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Motion of an incompressible solid with large deformations

Solide incompressible en grande déformation

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A R T I C L E   I N F O

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A B S T R A C T

We study the motion of a visco-elastic solid with large deformations. We prove the existence of a local-in-time motion and of a non-negative pressure, which is a measure reaction to the incompressibility condition.

R É S U M É

On étudie le mouvement d'un solide viscoélastique incompressible en grande déformation. On démontre l'existence d'un mouvement local en temps et d'une pression positive qui est une mesure, réaction à la condition d'incompressibilité.

Version française abrégée

Nous étudions le mouvement d'un solide viscoélastique incompressible en grande déformation. La condition d'incompressibilité (2) est unilatérale, car en traction des vides peuvent apparaître. Cette condition introduit une pression positive (voir (5) et (17)). Par une approximation de Moreau–Yosida de la fonction indicatrice de l'ensemble des matrices d'élongation qui vérifient la condition d'incompressibilité (2), on démontre l'existence d'un mouvement approché. On montre que ce mouvement approché a une limite et que la pression approchée a aussi une limite, qui est une mesure (Theorème 5.1). Cette mesure autorise des collisions, c'est-à-dire des discontinuités de vitesse, lors de la disparition de vides.

1. Introduction

We consider the motion between time 0 and time $\tilde{t} > 0$ of a solid located in a smooth bounded domain $\mathcal{D}_a \subset \mathbb{R}^3$. The position function is $a \in \mathcal{D}_a,\ t \in (0,\tilde{t}) \to \Phi(a,t)$, with $\Phi(a,0) = a$. For the sake of simplicity, we assume that the solid is in

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contact on a smooth part Γ_0^a of its boundary with an obstacle schematized by springs applying actions proportional to the gap $(\Phi - a)$ and its gradient. Besides the body force $\overrightarrow{f}$, no other external action is applied.

We denote by $\mathcal{M}$ the space of 3×3 matrices, endowed with the usual scalar product. The subspaces $\mathcal{S} \subset \mathcal{M}$ of the symmetric matrices and $\mathcal{A} \subset \mathcal{M}$ of the antisymmetric matrices are orthogonal. We introduce the set

$$C_\alpha = \left\{ \mathbf{B} \in \mathcal{M} \;\middle|\; \text{tr}\,\mathbf{B} \geq 3\alpha,\ \text{tr}(\text{cof}\,\mathbf{B}) \geq 3\alpha^2,\ \det\mathbf{B} \geq \alpha^3 \right\},\ \ 0 < \alpha < 1, \tag{1}$$

where α is the only physical parameter, the value of which we choose different from 1. We recall that for any position Φ that is kinematically admissible, i.e. differentiable with $\det(\text{grad}\,\Phi) > 0$, there exists a unique symmetric positive definite matrix $\mathbf{W}$, the stretch matrix, and a rotation matrix $\mathbf{R}$ with $\det \mathbf{R} = 1$, such that $\text{grad}\,\Phi = \mathbf{RW}$. With this decomposition, the local impenetrability condition is to require, for the stretch matrix, that $\mathbf{W} \in C_\alpha \cap \mathcal{S}$. Note in particular that the physical constant α quantifies the resistance of the material to crushing.

This model has been introduced in [1], [3], [6], and in [2] (with more analytical details concerning the existence of solutions). We refer the reader to these papers for further details in the derivation of the model and, for some auxiliary results, we will exploit in the sequel. Actually, let us point out that the main novelty of this paper consists in the fact that incompressibility is required as an unilateral internal constraint (see Sec. 2).

2. The incompressibility condition

The usual incompressibility condition is $\det\mathbf{W} = 1$. But let us consider experiments and remark that when tension is applied to a sample, some voids may appear during the evolution, mainly at the microscopic level, with a volume increase at the macroscopic level. Moreover a phase change may occur and eventually makes possible an increase of volume. This behaviour has been described a long time ago by Jean-Jacques Moreau to investigate cavitation in fluid mechanics, [7]. The water is incompressible, but bubbles may appear inside water at the microscopic level when pressure is null: this is the cavitation phenomenon responsible for water hammers. It results that the unilateral condition

$$\det\mathbf{W} \geq 1 \tag{2}$$

is possible. On the contrary, for an incompressible material, it is impossible to have interpenetration at the microscopic level. It results $\det\mathbf{W} < 1$ is impossible. Note that the word incompressible refers to the impossibility to modify the volume by compression. We are motivated to think that condition (2) is the condition that accounts for the actual mechanical behaviour. The set

$$K = \{\mathbf{B} \in \mathcal{M} \mid \mathbf{B} \in \mathcal{S} \cap C_0,\ \det\mathbf{B} \geq 1\} \tag{3}$$

is convex (C_0 is set C_α with $\alpha = 0$, $\mathcal{S} \cap C_0$ is the set of the semi-definite matrices). We denote by I_K the indicator function of set K in $\mathcal{M}$. Set K accounts for the two internal constraints: symmetry of stretch matrix and incompressibility.

