C. R. Acad. Sci. Paris, Ser. I 356 (2018) 433–438

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Probability theory/Calculus of variations

Expected Shortfall and optimal hedging payoff

Expected Shortfall *et* payoff *optimal de couverture*

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A R T I C L E   I N F O

Article history:
Received 9 January 2018
Accepted after revision 19 March 2018
Available online 23 March 2018

Presented by Olivier Pironneau

A B S T R A C T

By using variational techniques, we provide an optimal payoff written on a given random variable for hedging – in the sense of minimizing the Expected Shortfall at a given threshold – a payoff written on another random variable. In numerous financially relevant examples, our result leads to optimal payoffs in closed form. From a theoretical viewpoint, our result is also useful for providing bounds to the classical Expected Shortfall minimization problem with given financial instruments.

R É S U M É

En utilisant des techniques de calcul variationnel, nous obtenons un *payoff*, fonction d'une variable aléatoire fixée, permettant de couvrir optimalement – au sens de la minimisation de l'*Expected Shortfall* à un seuil donné – un *payoff* fonction d'une autre variable aléatoire. Dans de nombreux cas pertinents en finance, le résultat obtenu aboutit à des *payoffs* optimaux en formule fermée. Du point de vue théorique, le résultat obtenu fournit aussi des bornes pour le problème classique de la minimisation de l'*Expected Shortfall* avec des instruments financiers donnés.

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Version française abrégée

Considérons un espace mesurable $(\Omega, \mathcal{F})$ muni d'une mesure de probabilité $\mathbb{P}$. Soient X et Y deux variables aléatoires définies sur Ω et à valeurs dans $\mathbb{R}^d$ et $\mathbb{R}^k$ respectivement. On suppose que X a une densité p_X sous $\mathbb{P}$ et l'on note $\mathcal{S}_X$ son support. On introduit une mesure de probabilités $\mathbb{Q}$ absolument continue par rapport à $\mathbb{P}$ et l'on note q_X la densité de X par rapport à $\mathbb{Q}$.

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https://doi.org/10.1016/j.crma.2018.03.010

Pour une fonction borélienne $h : \mathbb{R}^k \to \mathbb{R}$ telle que $h(Y) \in L^1(\Omega, \mathbb{P})$ et un seuil d'*Expected Shortfall* $\alpha \in (0, 1)$ (voir Eq. (4) pour la définition de l'*Expected Shortfall*), le problème traité dans cet article est celui de la minimisation – en réalité trouver un minimiseur – de

$$g \in \mathfrak{P} \mapsto \mathrm{ES}^{\mathbb{P}}_{\alpha}(h(Y) - g(X)), \tag{1}$$

où

$$\mathfrak{P} = \{g : \mathbb{R}^d \to \mathbb{R} \text{ fonction borélienne} | g(X) \in L^1(\Omega, \mathbb{P}) \cap L^1(\Omega, \mathbb{Q}), \mathbb{E}^{\mathbb{Q}}[g(X)] = 0\}.$$

Ce problème est, en mathématiques financières, un problème de design optimal de *payoff* écrit sur X pour couvrir, au sens de la minimisation de l'*Expected Shortfall*, un *payoff* écrit sur Y. Il est intimement lié au problème plus classique de la couverture en *Expected Shortfall* (connu sous le nom de ES/CVaR-hedging), dans lequel les instruments de couverture sont, en revanche, fixés (voir [1] et [2] pour un résultat général d'existence et des méthodes numériques).

Le résultat général que nous obtenons stipule que le *payoff* optimal est relié à la *Value at Risk* sous la probabilité conditionnelle $\mathbb{P}(\cdot|X)$. Il est donné par le théorème suivant.

Théorème 0.1. *Supposons que*

- $h(Y)$ *n'a pas d'atome sous* $\mathbb{P}(\cdot|X = x)$ *pour tout* $x \in \mathcal{S}_X$,
- $\forall x \in \mathcal{S}_X, \alpha(x) := 1 - (1 - \alpha)\frac{q_X(x)}{p_X(x)} > 0$, *i.e.* $\frac{q_X(x)}{p_X(x)} < \frac{1}{1-\alpha}$,
- $VaR^{\mathbb{P}(\cdot|X)}_{\alpha(X)}(h(Y)) \in L^1(\Omega, \mathbb{P})$.

