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Algebraic geometry

Categorical characterization of quadrics

Caractérisation catégorique des quadriques

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ABSTRACT

We give a characterization of smooth quadrics in terms of the existence of full exceptional collections of certain type, which generalizes a result of C.Vial for projective spaces.

RÉSUMÉ

En généralisant un résultat de C. Vial pour l'espace projectif, on donne une caractérisation des quadriques lisses en termes d'existence de collections pleines exceptionnelles d'un certain type.

1. Introduction

Let X be a complex variety and let $D^b(X)$ be its bounded derived category. An object $E \in \mathcal{D}^b(X)$ is called exceptional if we have

$$\mathrm{Hom}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases}$$

An exceptional collection is a sequence $\{E_1 \cdots, E_n\}$ of exceptional objects which satisfy that $\mathrm{Hom}(E_j, E_i[l])$ vanishes for all $j > i$, $l \in \mathbb{Z}$. An exceptional collection $\{E_1 \cdots, E_n\}$ is full if $\mathcal{D}$ is generated by $\{E_i\}$.

In general, full exceptional collections that consist of coherent sheaves are rare. The existence of such collections would impose strong restrictions on the geometry of the variety. We now list some varieties with such collections.

- Projective spaces: for any $a \in \mathbb{Z}$, $\{\mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n)\}$ is a full exceptional collection for $\mathbb{P}^n$ (see [2]).
- Smooth quadrics $Q^n \subset \mathbb{P}^{n+1}$: M. Kapranov [6] shows that
  - if n is odd, for any $a \in \mathbb{Z}$, $\{S, \mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n-1)\}$ is a full exceptional collection for Q^n,

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– if n is even, for any $a \in \mathbb{Z}$, $\{S^-, S^+, \mathcal{O}(a), \mathcal{O}(a+1), \cdots, \mathcal{O}(a+n-1)\}$ is a full exceptional collection for Q^n where S, S^-, S^+ are certain spinor bundles.

A *minifold* is a smooth projective variety X whose $D^b(X)$ admits a full exceptional collection $\mathcal{C}$ of minimal possible length $\dim X + 1$. S. Galkin, L. Katzarkov, A. Mellit and E. Shinder classify *minifolds* up to dimension 4 (see [4]). If $\mathcal{C}$ consists only of line bundles, C. Vial proves that X is necessarily a projective space (see [12, Theorem 1.2]). In [8], S. Kobayashi and T. Ochiai show that if there exists an ample line bundle H on a smooth projective variety X with $c_1(X) \geq \dim(X) \cdot c_1(H)$, then X is isomorphic to a projective space or a smooth quadric. The purpose of this article is to generalize C.Vial's theorem and prove a categorical analog of S. Kobayashi and T. Ochiai's classification.

There is quite a long history and many excellent results about describing projective spaces and smooth quadrics since Mori's famous work [10], see [1] and the references there. In view of Kapranov's result, it is reasonable to consider full exceptional collections of length $\dim X + 1$ or $\dim X + 2$.

Theorem 1.1. *Let X be a smooth projective variety of dimension n ($n \geq 3$). Suppose that there exists a full exceptional collection $\mathcal{C}$ of $D^b(X)$ which consists of coherent sheaves and is of length $n+1$. If $\mathcal{C}$ contains a sub-collection $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_n\}$ where $\mathcal{L}_i$ $(1 \leq i \leq n)$ are line bundles, then X is isomorphic to $\mathbb{P}^n$ or Q^n. Moreover, if n is an even number, X is isomorphic to $\mathbb{P}^n$.*

Theorem 1.2. *Let X be a smooth projective variety of dimension n ($n \geq 3$). Suppose that there exists a full exceptional collection $\mathcal{C}$ of $D^b(X)$ that consists of coherent sheaves and is of length $n+2$. If $\mathcal{C}$ contains a sub-collection $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_n\}$ where $\mathcal{L}_i$ $(1 \leq i \leq n)$ are line bundles, then X is even-dimensional and is isomorphic to Q^n.*

To prove Theorem 1.1 and Theorem 1.2, we use the method from C. Vial's article [12] and some technical input (see Lemma 2.1).

Convention: In this article, a variety is an integral scheme of finite type defined over $\mathbb{C}$.