3. The constitutive laws

We derive the constitutive laws from volume free energy $\Psi(\mathbf{W}, \text{grad}\Delta\Phi, \|\text{grad}\mathbf{R}\|^2)$, surface free energy $\Psi_\Gamma(\Phi - a, \text{grad}\Phi - \mathbf{I})$ and volume pseudo-potential of dissipation $D(\dot{\mathbf{W}}, \text{grad}\,\Omega)$ with $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ and

$$\Psi(\mathbf{W}, \text{grad}\ \Delta\Phi, \|\text{grad}\,\mathbf{R}\|^2) = \frac{1}{2}\,\|\mathbf{W} - \mathbf{I}\|^2 + \frac{1}{2}\,\|\text{grad}\,\Delta\Phi\|^2 + \hat{\Psi}(\mathbf{W}) + I_K(\mathbf{W}) + \frac{1}{4}\,\|\text{grad}\,\mathbf{R}\|^2\,,$$
$$\Psi_\Gamma(\Phi - a, \text{grad}\ \Phi - \mathbf{I}) = \frac{1}{2}\int_{\Gamma_0^a} (\Phi - a)^2 \text{d}\Gamma + \frac{1}{2}\int_{\Gamma_0^a} (\text{grad}\ \Phi - \mathbf{I})^2 \text{d}\Gamma,$$

and

$$D(\dot{\mathbf{W}}, \text{grad}\,\Omega) = \frac{1}{2}\,\|\dot{\mathbf{W}}\|^2 + \frac{1}{4}\,\|\text{grad}\,\Omega\|^2\,,$$

where $\mathbf{W}$ is a matrix of $\mathcal{M}$, and $\|\mathbf{W}\|^2 = \mathbf{W} : \mathbf{W}$, $\|\text{grad}\,\Delta\Phi\|^2 = \Phi_{i,\alpha\beta\beta}\Phi_{i,\alpha\delta\delta}$. The function $I_K(\mathbf{W})$ is the indicator function of convex set K that insures the symmetry of matrix $\mathbf{W}$ and incompressibility.

The free energy accounts for the impenetrability condition. In particular, this constraint is related to the presence of function $\hat{\Psi}(\mathbf{W})$, which is a smooth approximation from the interior of the indicator function of the set C_α in $\mathcal{M}$ (see [1], [2], [3] and [6]).

The incompressibility constitutive law is

$$\mathbf{S}_{\text{reac}} + \mathbf{A}_{\text{reac}} \in \partial I_K(\mathbf{W}),$$

with symmetric and antisymmetric parts given by

$$\mathbf{S}_{\mathrm{reac}} \in \frac{\mathrm{d}\det\mathbf{W}}{\mathrm{d}\mathbf{W}}\partial I_+(\det\mathbf{W}-1) = -p(\mathrm{cof}\mathbf{W}),\ -p\in\partial I_+(\det\mathbf{W}-1),\ \mathbf{A}_{\mathrm{reac}}\in\partial I_{\mathcal{S}}(\mathbf{W}) = \mathcal{A}, \tag{4}$$

with

$$\partial I_+(\det\mathbf{W}-1) = \begin{cases}\mathbb{R}^-, & if\ \det\mathbf{W}-1=0,\\ \{0\}, & if\ \det\mathbf{W}-1>0.\end{cases}$$

Introducing reaction stress $\mathbf{A}_{\mathrm{reac}}$, which ensures that stretch matrix $\mathbf{W}$ is symmetric [1–3,6], and non-negative pressure p, which ensures incompressibility (I_+ is the indicator function of $\mathbb{R}^+$).

Position Φ, reaction stress $\mathbf{A}_{\mathrm{reac}}$, and pressure p are the unknowns of the problem.

