of the relation J and the same holds for $F(U)$. Then, on U the factor map π is a local isometry.

3. Proof of Theorem 1.2

Start with "Arnold's cat map" $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ on the torus given by

$$F(x, y) = (2x + y, x + y) (\mathrm{mod}\ 1). \tag{1}$$

Clearly F is hyperbolic with eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{-\sqrt{5}-1}{2}$, hence it has the periodic specification property.

Denote the Lebesgue measure on $\mathbb{T}^2$ by λ. F has a dense set of periodic points and $(\mathbb{T}^2, F, \lambda)$ is measure-theoretically conjugate to Bernoulli shift. Let $\pi\colon \mathbb{T}^2 \to \mathbb{S}^2$ be the quotient map from Section 2.4 and let G be the induced homeomorphism of $\mathbb{S}^2$. Since G is a factor of F, $(\mathbb{S}^2, G, \mu)$ is Bernoulli with respect to a fully supported measure μ which is a push-forward of λ by π (see [16, Theorem 4.29(ii)]).

Fix any $x \in \mathbb{T}^2 \setminus (\mathcal{C} \cup F^{-1}(\mathcal{C}))$ and identify it with $x \in [-1/2, 1/2]^2$ in the universal cover. For each v from the unit circle, denote the corresponding radial line emerging from x by $\mathcal{L}^x_v = \{x + tv : t > 0\} \subset \mathbb{R}^2$. If we view F as a linear map on the universal cover, then we clearly have that $F(\mathcal{L}^x_v) = \mathcal{L}^{F(x)}_w$ for $w = F(v)/||F(v)||$. Therefore, for sufficiently small open neighborhoods V, $W \subset \mathbb{T}^2$ of x and $F(x)$ there is no ambiguity in writing $F(\mathcal{L}^x_v \cap V) = \mathcal{L}^{F(x)}_w \cap W$. In other words, F locally preserves radial lines on $\mathbb{T}^2$ for points outside $\mathcal{C} \cup F^{-1}(\mathcal{C})$. But π is locally invertible on a small neighborhood of these points, hence also G preserves the radial lines locally on $\mathbb{S}^2$.

Fix a dense set of periodic points $\mathcal{O} \subset \mathrm{Per}(G)$ and such that $G(\mathcal{O}) = \mathcal{O}$, $\mathcal{O} \cap \pi(\mathcal{C}) = \emptyset$ and that $\mathrm{Per}(G) \setminus \bigcup \mathcal{O}$ is dense in $\mathbb{S}^2$. We decompose $\mathcal{O}$ into a union of p_n-periodic orbits O_n; i.e. $\mathcal{O} = \bigcup_{n \in \mathbb{N}} O_n$. We will modify G inductively, blowing up consecutive periodic orbits from $\mathcal{O}$. Since G is differentiable, this can be done by a standard procedure in differentiable dynamics (see, e.g., [5], p. 234), or topologically adopting [1] as follows. Take the periodic obit $O_1 \subset \mathcal{O}$, say $O_1 = \{c, G(c), \ldots, G^{p_1-1}(c)\}$. Since $\pi^{-1}(O_1) \cap \mathcal{C} = \emptyset$, there are open discs $D_0, \ldots, D_{p_1-1}$ such that $\pi(D_i) \cap \pi(D_j) = \emptyset$ for $i \neq j$ and π is 1–1 on each D_i. Let $U_i = \pi(D_i)$. By the discussion above, we have a natural decomposition of U_i into radial lines (induced locally from D_i) centered at $G^i(c)$, such that if $L \subset U_i$ is a sufficiently short radial line emerging from $G^i(c)$ then $G(L) \cap U_{i+1}$ is contained in the corresponding line.