2. A technical lemma

Firstly, we note an easy fact about series of numbers. We shall use this fact and the Riemann–Roch theorem to prove Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *Let $\{a_1, \cdots, a_n\}$ be a set of n distinct integers. Assume that the cardinality of the set $A = \{a_j - a_i | 1 \leq i < j \leq n\}$ is n. For the case $n \geq 5$, there exists an integer d such that the ordered series $(a_1, \cdots, a_n)$ is one of the followings:*

(1) *$(a_1, \cdots, a_1 + (k-1)d, a_1 + (k+1)d, \cdots, a_1 + nd)$ where $1 \leq k \leq n-1$ is an integer;*
(2) *$(a_1, a_1 + 2d, \cdots, a_1 + (n-1)d, a_1 + (n+1)d)$;*
(3) *$\sigma(a_1, a_1 + d, \cdots, a_1 + (n-1)d)$ where σ is a composition of disjoint permutations $(i_1, i_1+1) \cdots (i_l, i_l+1)$ for some $1 \leq l \leq n$.*

Proof. We firstly assume $a_1 < a_2 < \cdots < a_n$ and claim that $(a_1, \cdots, a_n)$ is of type (1) or (2).

We shall prove our claim by induction on the length n of the ordered series $(a_1, \cdots, a_n)$. Firstly, it is easy to verify our claim when n is 5. Then if $n > 5$, we consider the series $(a_1, \cdots, a_{n-1})$. Note that the cardinality $\#B$ of the set $B := \{a_j - a_i | 1 \leq i < j \leq n-1\}$ is $n-1$ or $n-2$, as $a_n - a_1 \in A$ is not an element of B.

If $\#B$ is $n-2$, then $(a_1, \cdots, a_{n-1})$ is arithmetic, and we denote $a_2 - a_1$ by an integer d. Then B is $\{d, 2d, \cdots, (n-2)d\}$. Since $\#A$ is n, it is easy to verify that a_n is $a_1 + nd$ and hence, $(a_1, \cdots, a_n)$ is of type (1).

If $\#B$ is $n-1$, by induction, $(a_1, \cdots, a_{n-1})$ is of type (1) or (2).

If $(a_1, \cdots, a_{n-1})$ is of type (2), i.e. there exists an integer d such that $(a_1, \cdots, a_{n-1})$ is $(a_1, a_1 + 2d, \cdots, a_1 + (n-2)d, a_1 + nd)$, then the set B is $\{d, 2d, \cdots, (n-2)d, nd\}$. We can assume $a_n = a_1 + md$ for some integer $m > n$. If m is $n+1$ or $n+2$, then $(n-1)d$ belongs to A. If $m > n+2$, then $(m-2)d$ belongs to A. Note that $md = a_n - a_1 \in A$ is not an element of B. It follows that in both situations, we have $\#A \geq \#B + 2 = n+1$. So we can exclude the possibility that $(a_1, \cdots, a_{n-1})$ is of type (2).

If $(a_1, \cdots, a_{n-1})$ is of type (1), i.e. there exists integers d and $1 \leq k \leq n-2$ such that $(a_1, \cdots, a_{n-1})$ is $(a_1, \cdots, a_1 + (k-1)d, a_1 + (k+1)d, \cdots, a_1 + (n-1)d)$. Then the set B is $\{d, 2d, \cdots, (n-1)d\}$. Since $\#A$ is n, if k is 1, a_n equals $a_1 + nd$ or $a_1 + (n+1)d$. If $k \geq 2$, then a_n equals $a_1 + nd$. Hence $(a_1, \cdots, a_n)$ is of type (1) or (2) and we prove our claim.

For an arbitrary series $(a_1, a_2, \cdots, a_n)$, there is a permutation α satisfying $a_{\alpha(1)} < a_{\alpha(2)} < \cdots < a_{\alpha(n)}$. Now consider the set $C := \{a_{\alpha(j)} - a_{\alpha(i)} | 1 \leq i < j \leq n\}$. It is easy to see the equality:

$$C = \{|a_j - a_i| \mid 1 \leq i < j \leq n\}. \tag{2.1}$$

So we have $\#C \leq \#A$, which means that $\#C$ is $n-1$ or n.