Soit $g \in \mathfrak{P}$ *définie par*

$$g(x) = \begin{cases} VaR^{\mathbb{P}(\cdot|X=x)}_{\alpha(x)}(h(Y)) - t^*, & si\ x \in \mathcal{S}_X \\ 0, & si\ x \notin \mathcal{S}_X, \end{cases}$$

où

$$t^* = \mathbb{E}^{\mathbb{Q}}\left[VaR^{\mathbb{P}(\cdot|X)}_{\alpha(X)}(h(Y))\right]. \tag{2}$$

On a $\forall \tilde{g} \in \mathfrak{P}$,

$$ES^{\mathbb{P}}_{\alpha}(h(Y) - g(X)) \leq ES^{\mathbb{P}}_{\alpha}(h(Y) - \tilde{g}(X)).$$

Dans le cas particulier où Y est à valeurs réelles et h croissant (le cas décroissant se traite aussi aisément), on a un résultat particulier donné par le corollaire suivant.

Corollaire 0.2. *Supposons que*

- Y *est à valeurs réelles (i.e.* $k = 1$*) et n'a pas d'atome sous* $\mathbb{P}(\cdot|X = x)$ *pour tout* $x \in \mathcal{S}_X$,
- $\forall x \in \mathcal{S}_X, \alpha(x) := 1 - (1 - \alpha)\frac{q_X(x)}{p_X(x)} > 0$, *i.e.* $\frac{q_X(x)}{p_X(x)} < \frac{1}{1-\alpha}$,
- h *est croissant (pas nécessairement strictement) et* $h\left(VaR^{\mathbb{P}(\cdot|X)}_{\alpha(X)}(Y)\right) \in L^1(\Omega, \mathbb{P})$.

Soit $g \in \mathfrak{P}$ *définie par*

$$g(x) = \begin{cases} h\left(VaR^{\mathbb{P}(\cdot|X=x)}_{\alpha(x)}(Y)\right) - \mathbb{E}^{\mathbb{Q}}\left[h\left(VaR^{\mathbb{P}(\cdot|X)}_{\alpha(X)}(Y)\right)\right], & si\ x \in \mathcal{S}_X \\ 0, & si\ x \notin \mathcal{S}_X. \end{cases}$$

On a $\forall \tilde{g} \in \mathfrak{P}$,

$$ES^{\mathbb{P}}_{\alpha}(h(Y) - g(X)) \leq ES^{\mathbb{P}}_{\alpha}(h(Y) - \tilde{g}(X)).$$

Ce résultat permet d'obtenir des formules explicites pour le *payoff* optimal de couverture dans des cas très généraux, par exemple des distributions marginales presque quelconques pour X et Y et une structure de dépendance de type copule Student (voir Eq. (9)) – ce cas contient en particulier le cas des vecteurs Student (voir Eq. (10)) et des vecteurs gaussiens (voir Eq. (11)).

1. Introduction

1.1. General notations and definitions

Let us consider a measurable space $(\Omega, \mathcal{F})$ equipped with a reference probability measure $\mathbb{P}$. All random variables are assumed to be defined on $(\Omega, \mathcal{F})$. For $d \in \mathbb{N}^*$, we denote by $L^0_d(\Omega) := L^0(\Omega, \mathcal{F}; \mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued random variables. For any probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$, we also define $L^1(\Omega, \widehat{\mathbb{P}}) = \left\{ Z \in L^0_1(\Omega), \mathbb{E}^{\widehat{\mathbb{P}}}[|Z|] < +\infty \right\}$.

We recall below the concepts of Value at Risk (VaR) and Expected Shortfall (ES). For that purpose, we consider a probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ and a threshold $\alpha \in (0, 1)$.

– For any $Z \in L^0_1(\Omega)$, we define the α-$\widehat{\mathbb{P}}$-quantile, or $\widehat{\mathbb{P}}$-VaR at threshold α, of Z, by[1]

$$\mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z) = \inf\left\{ t \in \mathbb{R} \,\middle|\, \widehat{\mathbb{P}}(Z \le t) \ge \alpha \right\}. \tag{3}$$

Remark 1. The infimum in the above definition is attained because $t \mapsto \widehat{\mathbb{P}}(Z \le t)$ is nondecreasing and right-continuous.