4. The weak formulation and a first approximated system

For a final time $T>0$, the spaces of the virtual linear and angular velocities are

$$\mathcal{V}(T) = L^2(0,T;H^{3+\delta}(\mathcal{D}_a))\cap H^1(0,T;L^2(\mathcal{D}_a)),\ \delta>0,$$
$$\mathcal{V}_{\mathrm{rv}}(T) = L^2(0,T;H^1(\mathcal{D}_a)).$$

Note that we need test functions with grad $V\in C^0([0,T];H^2(\mathcal{D}_a))$. For the sake of simplicity, we are using the same notation for a Banach space X and any power of it. We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between $\mathcal{V}(T)$ and its dual, while the duality between $\mathcal{V}_{\mathrm{rv}}(T)$ and its dual space is denoted by $\int_0^T\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the duality between H^1 and the dual. We denote by ∂I_+^ε the Moreau–Yosida approximation of ∂I_+, which is in particular a Lipschitz function, and we define the approximated pressure:

$$p_\varepsilon = -\left(\frac{\partial I_+^\varepsilon}{\partial x}(\det\mathbf{W}-1)\right) = \frac{(\det\mathbf{W}-1)^-}{\varepsilon}. \tag{5}$$

Now, we fix $\varepsilon>0$ and apply Theorem 12 in [2] ensuring that the resulting approximated problem admits a local in time solution, where the final time, denoted by T with $0<T\le\tilde{t}$, does not depend on ε. Thus, the variational formulation of the approximated problem on time interval $(0,T)$ reads as follows. We look for

$$\Phi\in L^\infty(0,T;H^3(\mathcal{D}_a))\cap H^1(0,T;H^1(\mathcal{D}_a))\cap W^{1,\infty}(0,T;L^2(\mathcal{D}_a)),$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\in\mathcal{V}'(T),\ \mathbf{A}\in\mathcal{V}'_{\mathrm{rv}}(T)\text{ such that }\forall\vec{V}\in\mathcal{V}(T)\text{ and }\forall\widehat{\Omega}\in\mathcal{V}_{\mathrm{rv}}(T),$$

$$\begin{aligned}&\langle\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2},\vec{V}\rangle\rangle + \int_0^T\int_{\mathcal{D}_a} -p_\varepsilon(\det\mathbf{W})\mathbf{R}\mathbf{W}^{-1}:\mathrm{grad}\,\vec{V}\,\mathrm{d}a\,\mathrm{d}\tau\\ &+\int_0^T\int_{\mathcal{D}_a}\mathbf{R}\left\{(\mathbf{W}-\mathbf{I})+\dot{\mathbf{W}}+\frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\mathbf{W})\right\}:\mathrm{grad}\,\vec{V}\,\mathrm{d}a\,\mathrm{d}\tau\\ &+\int_0^T\langle\mathbf{A},\mathbf{R}^{\mathsf{T}}\mathrm{grad}\,\vec{V}-\mathrm{grad}\,\vec{V}^{\mathsf{T}}\mathbf{R}\rangle\mathrm{d}\tau+\int_0^T\int_{\mathcal{D}_a}\mathrm{grad}\,\Delta\Phi:\mathrm{grad}\,\Delta\vec{V}\,\mathrm{d}a\,\mathrm{d}\tau\\ &+\int_0^T\int_{\Gamma_0^a}(\Phi-a)\cdot\vec{V}\,\mathrm{d}\Gamma\mathrm{d}\tau+\int_{\Gamma_0^a}(\mathrm{grad}\,\Phi-\mathbf{I}):\mathrm{grad}\,\vec{V}\,\mathrm{d}\Gamma\mathrm{d}\tau=\int_0^T\int_{\mathcal{D}_a}\vec{f}\cdot\vec{V}\,\mathrm{d}a\,\mathrm{d}\tau,\end{aligned} \tag{6}$$

$$\Omega=\dot{\mathbf{R}}\mathbf{R}^{\mathsf{T}}\in\mathcal{V}_{\mathrm{rv}}(T),\ \mathbf{R}\mathbf{W}=\mathbf{F}=\ \mathrm{grad}\,\Phi,\ \mathbf{R}(a,0)=\mathbf{I}, \tag{7}$$