If $\#C$ is $n-1$, then $(a_{\alpha(1)}, \cdots, a_{\alpha(n)})$ is arithmetic and we denote $a_{\alpha(2)} - a_{\alpha(1)}$ by an integer d. Then C is $\{d, 2d, \cdots, (n-1)d\}$. Note that for any integer $k \neq 1$, if kd belongs to A, then $\mathrm{sgn}(k) \cdot d$ belongs to A. It follows that for any $k > 1$, $kd \in A$ implies $-kd \notin A$, otherwise, by equality (2.1), we have $\#A \geq \#C + 2 = n+1$.

Now we may assume $2d \in B$, then $(a_{\alpha(1)}, a_{\alpha(1)}+2d, \cdots, a_{\alpha(1)}+2(l-1)d, \cdots)$, as well as $(a_{\alpha(1)}+d, a_{\alpha(1)}+3d, \cdots, a_{\alpha(1)}+(2l-1)d, \cdots)$, is an ordered subseries of $(a_1, \cdots, a_n)$. For any positive integer l, let a_{i_l} be $a_{\alpha(1)}+2(l-1)d$ and let a_{j_l} be $a_{\alpha(1)}+(2l-1)d$. If $j_1 > i_1$, then for any l, $a_{j_l} - a_{i_1} = (2l-1)d$ is an element of A. So the set $\{kd \mid 1 \le k \le n-1\}$ is contained in A. Note that $\#A$ is n, by equality (2.1), we have $A = \{-d, d, 2d, \cdots, (n-1)d\}$. If $j_1 < i_1$, then we have $-d \in A$. Since $n \ge 5$, $a_{i_3} - a_{j_1} = 3d$ is an element of A. So there is $j_2 > i_1$, otherwise, $-3d$ belongs to A. It follows that A is $\{-d, d, 2d, \cdots, (n-1)d\}$. So if $\#C$ is $n-1$, the series $(a_1, \cdots, a_n)$ is of type (3).

Now we consider the remaining case that $\#C$ is n. By our claim at the beginning, there exists an integer d such that $(a_{\alpha(1)}, \cdots, a_{\alpha(n)})$ is of type (1) or (2). For any integer $l > 0$, by equality (2.1), the numbers ld and $-ld$ cannot belong to A at the same time. So we may assume $d \in A$ and $-d \notin A$.

Assume that $(a_{\alpha(1)}, \cdots, a_{\alpha(n)})$ is of type (2). Note that $(a_{\alpha(2)}, \cdots, a_{\alpha(n-1)})$ is an ordered subseries of $(a_1, a_2, \cdots, a_n)$. Since $n \ge 5$, $2d$ belongs to A. So we have $\alpha(1) < \alpha(2)$ and $\alpha(n-1) < \alpha(n)$, which means $(a_1, \cdots, a_n) = (a_{\alpha(1)}, \cdots, a_{\alpha(n)})$. If $(a_{\alpha(1)}, \cdots, a_{\alpha(n)})$ is of type (1), by a very similar argument, one can also verify $(a_1, \cdots, a_n) = (a_{\alpha(1)}, \cdots, a_{\alpha(n)})$. □

In order to prove Theorem 1.1 and Theorem 1.2, we need consider the case $n = 4$. By a case-by-case analysis, the situation of $n = 4$ is slightly different from Lemma 2.1.

Remark 2.2. We keep the assumptions and notations of Lemma 2.1.

For the case $n = 4$, (a_1, a_2, a_3, a_4) is one of the followings:

(1) $(a, a+d, a+2d, a+4d)$ or $(a, a+2d, a+3d, a+4d)$ where d is an integer;
(2) $\sigma(a, a+d, a+2d, a+3d)$ where d is a positive integer and $\sigma \ne id$ is a permutation with $A = \{\pm d, \delta_1 \cdot 2d, \delta_2 \cdot 3d\}$ such that δ_i belongs to $\{\pm 1\}$;
(3) $\sigma(a, a+d, b, b+d)$ where d is a positive integer with $b > a+d$ and σ is a permutation.

3. Proof of Theorem 1.1 and Theorem 1.2

Let X be a smooth projective variety. Recall that, for any two objects E and F in $D^b(X)$, the Euler pairing χ is the integer $\chi(E, F) := \sum_l (-1)^l \dim_{\mathbb{C}} \mathrm{Hom}(E, F[l])$.

Definition 3.1. An object E is said to be numerically exceptional if $\chi(E, E) = 1$. A collection $\{E_1, \cdots, E_r\}$ of numerical exceptional objects is a numerical exceptional collection if $\chi(E_j, E_i)$ vanishes for any $j > i$.