– For any $Z \in L^1(\Omega, \widehat{\mathbb{P}})$, we define the $\widehat{\mathbb{P}}$-Expected Shortfall at threshold α of Z by

$$\mathrm{ES}^{\widehat{\mathbb{P}}}_\alpha(Z) = \mathbb{E}^{\widehat{\mathbb{P}}}[Z_\alpha], \tag{4}$$

where Z_α is a random variable with cumulative distribution function

$$t \mapsto \begin{cases} 0, & \text{if } t < \mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z), \\ \frac{\widehat{\mathbb{P}}(Z \le t) - \alpha}{1-\alpha}, & \text{if } t \ge \mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z). \end{cases}$$

Remark 2. If $\widehat{\mathbb{P}}(Z = \mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z)) = 0$ (no atom at $\mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z)$ under $\widehat{\mathbb{P}}$), then (see [5] for a proof)

$$\mathrm{ES}^{\widehat{\mathbb{P}}}_\alpha(Z) = \mathbb{E}^{\widehat{\mathbb{P}}}\left[Z | Z \ge \mathrm{VaR}^{\widehat{\mathbb{P}}}_\alpha(Z) \right].$$

1.2. The problem

We introduce two random variables $X \in L^0_d(\Omega)$ and $Y \in L^0_k(\Omega)$ $(d, k \in \mathbb{N}^*)$. We assume that X has a density p_X under the probability measure $\mathbb{P}$, and we denote by $\mathcal{S}_X$ its support. We also introduce a probability measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ and we denote by q_X the density of X under $\mathbb{Q}$.

We introduce a Borelian function $h : \mathbb{R}^k \to \mathbb{R}$ such that $h(Y) \in L^1(\Omega, \mathbb{P})$. The problem addressed in this paper is the minimization (in fact finding a minimizer if there is one), for a given $\alpha \in (0, 1)$, of

$$g \in \mathfrak{P} \mapsto \mathrm{ES}^{\mathbb{P}}_\alpha(h(Y) - g(X)), \tag{5}$$

where

$$\mathfrak{P} = \{ g : \mathbb{R}^d \to \mathbb{R} \text{ Borelian function} | g(X) \in L^1(\Omega, \mathbb{P}) \cap L^1(\Omega, \mathbb{Q}), \mathbb{E}^{\mathbb{Q}}[g(X)] = 0 \}.$$

In what follows, we shall assume that either Y or $h(Y)$ satisfies the (H) hypothesis, where a random variable satisfies the (H) hypothesis if it has no atom under the probability measures $\mathbb{P}(\cdot | X = x)$ for all $x \in \mathcal{S}_X$.

In mathematical finance, this problem corresponds to finding the optimal payoff $g(X)$ (written on the asset(s) modeled by X), for statically hedging, in the sense of minimizing the Expected Shortfall at threshold α, a short position in the payoff $h(Y)$ (written on the asset(s) modeled by Y), where $\mathbb{Q}$ is the pricing probability measure used by the market. In particular, the (H) hypothesis states that the assets modeled by X and Y are (really) different: X typically corresponds to tradable assets, while Y corresponds to non-tradable assets.

This problem is related to the classical problem of ES-hedging (better known as CVaR-hedging) in which, unlike in our case, the hedging instruments are given. A complete proof of the existence of an optimal ES/CVaR-hedging strategy, in both static and dynamic settings, is given in [2] with applications to the hedging of energy derivatives. Interesting numerical methods involving stochastic algorithms and quantization have also been proposed for solving this problem (see [1] and [2]).[2]

Remark 3. We restrict the minimization problem to payoffs with price 0, as we can always subtract from a payoff its expected value under $\mathbb{Q}$ without changing the Expected Shortfall of the resulting portfolio.

[1] In our definitions of Value at Risk and Expected Shortfall, Z stands for a loss.

[2] In [3], Frikha partially extends some of these results to more general risk measures.