Making this formal, let U_i be a small neighborhood of $G^i(c)$, and $\mathcal{F}_i = \{\mathcal{L}^{G^i(c)}_v \cap U_i : v \in \mathbb{S}^1\}$ be the family of lines emanating from the point $G^i(c)$ such that $\bigcup \mathcal{F}_i = U_i \setminus G^i(c)$. We remove O_1 and compactify each $\mathrm{cl}\, U_i \setminus \{G^i(c)\}$ by a topological copy S^i_c of the unit circle $\mathbb{S}^1$, adding, for each index i and $v \in \mathbb{S}^1 \cong S^i_c$, a point $\theta^i_v \in S^i_c$ compactifying the radial line $\mathcal{L}^{G^i(c)}_v \cap U_i$. That way, we obtain a p_1-punctured sphere S_1. We may easily extend G to a continuous map $H_1\colon S_1 \to S_1$ by setting $H_1(\theta^i_v) = \theta^j_w$, where $j \equiv i + 1 (\mathrm{mod}\ p_1)$ and $w = F(v)/||F(v)||$. Clearly, H_1 defined that way is invertible with a continuous inverse, so H_1 is a homeomorphism of S_1. We also have a projection $\pi_1 : S_1 \to S_0$ given by $\pi_1(S^i_c) = G^i(c)$ for each $i = 0, \ldots, p_1 - 1$ and π_1 is the identity on $S_1 \setminus \bigcup_{i=0}^{p_1} S^i_c$ up to natural identification of points outside of O_1. Observe that

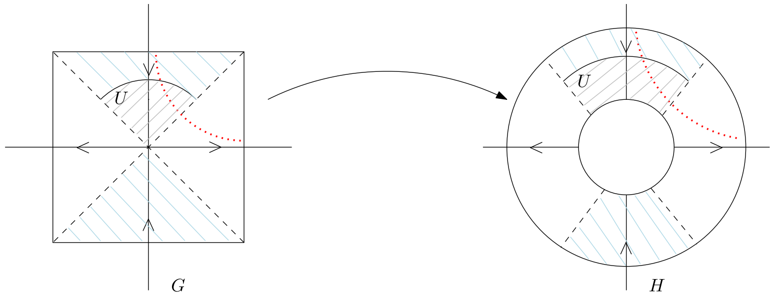

Fig. 1. Phase portrait for a hyperbolic point before and after a "blow-up".

the dynamics of all points outside O_1 under H_1 in S_1 is exactly the same as that of G on $\mathbb{S}^2$. Hence we can repeat the above procedure, puncturing S_1 and obtaining S_2 by replacing a periodic orbit $O_2 \subset \mathcal{O}$ of length p_2 (in S_1) by a periodic sequence of circles. Proceeding inductively, we obtain a sequence of punctured spheres S_n with $(\sum_{i=1}^{n} p_i)$-holes, homeomorphisms $H_n\colon S_n \to S_n$, and factor maps $\pi_n\colon S_n \to S_{n-1}$ that collapse newly introduced circles back to points of O_n, where again $S_0 = \mathbb{S}^2$ and $H_0 = G$. In other words, π_n reverts the modification made in step n and is defined analogously to π_1. Clearly, each π_n is a continuous onto map and $\pi_n \circ H_n = H_{n-1} \circ \pi_n$. By the choice of the set $\mathcal{O}$, we have $\overline{\bigcup_{n\in\mathbb{N}} O_n} = \mathbb{S}^2$.

Embed each S_n in $\mathbb{S}^2$ in a natural way, and extend π_n to a map $\eta_n\colon \mathbb{S}^2 \to \mathbb{S}^2$ in the following way. If D is an open disc bounded by S_n in $\mathbb{S}^2$, then we have two possibilities. If $\pi_n(\partial D)$ is a single point, then we fix any $y \in \partial D \cap S_n$ and define $\eta_n(x) = \pi_n(y)$ for every $x \in D$. In the second case, $\pi_n|_{\partial D}$ is the identity, so we can extend it to identity map on D. Now we shall define a 2-sphere Q_∞, and a Sierpiński curve $S_\infty \subseteq Q_\infty$. Denote by S_∞ the inverse limit of spaces S_n with bonding maps π_n and by Q_∞ the inverse limit of spheres $\mathbb{S}^2$ with η_n as bonding maps; i.e.