For Fano varieties of Picard number 1, we have the following result.

Theorem 3.2. *Let X be a smooth projective Fano variety of dimension n $(n \ge 3)$ whose* Pic(X) *is isomorphic to $\mathbb{Z}$. Assume that there is a numerical exceptional collection $\{\mathcal{L}_1, \cdots \mathcal{L}_n\}$ where $\mathcal{L}_i$ $(1 \le i \le n)$ are line bundles. Then X is isomorphic to $\mathbb{P}^n$ or a smooth quadric Q^n.*

Proof. Let H be an ample generator of Pic(X) and let λ be the index of X. The Euler characteristic $\chi(\mathcal{O}_X(aH))$ is a polynomial P with rational coefficients of degree n in the variable a. We write $\mathcal{L}_i$ as $\mathcal{L}_i = O_X(a_i H)$ for some integer a_i and we have $a_i \ne a_j$ for any $i \ne j$, as $\chi(\mathcal{O}_X)$ is 1. The equalities $\chi(\mathcal{L}_j, \mathcal{L}_i) = \chi(\mathcal{L}_i \otimes \mathcal{L}_j^{-1}) = P(a_i - a_j) = 0$ $(i < j)$ imply that $a_i - a_j$ $(i < j)$ are roots of $P(a) = 0$. So the cardinality μ of the set $\{a_j - a_i | 1 \le i < j \le n\}$ is $n-1$ or n.

If μ is $n-1$, the series $(a_1, a_2, \cdots, a_n)$ is arithmetic and we denote $a_2 - a_1$ by an integer d. The polynomial P vanishes at $-ld$ for any $1 \le l \le n-1$. Let the remaining root of $P(a) = 0$ be $-N$. By the Riemann–Roch theorem, we have

$$\begin{aligned} P(a) = \chi(\mathcal{O}_X(aH)) &= \frac{\deg(H^n)}{n!} a^n + \frac{\deg(H^{n-1} \cdot c_1(X))}{2(n-1)!} a^{n-1} + \cdots + \chi(\mathcal{O}_X) \\ &= \frac{\deg(H^n)}{n!} (a+d) \cdots (a+(n-1)d)(a+N). \end{aligned}$$

By comparing coefficients of the identities, we obtain two equalities:

$$\frac{\deg(H^n)}{n!} (n-1)! d^{n-1} N = 1 \qquad (1)$$

$$\frac{\deg(H^n)}{n!} \Big(N + \sum_{l=1}^{n-1} ld\Big) = \frac{\deg(H^{n-1} \cdot c_1(X))}{2(n-1)!}. \qquad (2)$$

One can deduce $d = 1$, $\deg H^n = 1$, $\lambda = n+1$ or $d = 1$, $\deg H^n = 2$, $\lambda = n$. Then by [8, Corollary to Theorem 2.1], the Fano variety X is isomorphic to $\mathbb{P}^n$ or Q^n.

If μ is n, we aim to show that X is isomorphic to $\mathbb{P}^n$. We firstly assume $n \geq 5$, then the ordered series $(a_1, a_2, \cdots, a_n)$ is classified in Lemma 2.1. If $(a_1, a_2, \cdots, a_n)$ is of type (1) as in Lemma 2.1, then there exists an integer d such that the polynomial $P(a)$ vanishes exactly at $-ld$ $(1 \leq l \leq n)$. By a similar argument, we have $\deg H^n = 1$, $d = 1$ and $\lambda = n+1$. If $(a_1, a_2, \cdots, a_n)$ is of type (2) as in Lemma 2.1, the roots of $P(a) = 0$ are $\{-d, -2d, \cdots, -(n-1)d, -(n+1)d\}$ for some integer d. Then we have $\frac{\deg(H^n)}{n!}(n-1)!d^n(n+1) = 1$, which is absurd as d is an integer. If $(a_1, a_2, \cdots, a_n)$ is of type (3) as in Lemma 2.1, the roots of $P(a) = 0$ are $\{\pm d, -2d, \cdots, -(n-1)d\}$ for some integer d. Then it is easy to deduce $\deg(H^n)d^n = -n$ and $\lambda = \frac{(n+1)(n-2)}{n} d \in \mathbb{Z}^+$, which is impossible.