2. Optimal payoff

2.1. General result

We start with a general theorem:

Theorem 2.1. *Assume that*

- $h(Y)$ *satisfies the (H) hypothesis,*
- $\forall x \in \mathcal{S}_X, \alpha(x) := 1-(1-\alpha)\frac{q_X(x)}{p_X(x)} > 0$, *i.e.* $\frac{q_X(x)}{p_X(x)} < \frac{1}{1-\alpha}$,
- $VaR_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(h(Y)) \in L^1(\Omega,\mathbb{P})$.

Let us define $g \in \mathfrak{P}$ *by*

$$g(x) = \begin{cases} VaR_{\alpha(x)}^{\mathbb{P}(\cdot|X=x)}(h(Y)) - t^*, & \text{if } x \in \mathcal{S}_X \\ 0, & \text{if } x \notin \mathcal{S}_X, \end{cases}$$

where

$$t^* = \mathbb{E}^{\mathbb{Q}}\left[VaR_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(h(Y))\right]. \tag{6}$$

Then, we have $\forall \tilde{g} \in \mathfrak{P}$,

$$ES_\alpha^{\mathbb{P}}(h(Y)-g(X)) \le ES_\alpha^{\mathbb{P}}(h(Y)-\tilde{g}(X)).$$

Proof. The proof is based on the variational characterization of Rockafellar and Uryasev (see [4] and [5]):

$$\forall Z \in L^1(\Omega,\mathbb{P}), \quad \mathrm{ES}_\alpha^{\mathbb{P}}(Z) = \min_t \left(t + \frac{1}{1-\alpha}\mathbb{E}^{\mathbb{P}}[(Z-t)_+]\right).$$

For any $\tilde{g} \in \mathfrak{P}$, we define the convex function

$$\mathfrak{S}^{\tilde{g}} : (t,\theta) \in \mathbb{R}\times[0,1] \mapsto t + \frac{1}{1-\alpha}\mathbb{E}^{\mathbb{P}}[(h(Y) - (\theta\tilde{g}(X) + (1-\theta)g(X)) - t)_+].$$

It is straightforward to verify that

$$\begin{aligned}&(h(Y) - (\theta\tilde{g}(X) + (1-\theta)g(X)) - t)_+ \\ &\ge (h(Y) - g(X) - t^*)_+ - (t-t^*)1_{h(Y)-g(X)\ge t^*} - \theta(\tilde{g}(X) - g(X))1_{h(Y)-g(X)\ge t^*}.\end{aligned}$$

By taking expectations and reorganizing the terms, we obtain:

$$\mathfrak{S}^{\tilde{g}}(t,\theta) \ge \mathfrak{S}^{\tilde{g}}(t^*,0) + (t-t^*)\left(1 - \frac{1}{1-\alpha}\mathbb{P}(h(Y)-g(X)\ge t^*)\right) - \frac{\theta}{1-\alpha}\mathbb{E}^{\mathbb{P}}\left[(\tilde{g}(X)-g(X))1_{h(Y)-g(X)\ge t^*}\right]. \tag{7}$$

By definition of g and t^*, because $h(Y)$ satisfies the (H) hypothesis, we have:

$$\mathbb{P}(h(Y)-g(X)\ge t^*|X) = \mathbb{P}\left(h(Y) \ge \mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(h(Y))\middle| X\right) = 1-\alpha(X) = (1-\alpha)\frac{q_X(X)}{p_X(X)}.$$

Therefore,

$$\mathbb{P}(h(Y)-g(X)\ge t^*) = (1-\alpha)$$

and

$$\mathbb{E}^{\mathbb{P}}\left[(\tilde{g}(X)-g(X))1_{h(Y)-g(X)\ge t^*}\right] = \mathbb{E}^{\mathbb{P}}\left[(\tilde{g}(X)-g(X))(1-\alpha)\frac{q_X(X)}{p_X(X)}\right] = (1-\alpha)\mathbb{E}^{\mathbb{Q}}\left[\tilde{g}(X)-g(X)\right] = 0.$$