$$Q_\infty = \{(z_0, z_1, \dots) : \eta_n(z_n) = z_{n-1}\},$$
$$S_\infty = \{(z_0, z_1, \dots) : \pi_n(z_n) = z_{n-1}\} \subset Q_\infty.$$

Since each η_n is a monotone map on a 2-manifold, and nondegenerate fibers of η_n are all closed disks, a result of Brown [7, Theorem 4] implies that Q_∞ is homeomorphic to $\mathbb{S}^2$. Observe that if we fix any $z \in O_n$, for some n, then the set B of all inverse sequences in Q_∞ with z on the first coordinate is homeomorphic to a disc. Simply, after dropping n first coordinates, we see that B is an inverse limit of a disk D with the identity as a unique bonding map. Therefore, S_∞ is obtained from Q_∞ by removing the interior of each element of a sequence of discs. But $\overline{\bigcup_{n\in\mathbb{N}} O_n} = \mathbb{S}^2$, hence S_∞ satisfies the conditions (S1)–(S3), and so it is a Sierpiński curve. Observe that if we put $H = H_1 \times H_2 \times \dots \times H_n \times \dots$ then $H(S_\infty) = S_\infty$, therefore H is a homeomorphism of the Sierpiński curve. Let $M = \mathbb{S}^2 \setminus \bigcup_{n=1}^{\infty} O_n$ and $M_\infty = \{(z_0, z_1, \dots) \in S_\infty : z_0 \in M\}$ be the set of all inverse sequences in S_∞ with the first coordinate in M. It follows directly from the construction that we can view $z \in M_\infty$ as $z = (x, x, x \dots)$ for some $x \in M$, and $H(z) = (G(x), G(x), \dots)$. Since periodic points of G in M are dense in $\mathbb{S}^2$, it is not hard to see that H has a dense set of periodic points. The set M_∞ is Borel, so for any Borel set $U \in S_\infty$ we can view $U \cap M_\infty$ as a Borel subset of M (by projection onto the first coordinate) and so we obtain a well-defined H-invariant Borel probability measure ν by putting $\nu(U) = \mu(U \cap M_\infty)$. The measure μ is ergodic, so we have $\mu(\mathbb{S}^2 \setminus M_\infty) = 0$, hence also $\nu(S_\infty \setminus M_\infty) = 0$, and so ν and μ are isomorphic, in particular (S_∞, H, ν) is measure-theoretically conjugate to a Bernoulli shift. Take any open set U in S_∞. We claim that $\nu(U) > 0$. Indeed, the basic open sets in S_∞ are given by $U_\infty = (\eta_1 \circ \dots \circ \eta_{i-1}(U_i), \dots, \eta_{i-1}(U_i), U_i, \eta_i^{-1}(U_i), \dots)$, where $U_i \subseteq S_i$ is open, for some $i \in \mathbb{N}$ (see, e.g., Theorem 3 on p. 79 in [10]). Since S_i is a sphere with a finite number of holes, the Lebesgue measure of U_i in S_i is positive and $\nu(U_\infty) = \nu(U_\infty \cap M_\infty) = \mu(U_i \cap M)$; therefore, U_∞ has positive product measure. This shows that ν has full support, which completes the proof of Theorem 1.2(1).