Now suppose $\mu = 3$ and $n = 3$. Then (a_1, a_2, a_3) is not arithmetic. By a simple calculation, we have $\lambda = \frac{4}{3}(a_3 - a_1) \geq 4$.

Now suppose $\mu = 4$ and $n = 4$. If (a_1, a_2, a_3, a_4) is of type (1) as in Remark 2.2, by a similar argument, we can deduce that λ is 5. If (a_1, a_2, a_3, a_4) is of type (2) as in Remark 2.2, then there exists an integer d such that the roots of $P(a) = 0$ are $\{d, -d, \delta_1 \cdot 2d, \delta_2 \cdot 3d\}$, where δ_i belongs to $\{\pm 1\}$. So we have $\frac{\deg(H^4)}{4} d^4 \cdot \delta_1 \cdot \delta_2 = -1$. Then $|d|$ is 1 and $\lambda = \frac{1}{2}(\delta_1 \cdot 2d + \delta_2 \cdot 3d)$ is not an integer, which is impossible. If (a_1, a_2, a_3, a_4) is of type (3) as in Remark 2.2, by a case-by-case analysis, the only possibility satisfying $\frac{\deg(H^4)}{4!} \prod_{i<j}(a_j - a_i) = 1$ and $\lambda > 0$ is $(a_1, a_2, a_3, a_4) = (a_1, a_1+1, a_1+3, a_1+4)$, hence λ is 5.

By all the above argument, if μ is n, then we have $\lambda \geq n+1$, hence X is isomorphic to $\mathbb{P}^n$. $\square$

We now prove a lemma about the Picard group of a variety. For similar results, we refer to [3, Theorem 3.4] and [12, Lemma 2.6].

Lemma 3.3. *Let X be a smooth projective variety of dimension n. Suppose that there exists a full exceptional collection $\mathcal{C}$ of $D^b(X)$. If the length of $\mathcal{C}$ is $n+1$, then* Pic(X) *is isomorphic to $\mathbb{Z}$. If the length of $\mathcal{C}$ is $n+2$ $(n \geq 3)$, then n is an even number and* Pic(X) *is isomorphic to $\mathbb{Z}$.*

Proof. For any smooth projective variety X, the existence of a full exceptional collection of $D^b(X)$ whose length is m implies that the Grothendieck group $K_0(X)$ of $D^b(X)$ is isomorphic to $\mathbb{Z}^m$. Moreover, as a consequence of the Grothendieck–Riemann–Roch theorem, we have the identity:

$$m = \operatorname{rk} K_0(X) = \sum_{i=0}^{n} \operatorname{rk} \mathrm{CH}^i_{\mathbb{Q}}(X)$$

where every $\operatorname{rk} \mathrm{CH}^i_{\mathbb{Q}}(X)$ is non-zero. If m equals $n+1$, then Pic(X) is of rank 1. If m equals $n+2$, by [7] or [11, Proposition 3.10], the cycle class maps $cl_i : \mathrm{CH}^i_{\mathbb{Q}}(X) \to H^{2i}(X, \mathbb{Q})$ $(0 \leq i \leq n)$ are all isomorphisms. So for any $0 \leq i \leq n$, there exists a perfect pairing $\mathrm{CH}^i_{\mathbb{Q}}(X) \times \mathrm{CH}^{n-i}_{\mathbb{Q}}(X) \to \mathbb{Q}$ induced by the Poincaré duality. It follows $\operatorname{rk} \mathrm{CH}^i_{\mathbb{Q}}(X) = \operatorname{rk} \mathrm{CH}^{n-i}_{\mathbb{Q}}(X) = 1$ for any $i \neq n-i$. So n is an even number and X is of Picard number 1. By [5, Lemma 2.2] or [12, Lemma 2.6], the Chow group $\mathrm{CH}^1(X)$ is torsion free, so Pic(X) is isomorphic to $\mathbb{Z}$. $\square$

Proof of Theorem 1.1. By [3, Proposition 3.2], the variety X is Fano. Then Theorem 1.1 is a direct corollary of Theorem 3.2 and Lemma 3.3. $\square$