Eq. (7) states therefore that $\mathfrak{S}^{\tilde{g}}(t,\theta) \ge \mathfrak{S}^{\tilde{g}}(t^*,0)$. By taking $\theta = 1$, we obtain the result:

$$\mathrm{ES}_\alpha^{\mathbb{P}}(h(Y)-\tilde{g}(X)) = \min_t \mathfrak{S}^{\tilde{g}}(t,1) \ge \mathfrak{S}^{\tilde{g}}(t^*,0) \ge \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y)-g(X)). \qquad \square$$

2.2. Special result in the one-dimensional case

In the case where Y is real-valued (i.e. $k=1$), if h is a continuous and increasing function,[3] the assumption that $h(Y)$ satisfies (H) in Theorem 2.1, can be replaced by that of Y satisfying (H). Furthermore, in that case, Eq. (6) writes:

$$g(x) = \begin{cases} h\left(\mathrm{VaR}_{\alpha(x)}^{\mathbb{P}(\cdot|X=x)}(Y)\right) - \mathbb{E}^{\mathbb{Q}}\left[h\left(\mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(Y)\right)\right], & \text{if } x \in \mathcal{S}_X \\ 0, & \text{if } x \notin \mathcal{S}_X. \end{cases} \tag{8}$$

The strict monotonicity assumption for obtaining Eq. (8) can in fact be relaxed by using approximation techniques. This leads to the following corollary.

Corollary 2.2. *Assume that*

- *Y is real-valued (i.e. $k=1$) and satisfies the (H) hypothesis,*
- *$\forall x \in \mathcal{S}_X, \alpha(x) := 1-(1-\alpha)\frac{q_X(x)}{p_X(x)} > 0$, i.e. $\frac{q_X(x)}{p_X(x)} < \frac{1}{1-\alpha}$,*
- *h is nondecreasing and $h\left(\mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(Y)\right) \in L^1(\Omega, \mathbb{P})$.*

Let us define $g \in \mathfrak{P}$ by

$$g(x) = \begin{cases} h\left(\mathrm{VaR}_{\alpha(x)}^{\mathbb{P}(\cdot|X=x)}(Y)\right) - \mathbb{E}^{\mathbb{Q}}\left[h\left(\mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(Y)\right)\right], & \text{if } x \in \mathcal{S}_X \\ 0, & \text{if } x \notin \mathcal{S}_X. \end{cases}$$

We have $\forall \tilde{g} \in \mathfrak{P}$,

$$\mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - g(X)) \le \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - \tilde{g}(X)).$$

Proof. Let us introduce a sequence of increasing functions $(h_n)_n$ by $h_n(y) = h(y) + \frac{1}{(n+1)\pi}\arctan(y)$. It approximates h in the sense that $\|h_n - h\|_\infty = \frac{1}{n+1} \to_{n\to+\infty} 0$. By Theorem 2.1 and Eq. (8), we know that if we define $g_n \in \mathfrak{P}$ by

$$g_n(x) = \begin{cases} h_n\left(\mathrm{VaR}_{\alpha(x)}^{\mathbb{P}(\cdot|X=x)}(Y)\right) - \mathbb{E}^{\mathbb{Q}}\left[h_n\left(\mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(Y)\right)\right], & \text{if } x \in \mathcal{S}_X \\ 0, & \text{if } x \notin \mathcal{S}_X, \end{cases}$$

then we have $\|g_n - g\|_\infty = \frac{2}{n+1} \to_{n\to+\infty} 0$ and $\forall \tilde{g} \in \mathfrak{P}$,

$$\begin{aligned} &\mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - g(X)) \\ &\le \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - h_n(Y)) + \mathrm{ES}_\alpha^{\mathbb{P}}(h_n(Y) - g_n(X)) + \mathrm{ES}_\alpha^{\mathbb{P}}(g_n(X) - g(X)) \\ &\le \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - h_n(Y)) + \mathrm{ES}_\alpha^{\mathbb{P}}(h_n(Y) - \tilde{g}(X)) + \mathrm{ES}_\alpha^{\mathbb{P}}(g_n(X) - g(X)) \\ &\le \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - h_n(Y)) + \mathrm{ES}_\alpha^{\mathbb{P}}(h_n(Y) - h(Y)) + \mathrm{ES}_\alpha^{\mathbb{P}}(h(Y) - \tilde{g}(X)) + \mathrm{ES}_\alpha^{\mathbb{P}}(g_n(X) - g(X)). \end{aligned}$$