$$\begin{aligned}&\int_0^T\int_{\mathcal{D}_a}(\mathrm{grad}\,\mathbf{R})\mathbf{R}^{\mathsf{T}}:\mathrm{grad}\,\widehat{\Omega}+\mathrm{grad}\,\Omega:\mathrm{grad}\,\widehat{\Omega}\,\mathrm{d}a\,\mathrm{d}\tau\\ &=\int_0^T\int_{\mathcal{D}_a}\mathbf{R}\left\{\dot{\mathbf{W}}\mathbf{W}-\mathbf{W}\dot{\mathbf{W}}\right\}\mathbf{R}^{\mathsf{T}}:\widehat{\Omega}\,\mathrm{d}a\,\mathrm{d}\tau+\int_0^T\langle\mathbf{A},\mathbf{R}^{\mathsf{T}}\widehat{\Omega}\mathbf{R}\mathbf{W}+\mathbf{W}\mathbf{R}^{\mathsf{T}}\widehat{\Omega}\mathbf{R}\rangle\mathrm{d}\tau.\end{aligned} \tag{8}$$

The variational formulations are completed with the initial conditions

$$\Phi(a,0)=a,\ \ \dot{\Phi}(a,0)=0, \tag{9}$$

giving $I_+(\det \mathbf{W}(0)-1)=I_+^{\varepsilon}(\det \mathbf{W}(0)-1)=0$.

Note that here we have, in addition, the boundary contributions that were not present in [2]. Actually, this choice does not create further difficulties in the mathematical treatment. The reader may refer to the arguments that we will detail to perform the passage to the limit as $\varepsilon \searrow 0$.

Following [1,2], we prove the following existence (and stability) result.

Theorem 4.1. *The approximated problem has local in time solution* $(\Phi_\varepsilon, \mathbf{A}_\varepsilon, p_\varepsilon)$ *on* $(0,T)$*. The final time* T *is independent of* ε *and satisfies* $0<T\le\tilde{t}$*. Moreover, the solution satisfies*

$$\|\dot{\Phi}_\varepsilon\|_{L^2(0,T;H^1(\mathcal{D}_a))\cap L^\infty(0,T;L^2(\mathcal{D}_a))}+\|\Phi_\varepsilon\|_{L^\infty(0,T;H^3(\mathcal{D}_a))}+\int\limits_{\mathcal{D}_a} I_+^{\varepsilon}(\mathbf{W}_\varepsilon(T))\,\mathrm{d}a$$

$$+\|\dot{\mathbf{W}}_{\boldsymbol{\varepsilon}}\|_{L^2(0,T;L^2(\mathcal{D}_a))}+\|\mathbf{R}_{\boldsymbol{\varepsilon}}\|_{L^\infty(0,T;H^1(\mathcal{D}_a))}+\|\Omega_\varepsilon\|_{\mathcal{V}_{\mathrm{rv}}(T)}+\|\mathbf{A}_\varepsilon\|_{\mathcal{V}'_{\mathrm{rv}}(T)}\le c$$

where c *does not depend on* ε.

The proof of the theorem is performed by use of a Galerkin approximation combined with a priori estimates and passage to the limit procedure. In particular, it is required a Lipschitz regularity for the involved nonlinearities. The fact that the existence is proved just locally in time comes from the fact that the motion may be interrupted by crushing resulting in a discontinuity of velocity (i.e. an internal collision, [5], [6]). The existence is proved for a suitably introduced weak formulation, we will clarify in the following the statement of the existence of a solution to our problem. Let us point out that the above estimates are recovered by the Theorem of kinetic energy. From the analytical point of view, they are obtained testing the equations by the actual velocities $\mathrm{d}\Phi/\mathrm{d}t$, Ω. Hence, the reaction $\mathbf{A}_\varepsilon$ is directly estimated by a comparison in the equation due to the fact that $\mathbf{W}_\varepsilon$ belongs to C_α. Finally, the terms involving $\hat{\Psi}$ are estimated exploiting the smoothness of the function $\hat{\Psi}$ in the interior of C_α.

5. The limit process as $\varepsilon \searrow 0$

In this section, we perform the passage to the limit for solutions to the approximated problem as $\varepsilon \searrow 0$. Actually, let us point out that we can pass to the limit for a weaker version of the problem, which we are going to clarify.