In order to prove Theorem 1.2, we need the concept of the anticanonical pseudoheight, which has been introduced by A. Kuznetsov. For an exceptional collection $\mathcal{C} = \{E_1, \cdots, E_n\}$, the anticanonical pseudoheight $ph_{\mathrm{ac}}(\mathcal{C})$ of $\mathcal{C}$ is defined as

$$ph_{\mathrm{ac}}(\mathcal{C}) := \min_{1 \leq a_0 < \cdots < a_p \leq n} (e(E_{a_0}, E_{a_1}) + \cdots + e(E_{a_{p-1}}, E_{a_p}) + e(E_{a_p}, E_{a_0} \otimes K_X^{-1}) - p)$$

where for any F, $F' \in D^b(X)$, the relative height $e(F, F')$ is defined as $\min\{k | Ext^k(F, F') \neq 0\}$. A. Kuznetsov shows in [9, Corollary 6.2] that if there is $ph_{\mathrm{ac}}(\mathcal{C}) > -\dim X$, then $\mathcal{C}$ is not full.

Proof of Theorem 1.2. By Lemma 3.3, we have Pic$(X) \simeq \mathbb{Z}H$ for some ample line bundle H and $\operatorname{rk} K_0(X)$ is $n+2$, which implies that X is even-dimensional. We keep using the notations in the proof of Theorem 3.2 and aim to show that X is Fano. If $n > 5$, by a very similar argument as in the proof of Theorem 3.2, we can list all the possibilities in the following diagram:

$(a_1, \cdots, a_n)$	Roots of $P(a)$	λ	X
(1) $(a_1, a_1+1, \cdots, a_1+(n-1))$	$\{-k \mid 1 \leq k \leq n-1\}$	n	Q^n
(2)$(a_1, a_1+1, \cdots, a_1+(n-1))$	$\{-k \mid 1 \leq k \leq n\}$	$n+1$	$\mathbb{P}^n$
(3) $(a_1, a_1-1, \cdots, a_1-(n-1))$	$\{k \mid 1 \leq k \leq n-1\}$	$-n$	general type
(4) $(a_1, a_1-1, \cdots, a_1-(n-1))$	$\{k \mid 1 \leq k \leq n\}$	$-n-1$	general type
(5) $(a_1, \cdots, a_1-k+1, a_1-k-1 \cdots, a_1-n)$	$\{k \mid 1 \leq k \leq n\}$	$-n-1$	general type

By the Kodaira vanishing theorem, for line bundles $\mathcal{L}$, $\mathcal{L}'$ with $\mathcal{L} \otimes \mathcal{L}'^{-1}$ ample, the relative height $e(\mathcal{L}, \mathcal{L}')$ is bigger than n. Note that for all coherent sheaves F, F', $e(F, F')$ is non-negative. It easily follows that in cases (3) (4) (5) of our diagram, there is $ph_{\mathrm{ac}}(\mathcal{C}) > -n$, which contradicts our assumption.

If n is 4, let μ be the cardinality of the set $\{a_j - a_i | 1 \leq i < j \leq 4\}$. For the cases $\mu = 3$ or $\mu = 4$ and (a_1, a_2, a_3, a_4) of type (1) as in Remark 2.2, by a similar argument as in the proof of Theorem 3.2, one can easily deduce that X is Fano or is of general type. But if X is of general type, i.e. K_X is ample, there is $ph_{\mathrm{ac}}(\mathcal{C}) > -4$. So X is Fano.

By a similar argument as in the proof of Theorem 3.2, (a_1, a_2, a_3, a_4) cannot be of type (2); otherwise, the index λ will not be an integer. Now suppose (a_1, a_2, a_3, a_4) is of type (3). Note that if $\chi(\mathcal{L}_j, \mathcal{L}_i)$ vanishes, by the Serre duality, $\chi(\mathcal{L}_i, \mathcal{L}_j \otimes K_X)$ also vanishes. It follows that for any $i < j$, $a_j - a_i - \lambda$ is also a root of $P(a) = 0$. For example, the series $(a_1, a_2, a_3, a_4) = (a + 3, a, a + 4, a + 1)$ does not satisfy this property, although here the cardinality $\#A$ is 4. Then, by a case by case analysis, the only possibility for (a_1, a_2, a_3, a_4) is $(a, a + 1, a + 3, a + 4)$ with $a \in \mathbb{Z}$, which also implies that X is Fano. Note that $K_0(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}^{n+1}$, by Theorem 3.2, X is isomorphic to Q^n. □

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