The result is then obtained by taking the limit $n \to +\infty$ since $\mathrm{ES}_\alpha^{\mathbb{P}}$ is $\frac{1}{1-\alpha}$-Lipschitz with respect to the L^1 norm and since, we have $h_n(Y) \xrightarrow[n\to+\infty]{L^1(\Omega,\mathbb{P})} h(Y)$ and $g_n(X) \xrightarrow[n\to+\infty]{L^1(\Omega,\mathbb{P})} g(X)$. □

2.3. Comments and examples

Theorem 2.1 states that for finding a function g that minimizes (5), one simply needs to compute conditional quantiles (this is a classical problem in statistics, at least when $p_X = q_X$, usually solved numerically through quantile regressions on a basis of splines). Interestingly, thanks to Corollary 2.2, the problem can in fact be solved analytically in numerous cases of interest in finance. If, under $\mathbb{P}$, the cumulative distribution functions of X and Y are respectively F_X and F_Y, both assumed to be continuous and strictly increasing, and if the dependence structure of (X, Y) is that of a Student copula with "correlation" parameter ρ, and degree of freedom $\nu > 1$, then[4]

$$\mathrm{VaR}_{\alpha(X)}^{\mathbb{P}(\cdot|X)}(Y) = F_Y^{-1}\left(t_\nu\left(\rho t_\nu^{-1}(F_X(X)) + \sqrt{1-\rho^2}\sqrt{\frac{\nu + t_\nu^{-1}(F_X(X))^2}{\nu+1}}\, t_{\nu+1}^{-1}(\alpha(X))\right)\right), \tag{9}$$

[3] Similar results can be obtained for decreasing functions. The only difference is a change in the threshold in the Value at Risk from $\alpha(X)$ to $1-\alpha(X)$.

[4] We implicitly assume that $\alpha(x) > 0, \forall x \in \mathbb{R}$.

where t_r is the cumulative distribution function of a standard Student variable with r degrees of freedom. In particular, if (X, Y) is a Student vector $t\left(\begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix}, \begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}, \nu\right)$, and if $p_X = q_X$, then the optimal payoff $g(X)$ for hedging a call with strike K on Y is (up to an additive constant) equal to:

$$\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X) + \sqrt{1-\rho^2}\sigma_Y\sqrt{\frac{\nu + \frac{(X-\mu_X)^2}{\sigma_X^2}}{\nu+1}}t_{\nu+1}^{-1}(\alpha) - K\right)_+ . \tag{10}$$

Interestingly, Eq. (10) simplifies to

$$\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X) + \sqrt{1-\rho^2}\sigma_Y N^{-1}(\alpha) - K\right)_+ , \tag{11}$$

where N is the cumulative distribution functions of a standard Gaussian distribution, in the limit case $\nu \to +\infty$ corresponding to a Gaussian vector (X, Y). If $\rho > 0$, then this corresponds to the payoff associated with $\rho\frac{\sigma_Y}{\sigma_X}$ calls on X, with strike $K_X = \mu_X + \frac{K-\mu_Y-\sqrt{1-\rho^2}\sigma_Y N^{-1}(\alpha)}{\rho\frac{\sigma_Y}{\sigma_X}}$.

Acknowledgements

This research has been conducted with the support of the Research Initiative "Modélisation des marchés actions, obligations et dérivés" financed by HSBC France under the aegis of the Europlace Institute of Finance. The author would like to thank Arthur Charpentier (UQAM and "Université de Rennes-1"), Nicolas Grandchamp des Raux (HSBC), Sophie Laruelle ("Université Paris-Est Créteil"), Jean-Michel Lasry ("Institut Louis-Bachelier"), Gilles Pagès ("Université Pierre-et-Marie-Curie"), Jiang Pu ("Université Paris-Diderot" and "Institut Europlace de finance"), and Mohamed Sassi (HSBC) for the conversations he had with them on the topic addressed in this note and on related topics.

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