5.1. The pressure a priori estimate

Now, we aim to estimate the term p_ε independently of ε. To this aim, we take as a test function $\Phi_\varepsilon \in L^\infty(0,T;H^3(\mathcal{D}_a))$ in the variational formulation of the approximated problem; using $\mathbf{A}_{\boldsymbol{\varepsilon}}:\mathbf{W}_{\boldsymbol{\varepsilon}}=\mathbf{0}$, we get

$$\int\limits_0^T\int\limits_{\mathcal{D}_a} p_\varepsilon \mathbf{R}_\varepsilon \operatorname{cof}\mathbf{W}_\varepsilon : \operatorname{grad}\Phi_\varepsilon\,\mathrm{d}a\,\mathrm{d}\tau=\int\limits_0^T\int\limits_{\mathcal{D}_a} 3p_\varepsilon\,(\det\mathbf{W}_\varepsilon)\,\mathrm{d}a\,\mathrm{d}\tau\le c. \tag{10}$$

Because pressure p_ε is non negative and $\det\mathbf{W}_\varepsilon\ge\alpha^3$, one has

$$\|p_\varepsilon\|_{L^1(Q)}\le c, \tag{11}$$

independently of ε, where $Q=(0,T)\times\mathcal{D}_a$. The estimate (11) combined with the regularity of $\mathbf{W}_\varepsilon$ and $\mathbf{R}_\varepsilon$ ensures that the operator

$$\langle\langle\,\mathcal{P}_\varepsilon, V\rangle\rangle=\int\limits_0^T\int\limits_{\mathcal{D}_a} p_\varepsilon(\det\mathbf{W}_\varepsilon)\mathbf{R}_\varepsilon\mathbf{W}_\varepsilon^{-1}:\operatorname{grad}V\,\mathrm{d}a\,\mathrm{d}\tau$$

is bounded in $\mathcal{V}(T)'$.

5.2. The limit of pressure

First, we point out that pressure p_ε converges weakly star (for topology $\sigma(M(Q),C(Q))$ to a measure $p\in M(Q)$. Following the arguments introduced in [4] considering the boundedness of the operator

$$\|\mathcal{P}_\varepsilon\|_{\mathcal{V}(\hat{t})'} \le c, \tag{12}$$

we can also deduce a convergence in the dual space. Then, passing to the limit and exploiting the weak and weak star convergence results detailed in [2], we can rewrite the variational formulation of the term

$$< p\,(\det \mathbf{W})\,\mathbf{R}\mathbf{W}^{-1} : \operatorname{grad} V >_\sigma = < p\mathbf{I} : (\det \mathbf{W})\,(\operatorname{grad} V)\mathbf{W}^{-1}\mathbf{R}^\mathsf{T} >_\sigma, \tag{13}$$

where $p\,(\det \mathbf{W})\,\mathbf{R}\mathbf{W}^{-1}$ is the product of measure p with continuous function $(\det \mathbf{W})\,\mathbf{R}\mathbf{W}^{-1}$ and $< \cdot : \cdot >_\sigma$ stands for the duality between measures and smooth functions.

Now, our aim is to show that $p \in \partial I_+$, at least when it is sufficiently regular. To this aim, we prove a lower semicontinuity property, directly passing to the limit in the equations and using the strong convergence of $\mathbf{R}$ and $\mathbf{W}$ in $C(Q)$. We choose the gradient of the virtual velocities $\operatorname{grad} V$ continuous in Q. This is the reason why we have considered a more regular set of test functions with respect to the existence result detailed in [2]. To get the properties of pressure, we have

$$\int_0^T \int_{\mathcal{D}_a} -p_\varepsilon(\det \tilde{\mathbf{W}} - \det \mathbf{W}_\varepsilon)\,\mathrm{d}a\,\mathrm{d}\tau + \int_0^T \int_{\mathcal{D}_a} I_+^\varepsilon(\det \mathbf{W}_\varepsilon - 1)\,\mathrm{d}a\,\mathrm{d}\tau \le \int_0^T \int_{\mathcal{D}_a} I_+^\varepsilon(\det \tilde{\mathbf{W}} - 1)\,\mathrm{d}a\,\mathrm{d}\tau. \tag{14}$$