It remains to prove (3). Assume, on the contrary, that H has the specification property. Since the specification property is preserved under higher iterations, H^{p_1} has the specification property, where p_1 is the period of O_1. For simplicity of notation, replace H^{p_1} by H and A^{p_1} by A. By [15, Theorem 2.1] the specification property implies that, for every invariant measure $\mu \in M_T(S_\infty)$, there exists a sequence of ergodic measures such that $\mu_n \to \mu$, when $n \to \infty$, in the Lévy–Prokhorov metric. Since we blew up a hyperbolic periodic point c in O_1 in the first step of our construction, after passing from A to the coordinates giving its diagonalization, we have locally a phase portrait (for G and H) as in Fig. 1. Let us start with the following observation. Consider the hyperbolic linear map $f(x, y) = (ax, by)$, where $0 < a < 1 < b$ and $ab = 1$. Let $D = [-\varepsilon, \varepsilon]^2$ for some small $\varepsilon > 0$. Now let $z = (p, q) \in D$ with $|p| \geq |q|$ and assume that the trajectory of z is not fully contained in D. Then there exists a minimal $m \geq 1$ such that $a^m|p| \geq b^m|q|$ and $a^{m+1}|p| < b^{m+1}|q|$. Observe that $b^{2m}|q| < \varepsilon$ as otherwise $\varepsilon \leq b^{2m}|q| = a^{-m}b^m|q| \leq |p|$ and so $(p, q) \notin D$, which is a contradiction. Now, let v be a compactification of the line representing the stable direction for the hyperbolic point c, and take a small neighborhood U of v. Let $U' = \pi(U)$, where π is the natural factor map $\pi\colon (S_\infty, H) \to (\mathbb{S}^2, G)$. If U is sufficiently small, then $\pi(U) \subset D$ and furthermore, if $(p, q) \in \pi(U)$ then $|p| \geq |q|$, see Fig. 1.

Fix any periodic point $u \in S_\infty$, say of period s, and consider the invariant measure $\hat{\mu} = (1-\alpha)\delta_c + \frac{\alpha}{s}\sum_{i=0}^{s-1} \delta_{H^i(u)}$ with a small α, say $\alpha < \frac{1}{10}$. Assume also that $\pi(u) \notin D$. Take $0 < \gamma < \alpha/2s$ small enough, so that $\mathrm{dist}(c, \{u, H(u), \dots, H^{s-1}(u)\}) >$

3γ and denote $W = B(u, 2\gamma)$. We may also assume that U and γ are such small that $\overline{W} \cap \overline{U} = \emptyset$, $\pi(W) \cap D = \emptyset$, and $B(c, 4\gamma) \subset U$. Denote $V = B(c, 2\gamma)$.

By the results of [15] mentioned earlier, there exists an ergodic measure $\hat{\nu}$ such that $\rho(\hat{\nu}, \hat{\mu}) < \gamma$. This implies that

$$\hat{\nu}(U) \geq \hat{\nu}(V^{\gamma}) \geq \hat{\mu}(V) - \gamma \geq (1-\alpha) - \gamma > 4/5$$

and

$$\hat{\nu}(W) \geq \hat{\nu}(\{u\}^{2\gamma}) \geq \hat{\mu}(\{u\}^{\gamma}) - \gamma > 0.$$

By the Birkhoff ergodic theorem, there exists $x \in S_\infty$ such that $\lim_{n\to\infty} \frac{1}{n}|\{j < n : H^j(x) \in U\}| = \hat{\nu}(U)$ and $\lim_{n\to\infty} \frac{1}{n}|\{j < n : H^j(x) \in W\}| = \hat{\nu}(W)$. Since $\hat{\nu}(W) > 0$ there exists an increasing sequence k_i such that $H^{k_i}(x) \in W$ for every i. Let us estimate the number of iterations $k_i \leq j < k_{i+1}$ such that $H^j(x) \in U$. Observe that $\pi(H^{k_i}(x)) \notin D$ and $\pi(H^{k_{i+1}}(x)) \notin D$; therefore, by the earlier analysis, we see that no more than half of iterations $H^j(x)$ for $j = k_i + 1, \ldots, k_{i+1}$ can visit U. This implies that $\limsup_{i\to\infty} \frac{1}{k_i}|\{j < k_i : H^j(x) \in U\}| \leq 1/2$. By the choice of x, we obtain that $\hat{\nu}(U) \leq 1/2 < 4/5 < \hat{\nu}(U)$, which is a contradiction. This shows that (S_∞, H) does not have the specification property, completing the proof.

Remark 1. Similar construction works also if we start with other orientable closed surfaces, as any of them admits a branched covering onto $\mathbb{S}^2$ [2].

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