Let us choose $\tilde{\mathbf{W}} \in \mathcal{S}$ satisfying $\det \tilde{\mathbf{W}} \ge 1$. Then, we have $I_+^\varepsilon(\det \tilde{\mathbf{W}} - 1) = 0$ and

$$\int_0^T \int_{\mathcal{D}_a} -p_\varepsilon \det \tilde{\mathbf{W}}\,\mathrm{d}a\,\mathrm{d}\tau - \int_0^T \int_{\mathcal{D}_a} -p_\varepsilon \det \mathbf{W}_\varepsilon\,\mathrm{d}a\,\mathrm{d}\tau + \int_0^T \int_{\mathcal{D}_a} I_+^\varepsilon(\det \mathbf{W}_{\boldsymbol{\varepsilon}} - \mathbf{1})\,\mathrm{d}a\,\mathrm{d}\tau \le 0, \tag{15}$$

and

$$\int_0^T \int_{\mathcal{D}_a} -p_\varepsilon \det \tilde{\mathbf{W}}\,\mathrm{d}a\,\mathrm{d}\tau - \int_0^T \int_{\mathcal{D}_a} -p_\varepsilon\,(\det \mathbf{W}_\varepsilon)\,\mathrm{d}a\,\mathrm{d}\tau \le 0, \tag{16}$$

giving

$$< -p : (\det \tilde{\mathbf{W}} - 1) >_\sigma - < -p : (\det \mathbf{W} - 1) >_\sigma \le 0. \tag{17}$$

Note that, in case pressure is sufficiently regular (here we have just an L^1 estimate), due to the fact that $\det \mathbf{W} \ge 1$, by the previous argument and the definition of the subdifferential of convex functions, we can deduce $-p \in \partial I_+(\det \mathbf{W} - 1)$, in term of the duality between $\mathcal{V}(\hat{t})$ and $\mathcal{V}(\hat{t})'$.

Note that, dealing with measures and the abstract operator $\mathcal{P}_\varepsilon$, we cannot in general identify the pressure a.e. as an element of the subdifferential. However, the weak formulation of the problem we are solving and (17) correspond to the weak formulation of the constraint.

5.3. The limit of the acceleration

By a comparison in the equations, we can estimate the acceleration term $\|\frac{\mathrm{d}^2\Phi_\varepsilon}{\mathrm{d}t^2}\|_{\mathcal{V}(\hat{t})'} \le c$. Then the approximated acceleration converges weakly star in this dual space. Note that velocity $\dot{\Phi}$ is expected not to be continuous with respect to the time and to present jumps. From a physical point of view, this lack of regularity, related to the presence of the internal non-smooth constraint represented by pressure, corresponds to the fact that collisions may occur when voids are closing [5], [6]. For the initial condition, we study the evolution before time $t = 0$ with all the external actions null and solution $\Phi = a$ when $t < 0$ and all the solutions are in $L^2(d, T)$, the beginning of the evolution being at time $d < 0$. One may also remark that velocity $\dot{\Phi}^- = \lim_{t \to 0^-} \dot{\Phi}(t)$, the velocity before a possible collision at initial time $t = 0$, is given.

The existence result is

Theorem 5.1. *Under the assumptions we have specified, there exists a solution to (6)–(9) in the time interval $(0, T)$ with $0 < T \le \tilde{t}$. The incompressibility condition $\det \mathbf{W} \ge 1$ is satisfied, the reaction pressure p is a positive measure satisfying (17), and the terms involving p are understood in the duality.*

References

[1] E. Bonetti, P. Colli, M. Frémond, The motion of a solid with large deformations, C. R. Acad. Sci. Paris, Ser. I 351 (2013) 579–583.
[2] E. Bonetti, P. Colli, M. Frémond, 2D motion with large deformations, Boll. UMI 7 (2014) 19–44.
[3] E. Bonetti, P. Colli, M. Frémond, The 3D motion of a solid with large deformations, C. R. Acad. Sci. Paris, Ser. I 352 (2014) 183–187.
[4] E. Bonetti, E. Rocca, R. Scala, G. Schimperna, On the strongly damped wave equation with constraint, Commun. Partial Differ. Equ. 42 (7) (2017) 1042–1064.

[5] M. Frémond, Collisions, Edizioni del Dipartimento di Ingegneria Civile dell'Università di Roma Tor Vergata, Roma, ISBN 978-88-6296-000-7, 2007.
[6] M. Frémond, Virtual Work and Shape Change in Solid Mechanics, Springer Ser. Solid Struct. Mech., vol. 7, Springer-Verlag, Berlin, Heidelberg, Germany, 2017.
[7] J.J. Moreau, Principes extrémaux pour le problème de la naissance de la cavitation, J. Méc. 5 (1966) 439–470.